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Nonparametric Estimation for Locally Stationary GARCH Processes with Laplace-Distributed Noise

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Abstract

This paper investigates the Quasi-Maximum Likelihood Estimation (QMLE) method for time-varying GARCH models under the assumption that the innovations follow the standard Laplace distribution. We consider a locally stationary framework. The estimation process is based on a kernel-weighted likelihood approach. We establish the consistency and asymptotic normality of the proposed estimators under regularity assumptions. The efficacy of the method is demonstrated through simulation studies and an application to real-world financial data, highlighting the practical advantages of modeling heavy-tailed innovations with the Laplace distribution.

Keywords: quasi-maximum likelihood estimator, Laplace distribution, strong consistency, asymptotic normality, tvGARCH.

1. Introduction

Locally stationary processes extend classical time series models by allowing statistical properties to evolve over time, making them well-suited for capturing dynamic patterns beyond stationarity. Within this framework, the Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model, originally designed for stationary data to model volatility clustering, has been adapted to time-varying and locally stationary settings, enhancing its ability to capture evolving volatility.

The GARCH model, introduced by Bollerslev (1986), along with its time-varying extensions, has been extensively studied since its inception and has quickly become a staple for modeling volatility in financial time series. Building on this, Hansen and Lunde (2005) evaluated the performance of these models in forecasting financial volatility. Dahlhaus (1997) laid the theoretical groundwork for local-stationary processes, inspiring subsequent extensions of GARCH to non-stationary settings. Dahlhaus and Polonik (2006) further developed the concept by exploring the asymptotic properties of the QMLE for time-varying GARCH (tvGARCH) processes with Gaussian errors. Djaballah and Kerrar (2021) provided key contributions to the QMLE for GARCH models with non-Gaussian noise. Fryzlewicz and van Bellegem (2011) introduced methodologies for decomposition time series into locally stationary components. More recently, Ahlgren, Alexander, and Teräsvirta (2024) introduced a new formulation of the tvGARCH model. These studies collectively highlight the GARCH model, its non-stationary version, and the estimation of its parameters using maximum likelihood estimation.

In this paper, we propose a novel extension to the time-varying GARCH (tvGARCH) framework by incorporating Standard Laplace distributed errors. While previous studies have primarily focused on classical models under the assumption of Gaussian innovations, Bollerslev (1986) utilized Gaussian Maximum Likelihood Estimation (MLE) to establish the foundational GARCH model, and subsequent works have largely retained this framework due to its analytical simplicity. Recently, Bardet, Doukhan, and Wintenberger (2022) investigated the M-estimator for causal affine processes within the Gaussian framework. Also, Bardet (2025) assumes Gaussian errors in the case of time-varying GARCH. However, the Gaussian assumption often fails to adequately capture the heavy-tailed nature of many real-world datasets, particularly in financial time series. This limitation aligns with the findings of Mandelbrot (1963), who demonstrated that the first differences of the logarithm of cotton and common stock prices exhibit fatter tails than those compatible with the normal distribution.

The Laplace distribution, while resembling the normal distribution in density, fundamentally differs by replacing the squared term $(x-\mu)^2$ with the absolute term $|x-\mu|$. This yields a sharper peak at the mean and heavier tails than both Gaussian and Student distributions, enabling more accurate modeling of white noise in financial time series and capturing extreme events beyond the reach of Gaussian-based models. Such properties enhance the applicability of the tvGARCH framework to non-stationary and locally stationary processes with heavy-tailed innovations, offering deeper insights into conditional distributions under non-Gaussian assumptions and supporting applications in derivative pricing and risk management. Its robustness to outliers further underpins its use in hydrology for extremes such as annual maximum rainfall and river discharge, and in navigation for modeling position errors in complex aircraft or vessel systems Hsu (1979). In machine learning and big data, the Laplace distribution's ability to accommodate variability and heterogeneity strengthens predictive accuracy, and its utility in modeling irregular data patterns continues to expand across applied sciences and engineering.

The remainder of the paper is organized as follows. Section 2 presents the main theoretical results, including the consistency and asymptotic normality of the estimator. Section 3 reports numerical experiments on simulated and financial data to evaluate our estimator's performance, together with comparisons to classical models. Section 4 contains the proofs of the main results. Section 5 concludes the paper.

2. Main results

2.1. Model, estimator, and assumptions

Our methodology hinges on the notion of local-stationarity, which is characterized by the existence of a function that governs the behavior of the process in local neighborhoods. Leveraging this concept, we use a kernel-based estimator that minimizes a localized contrast function, ensuring robustness and efficiency in estimation. Importantly, we establish the consistency and asymptotic normality of this estimator.

The GARCH(p,q) process is given as the stationary solution for the system of equations:

$$X_t = \eta_t \sigma_t,$$
$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2,$$

where $\{\eta_t\}$ are independent and identically distributed (i.i.d.) random variables following a Laplace distribution. The standard GARCH model may fail to capture the evolving nature of volatility. To address this, we replace the parameters ω, $\{\alpha_i\}$, and $\{\beta_j\}$ with time-varying functions $\omega(\cdot)$, $\{\alpha_i(\cdot)\}$, and $\{\beta_j(\cdot)\}$, leading to a time-varying GARCH (tvGARCH) model as described in Dahlhaus (2012). We consider below the tvGARCH(p, q) processes defined by:

$$\begin{aligned} X_{t,N} &= \eta_t \sigma_{t,N}, \\ \sigma_{t,N}^2 &= \omega\left(\frac{t}{N}\right) + \sum_{i=1}^{p} \alpha_i \left(\frac{t}{N}\right) X_{t-i,N}^2 + \sum_{j=1}^{q} \beta_j \left(\frac{t}{N}\right) \sigma_{t-j,N}^2, \end{aligned} \tag{1}$$

where $\{\eta_t\}$ are independent and identically distributed (i.i.d.) random variables with a Laplace distribution, and N represents the sample size. At a fixed time u_0 (denote $u_0 = \frac{t_0}{N}$, rescaling time makes it possible to establish a framework for a meaningful asymptotic theory), the corresponding stationary approximation of the process $X_{t,N}$ is given by:

$$\tilde{X}_t(u_0) = \eta_t \sigma_t(u_0),$$
$$\tilde{\sigma}_t(u_0)^2 = \omega(u_0) + \sum_{i=1}^{p} \alpha_i(u_0) \tilde{X}_{t-i}^2 + \sum_{j=1}^{q} \beta_j(u_0) \tilde{\sigma}_{t-j}^2(u_0).$$

Under the condition that $\sup_u \left(\sum_{i=1}^{p} \alpha_i(u) + \sum_{j=1}^{q} \beta_j(u)\right) < 1$, Dahlhaus and Rao (2006) proved that $X_{t,N}^2 = \tilde{X}_t(u_0)^2 + O_p\left(\left|\frac{t}{N} - u_0\right| + \frac{1}{N}\right)$ (see also Subba Rao (2006)).

To estimate the parameter vector $\boldsymbol{\theta} = (\omega(t), \alpha_1(t), \ldots, \alpha_p(t), \beta_1(t), \ldots, \beta_q(t))'$ in (1), whose true value is denoted by $\boldsymbol{\theta}_0$, we use an approximation of the conditional quasi-likelihood (4), constructed under the assumption that the innovations η_t follow the standard Laplace distribution. By assumption, the Laplace(0,1) distribution has the density $f(x) = 0.5 \times e^{-|x|}$. Thus, the logarithm of the conditional quasi likelihood is given by:

$$q_{t,N} = q_{t,N} = \log \sigma_{t,N}(\boldsymbol{\theta}) + \frac{|X_t|}{\sigma_{t,N}(\boldsymbol{\theta})}.$$

Based on Dahlhaus's framework, multiplying by the kernel function essentially smooths the estimation process by giving more importance to the data points that are closer to the target point u_0. This approach allows the likelihood function to better capture the local behavior of the time series around u_0, which leads to more accurate and adaptive estimates of the parameters. Therefore, the quasi-maximum likelihood estimator at the point u_0 is as follows:

$$\widehat{\boldsymbol{\theta}}_N(u_0) := \arg\min_{\boldsymbol{\theta} \in \Theta} \mathcal{L}_N(u_0, \theta). \tag{2}$$

Where :

$$\mathcal{L}_N(u_0, \boldsymbol{\theta}) = \frac{1}{bN} \sum_{t=1}^{N} K\left(\frac{u_0 - \frac{t}{N}}{b}\right) q_{t,N}(u_0, \boldsymbol{\theta}) \tag{3}$$

with $K(x)$ is a symmetric kernel function of bounded variation satisfying $\int_{-1/2}^{1/2} K(x)dx = 1$ and $\int_{-1/2}^{1/2} xK(x)dx = 0$. Here, b is referred to as the bandwidth, a smoothing parameter. The likelihood function can be expressed as (4), redefined using the stationary approximation

$\tilde{X}_t(u_0)$ of the process $X_{t,N}$ to simplify analysis, ensure asymptotic validity, and facilitate parameter estimation by leveraging the locally stationary behavior of the process:

$$\widetilde{\mathcal{L}}_N(u_0, \boldsymbol{\theta}) = \frac{1}{bN} \sum_{t=1}^{N} K\left(\frac{u_0 - \frac{t}{N}}{b}\right) \widetilde{q}_{t,N}(u_0, \boldsymbol{\theta}) \tag{4}$$

with $\widetilde{q}_{t,N}(u_0, \boldsymbol{\theta}) = \log \widetilde{\sigma}_t(u_0) + \frac{|\tilde{X}_t(u_0)|}{\widetilde{\sigma}_t(u_0)}$.

Next, we delve into the asymptotic behavior of the maximum likelihood estimator defined in (2) for the time-varying generalized autoregressive conditional heteroscedastic (tvGARCH (1)) model with Laplace errors. A key distinction within the framework of asymptotic normality is the difference in convergence rates between localized and stationary QMLE. Specifically, the convergence rate is $\sqrt{n}$ in the stationary case, whereas it becomes $\sqrt{Nb}$ in the non-stationary setting, reflecting the influence of local variations. To establish these results rigorously, we introduce the following assumptions (H) as the foundation for the analysis.

Assumptions (H)

- H_1 : The true parameter vector $\boldsymbol{\theta_0} \in \Theta$, where $\boldsymbol{\theta}$ is a compact set in $\mathbb{R}^d$, and $\boldsymbol{\theta_0}$ lies in the interior of Θ. This ensures that the parameter space is bounded and excludes boundary points, which simplifies the mathematical treatment of consistency and asymptotic normality.

- H_2 : The sequence of stochastic processes $\{X_{t,N} : t = 1, ..., N\}$ has a time-varying GARCH representation defined in (2) where the parameters satisfy the following properties.

  There exist a positive constant ρ, $C < \infty$, Q, ν (with $0 < \nu < 1$) and a positive sequence $\{\ell(j)\}$ such that: $\inf_u \alpha_0(u) > \rho$ and

  $$\sup_u \alpha_j(u) \leq \frac{Q}{\ell(j)},$$

  $$Q \sum_{j=1}^{\infty} \frac{1}{\ell(j)} \leq (1 - \nu),$$

  $$|\alpha_j(u) - \alpha_j(\nu)| \leq C \frac{|u - \nu|}{\ell(j)},$$

  where $\{\ell(j)\}$ satisfies

  $$\sum_{j \geq 1} \frac{j}{\ell(j)} < \infty$$

  ensuring the decay is rapid enough for certain summation bounds to hold.

  Notation: $\mathrm{Lip}_p(\Theta)$ denotes the class of functions $f : \Theta \to \mathbb{R}$ that are Lipschitz continuous with respect to the L_p norm. That is, $f \in \mathrm{Lip}_p(\Theta)$ if there exists a constant $L < \infty$ such that

  $$|f(\theta_1) - f(\theta_2)| \leq L \,\|\theta_1 - \theta_2\|_p, \quad \forall\, \theta_1, \theta_2 \in \Theta.$$

- H_3 : Let Θ be the compact set

  $$\boldsymbol{\theta} = \Big\{\boldsymbol{\theta} = \Big(\omega, \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q\Big) : \\ \sum_{j=1}^{q} \beta_j + \sum_{i=1}^{p} \alpha_i \leq 1,\ \ \rho_1 \leq \omega \leq \rho_2,\ \ \rho_1 \leq \alpha_i,\ \ \rho_1 \leq \beta_j\Big\},$$

where $0 < \rho_1 < \rho_2 < \infty$. For each $u \in (0,1)$, we assume $\theta_u \in \mathrm{Int}(\boldsymbol{\theta})$, where $\theta(u) = (\omega(u), \alpha_1(u), \ldots, \beta_1(u), \ldots)$. This immediately yields to:

$$\frac{\partial \sigma_{t,N}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \frac{1}{\sigma_{t,N}(\boldsymbol{\theta})} \leq \frac{1}{\rho_1} \text{ and } \frac{\partial \tilde{\sigma}_{t,N}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \frac{1}{\tilde{\sigma}_{t,N}(\boldsymbol{\theta})} \leq \frac{1}{\rho_1}.$$

This guarantee smooth variation in the conditional variance.

- H_4 (Locally-stationary assumption): There exist $K_{\boldsymbol{\theta}} > 0$ and a continuous function $\boldsymbol{\theta}_0 : u \in [0,1] \mapsto \boldsymbol{\theta}_0(u) \in \mathbb{R}^d$, such that

$$\|\boldsymbol{\theta}_N - \boldsymbol{\theta}_0(u)\| \leq K_{\boldsymbol{\theta}} \left| u - \frac{t}{N} \right|^{\gamma}, \qquad \text{for any } N \in \mathbb{N}^* \text{ and } 1 \leq t \leq N.$$

This condition describes a Hölder-type behavior for the approximation of $\boldsymbol{\theta}_N$ by the function $\boldsymbol{\theta}_0$. When $t/N \underset{N\to+\infty}{\longrightarrow} u$ then $\boldsymbol{\theta}_N \underset{N\to+\infty}{\longrightarrow} \boldsymbol{\theta}_0(u)$ and the behavior of $(X_{t,N})$ is similar to its so-called stationary version.

2.2. Consistency

Lemma 2.1. *Suppose $\{X_{t,N} : t = 1, \ldots, N\}$ is a* tvGARCH(p,q) *process satisfying Assumptions H_2 and H_3. Let $\tilde{\mathcal{L}}_N(u_0, \boldsymbol{\theta})$, be the likelihood function defined above in (4), and $\mathcal{L}(u_0, \boldsymbol{\theta}) := \mathbb{E}(\tilde{q}_0(u_0, \boldsymbol{\theta}))$, and let the bias caused by the deviation from stationarity $\mathcal{B}_{u_0,N}$ defined as:*

$$\mathcal{B}_{u_0,N} = \mathcal{L}_N(u_0, \boldsymbol{\theta}) - \tilde{\mathcal{L}}_N(u_0, \boldsymbol{\theta}).$$

Then

$$\sup_{\boldsymbol{\theta}\in\Theta} \left| \tilde{\mathcal{L}}_N(u_0, \boldsymbol{\theta}) - \mathcal{L}(u_0, \boldsymbol{\theta}) \right| \xrightarrow{\mathbb{P}} 0,$$
$$\sup_{\boldsymbol{\theta}\in\Theta} |\mathcal{B}_{u_0,N}(\boldsymbol{\theta})| \xrightarrow{\mathbb{P}} 0,$$

where $b \to 0$ and $bN \to \infty$ as $N \to \infty$.

Corollary 2.2. *Let $\mathcal{L}_{t,N}(u_0, \boldsymbol{\theta}) = \frac{1}{bN} \sum_{t=1}^{N} K\left(\frac{u_0 - \frac{t}{N}}{b}\right) q_{t,N}(u_0, \boldsymbol{\theta})$. Under the conditions in Lemma 2.1, we have*

$$\sup_{\boldsymbol{\theta}\in\Theta} |\mathcal{L}_N(u_0, \boldsymbol{\theta}) - \mathcal{L}(u_0, \boldsymbol{\theta})| \xrightarrow{\mathbb{P}} 0. \tag{5}$$

Theorem 2.3. *Let $X_{t,N}$ be the solution of a locally stationary model satisfying the assumptions above. Then for any $u \in (0,1)$, the estimator $\hat{\boldsymbol{\theta}}(u)$ consistently estimates $\boldsymbol{\theta}_{\mathbf{0}}(u)$*

$$\widehat{\boldsymbol{\theta}}(u) \underset{N\to+\infty}{\xrightarrow{\mathbb{P}}} \boldsymbol{\theta}_{\mathbf{0}}(u), \quad \text{if} \quad b \underset{N\to+\infty}{\longrightarrow} 0 \quad \text{and} \quad bN \underset{N\to+\infty}{\longrightarrow} \infty.$$

Moreover, if $p > 1$ and $N^{1-1/p}b \underset{N\to+\infty}{\longrightarrow} \infty$ then for any $\varepsilon > 0$ then $\widehat{\boldsymbol{\theta}}$ uniformly consistently estimates $\boldsymbol{\theta}_0$:

$$\sup_{u\in[0,1]} \left\| \widehat{\boldsymbol{\theta}}(u) - \boldsymbol{\theta}_0(u) \right\| \underset{N\to+\infty}{\xrightarrow{\mathbb{P}}} 0.$$

Having established the consistency of the estimator $\widehat{\boldsymbol{\theta}}(u)$, we now turn to its asymptotic distribution to demonstrate asymptotic normality under the same set of assumptions.

2.3. Asymptotic normality

Theorem 2.4. *Under Assumptions (H), and* $\mathbb{E}\left\|\dfrac{\partial\tilde{\mathcal{L}}_N\left(u_0,\boldsymbol{\theta}\right)}{\partial\boldsymbol{\theta}}\right\|^2<\infty$*, let* $b_N>0$ *be a sequence such that* $bN\underset{N\to+\infty}{\longrightarrow}\infty$ *and* $b^{1+2\gamma}N\underset{N\to+\infty}{\longrightarrow}0$*. Then, for any* $u\in(0,1)$*:*

$$\sqrt{Nb}\left(\widehat{\boldsymbol{\theta}}(u)-\boldsymbol{\theta}_0(u)\right)\underset{N\to+\infty}{\overset{\mathcal{L}}{\longrightarrow}}\mathcal{N}_d\left(0,\Gamma^{-1}\left(\boldsymbol{\theta}_0(u)\right)\Sigma\left(\boldsymbol{\theta}_0(u)\right)\Gamma^{-1}\left(\boldsymbol{\theta}_0(u)\right)\right),$$

$\Sigma\left(\boldsymbol{\theta}_0(u)\right)$ *and* $\Gamma\left(\boldsymbol{\theta}_0(u)\right)$ *are positive definite matrices, with*

$$\Sigma\left(\boldsymbol{\theta}_0(u)\right)=\int_{\mathbb{R}}K^2(x)dx\cdot\mathbb{E}\left[\left(\frac{\partial\widetilde{q}_{t,N}(u_0,\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}}\right)\left(\frac{\partial\widetilde{q}_{t,N}(u_0,\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}}\right)'\right],$$

and

$$\Gamma\left(\boldsymbol{\theta}_0(u)\right)=\mathbb{E}\left[\frac{\partial^2\widetilde{q}_{t,N}(u_0,\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}^2}\right]=\mathbb{E}\left(\frac{1}{\sigma_t^4\left(u_0\right)}\frac{\partial\sigma_t^2\left(u_0\right)}{\partial\boldsymbol{\theta}}\frac{\partial\sigma_t^2\left(u_0\right)}{\partial\boldsymbol{\theta}'}\right).$$

Remark 1. The absolute value function satisfies $\frac{\partial}{\partial x}|x|=\operatorname{sign}(x)$, and $\frac{\partial}{\partial x}\operatorname{sign}(x)=2\,\delta(x)$, where $\delta(\cdot)$ denotes the Dirac delta function. Since $\mathbb{P}(X_t=0)=0$, these generalized derivatives are valid for asymptotic analysis, following Boularouk (2023).

Remark 2. This work is related to a broader line of research, including Bardet *et al.* (2022), which addresses similar topics. However, it is not the same, as Bardet assumes Gaussianity in the examples he studies. In contrast, the natural convergence associated with the Laplace distribution exhibits unique features, particularly in terms of differentiability properties.

We also mention other related works, such as Bendjeddou and Sadoun (2025), who studied a GARCH model with negative-binomial errors, and Ahlgren *et al.* (2024), who proposed a new class of time-varying GARCH models by extending the framework to the case where the variance σ^2 in Equation (1) is replaced by $\sigma^2+g(t)$, with g a continuous bounded function.

In Xu, Sun, Aslam, and Sun (2017), the stationary case of the GARCH model with Laplace$(1,1)$ errors was addressed.

3. Numerical experiments

In this section, we use MATLAB to apply the kernel-based estimator to several cases of locally stationary GARCH models with Standard Laplace errors. We then apply this method to real financial data (the N225 index). The MATLAB codes and simulation scripts used in this section are available at: `https://github.com/LSGARCH/QMLEtvGARCH/tree/Laplace`.

We follow the same Monte Carlo simulation steps as in Bardet, Doukhan, and Wintenberger (2020). Specifically, we apply our estimator to simulated data for $N=1000,2000$ and 5000 and computed the maximum likelihood estimators $\widehat{\boldsymbol{\theta}}$. We compare the square root of the mean integrated squared error of the estimator with that of the Gaussian estimator. Finally, we use the Gaussian kernel to simulated data $G(x)=\frac{1}{\sqrt{\pi}}e^{-\frac{x^2}{2}}$. We also use two well known kernels for real data, the uniform kernel $U(x)=\frac{1}{2}\mathbb{I}_{x\in[-1,1]}$ and the Epanechnikov one $E(x)=\frac{3}{4}\left(1-x^2\right)\mathbb{I}_{x\in[-1,1]}$.

The window bandwidth b is a tuning parameter that requires to be chosen. In order to neglect the bias we chose $b=N^{-\lambda}$ with $\lambda=0.35$, inducing $Nb^3\underset{n\to\infty}{\longrightarrow}0$, which is the consistency and the asymptotic normality condition required for $\delta=1$-Hölder function $\boldsymbol{\theta}_0$.

It should be noted that in Bardet *et al.* (2020) and Bardet *et al.* (2022), both in simulations and real data applications, were based on the Gaussian assumption. A comparison between the results of the Gaussian estimator and the Laplace estimator we used can be found in the tables below. Another comparison between the tvGARCH model and the classical GARCH model with Laplace errors was conducted based on AIC and BIC.

3.1. Monte Carlo simulations

Example 1. *tvGARCH(1,0) which means tvARCH(1)*

We assume the following real parameters:

$$\omega\left(\frac{t}{N}\right) = 1 + 0.5\sin\left(5\frac{t}{N}\right), \quad \alpha_1\left(\frac{t}{N}\right) = 0.1 + 0.4\cos^2\left(4\frac{t}{N}\right).$$

For all $1 \le t \le N$ and $n \in \mathbb{N}^*$. Clearly, $\omega(u) = 1 + 0.5\sin(5u)$ and $\alpha_1(u) = 0.1 + 0.4\cos^2(4u)$. Moreover, we assume that (η_t) is modeled as a sequence of independent and identically distributed random variables following Laplace distribution.

To analyze the behavior of our estimator, we generate a dataset of size 1000, 2000, and 5000 in MATLAB using the specified parameters described above. Figure 1 illustrates the simulated data. Subsequently, we applied our estimator to this dataset to evaluate its performance under the given time-varying conditions.

Figure 2 illustrates the paths of the real functions ω and α_1 (in blue), alongside their estimated counterparts $\hat{\omega}$ and $\hat{\alpha}_1$ (in red) for a sample size of $N = 1000, 2000$ and 5000. Table 1 presents the square root of the mean integrated squared error for time-varying autoregressive conditional heteroscedastic processes of order one (tvARCH(1)) with maximum likelihood estimation based on the Laplace distribution for sample sizes $N = 1000$, 2000, and 5000, compared with the maximum likelihood estimation based on the Gaussian distribution.

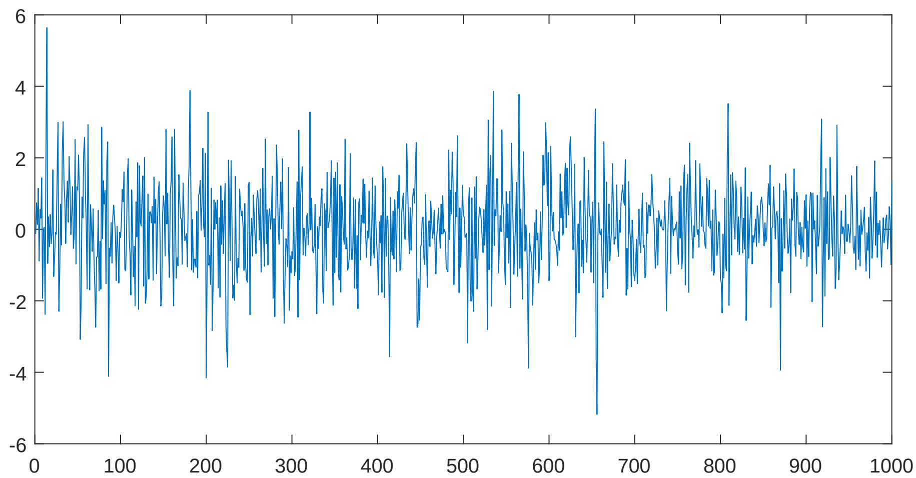

Figure 1: Trajectory of a tvARCH(1)-process with Laplace errors for $N = 1000$

Example 2. *tvGARCH(1,1)*

We assume the following real parameters for the tvGARCH(1,1) model:

$$\omega\left(\frac{t}{N}\right) = 1{+}0.5\sin\left(5\frac{t}{N}\right), \quad \alpha_1\left(\frac{t}{N}\right) = 0.1{+}0.4\cos^2\left(4\frac{t}{N}\right), \quad \beta_1\left(\frac{t}{N}\right) = 0.01{+}0.4\left(\frac{t}{N}\right).$$

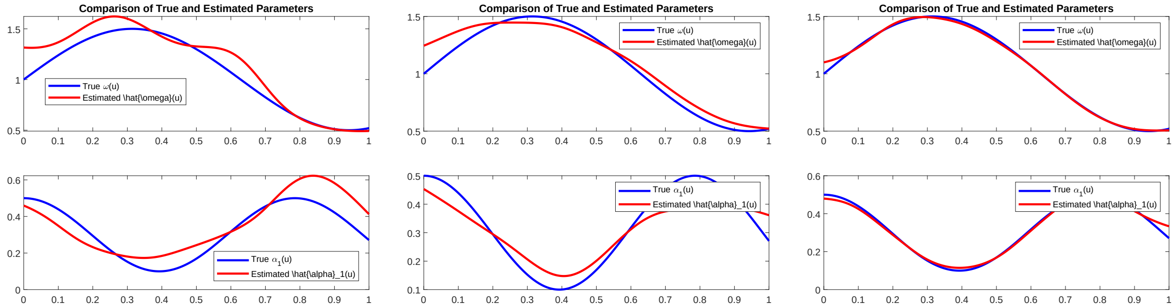

Figure 2: Paths of real functions ω, α_1 (in blue), with estimated $\hat{\omega}$, $\hat{\alpha}_1$ (in red) for N = 1000, 2000, and 5000 respectively

Table 1: Square root of the MISE for tvARCH(1) processes with Laplace MLE for $N = 1000$, 2000, and 5000, compared with the Gaussian MLE

	$\widehat{\omega}$		$\widehat{\alpha}_1$	
N	*Laplace*	$\mathcal{N}$	*Laplace*	$\mathcal{N}$
1000	**0.294**	0.3011	**0.4550**	1.2896
2000	**0.261**	0.4134	**0.2253**	1.1198
5000	**0.144**	0.3560	**0.219**	1.1128

Figure 3 illustrates the simulated data. Figure 4 illustrates the paths of the real functions ω, α_1 and β_1 (in blue), alongside their estimated counterparts $\hat{\omega}$ and $\hat{\alpha}_1$ and $\hat{\beta}_1$ (in red) for a sample size of $N = 1000$, 2000 and 5000.

Table 2 presents the square root of the mean integrated squared error for tvGARCH(1, 1) with maximum likelihood estimation based on the Laplace distribution for sample sizes $N = 1000$, 2000, and 5000, compared with the maximum likelihood estimation based on the Gaussian distribution.

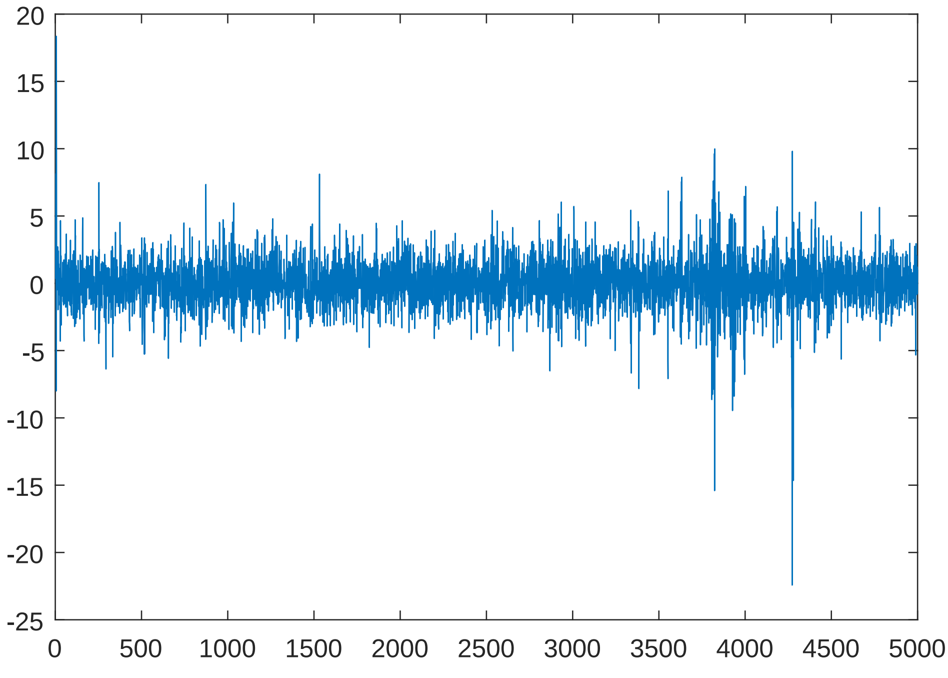

Figure 3: Trajectory of a tvGARCH(1,1) process with Laplace errors for N = 5000

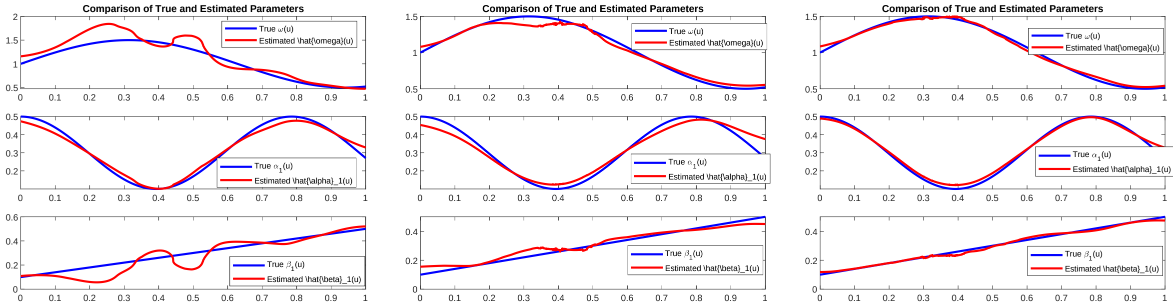

Figure 4: Paths of real functions ω, α_1 (in blue), with estimated $\hat{\omega}$, $\hat{\alpha}_1$ (in red) for N = 1000, 2000, and 5000 respectively

Table 2: Square root of the MISE for tvGARCH(1,1) processes with Laplace MLE for $N =$ 1000, 2000, and 5000, compared with Gaussian MLE

	$\widehat{\omega}$		$\widehat{\alpha}_1$		$\widehat{\beta}_1$	
N	Laplace	$\mathcal{N}$	Laplace	$\mathcal{N}$	Laplace	$\mathcal{N}$
1000	**0.03**	0.512	**0.0932**	0.109	**0.1058**	0.702
2000	**0.0015**	0.422	**0.0110**	0.0125	**0.0301**	0.1121
5000	**1.6585e-04**	0.223	**3.1182e-04**	0.0320	**0.0064**	0.09

Example 3. *tvGARCH(2,1)*

For the tvGARCH(2,1) process, we assume the following real parameters:

$$\omega\left(\frac{t}{N}\right) = 1 + 0.5\sin\left(5\frac{t}{N}\right), \quad \alpha_1\left(\frac{t}{N}\right) = 0.1 + 0.4\cos^2\left(4\frac{t}{N}\right),$$

$$\alpha_2\left(\frac{t}{N}\right) = 0.05 + 0.2 \times (\frac{t}{N})^2, \quad \beta_1\left(\frac{t}{N}\right) = 0.1 + 0.4\left(\frac{t}{N}\right).$$

Figure 5 presents the paths of real functions $\omega(t)$, $\alpha_1(t)$, $\alpha_2(t)$ and $\beta_1(t)$, compared with estimated paths $\hat{\omega}(t)$, $\hat{\alpha}_1(t)$, $\hat{\alpha}_2(t)$ and $\hat{\beta}_1(t)$ for $N = 1000$, 2000, and 5000.

Tables 3, 4 and 5 contain square root of the MISE for tvGARCH(2,1) processes with laplace QMLE for N = 1000, 2000 and 5000, compared with gaussian QMLE.

Table 3: Square root of the MISE for the tvGARCH(2,1) process with $N = 1000$

	ω	α_1	α_2	β_1
Lap	**0.0245**	**0.0694**	**0.0114**	**0.0080**
$\mathcal{N}(0,1)$	0.3011	0.0699	0.072	1.0805

Table 4: Square root of the MISE for the tvGARCH(2,1) process with $N = 2000$

	ω	α_1	α_2	β_1
$Lap(0,1)$	**7.6553e-04**	**0.0255**	**0.0038**	**0.0022**
$\mathcal{N}(0,1)$	0.0529	0.0538	0.1846	0.108

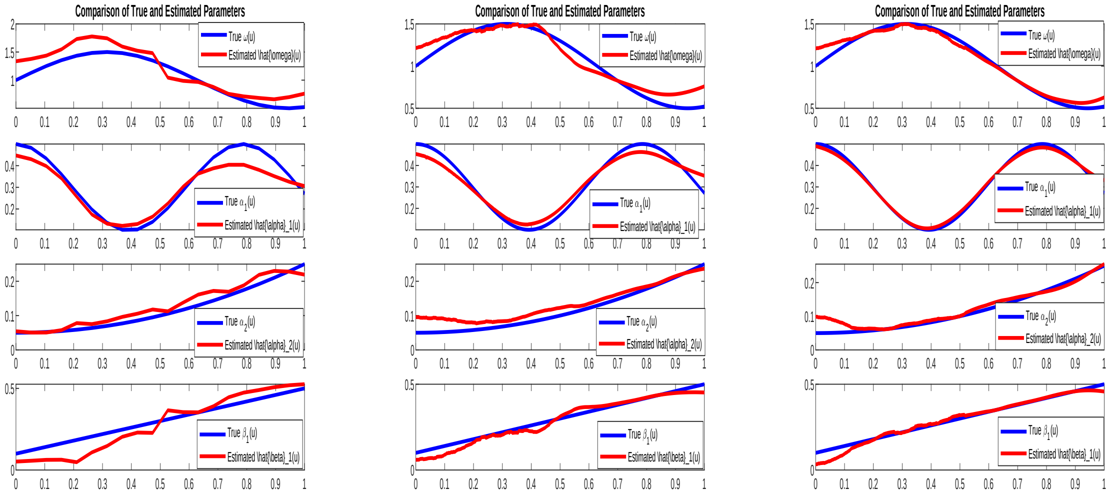

Figure 5: Paths of real functions $\omega(t)$, $\alpha_1(t)$, $\alpha_2(t)$ and $\beta_1(t)$ (in blue), with estimated paths (in red) $\hat{\omega}(t)$, $\hat{\alpha}_1(t)$, $\hat{\alpha}_2(t)$ and $\hat{\beta}_1(t)$ for N = 1000, 2000, and 5000 respectively

Table 5: Square root of the MISE for tvGARCH(2,1) process with $N = 5000$

	ω	α_1	α_2	β_1
$Lap(0,1)$	**1.4773e-04**	**0.0073**	**0.0024**	**8.5919e-05**
$\mathcal{N}(0,1)$	0.187	0.049	0.03	0.591

Conclusion. The results of the simulations demonstrate that the estimators are indeed consistent, and the square root of the MISE reveals that the Laplace QMLE outperforms the Gaussian QMLE. This highlights the robustness of the Laplace-based approach in capturing the dynamics of time-varying GARCH models with heavy-tailed noise.

3.2. Application to financial data

We apply our local non-parametric estimator to financial data, specifically the log-returns of the daily closing values of the N225 index. Specifically, we analyze the log-returns of the daily closing values of the N225 index from January 2009 to December 2019, resulting in a dataset of 2689 observations (see Figure 6). As shown in the same figure, the high kurtosis and significant p-value from the Jarque-Bera test suggest that the distribution has heavier tails and is not normally distributed.

The ARCH test result obtained using MATLAB (Table 6) shows a p-value of 1.1102×10^{-16}, decisively rejecting the null hypothesis of no ARCH effect. This result, combined with the critical value of 3.8415, provides strong evidence of heteroskedasticity in the sequence.

To verify that the data follows a Laplace distribution, we conduct a Kolmogorov-Smirnov test. We obtained the following results: Kolmogorov-Smirnov statistic $D = 0.0147$, p-value $= 0.6863$. p-value $= 0.6863$). The small D statistic and high p-value, compared with the significance level of 0.05, suggest that the data are consistent with a Laplace distribution.

We found the following results after applying our estimator to the data, as shown in Figures 7 and 8.

Conclusion. We observe a clear similarity between the graphs of estimated conditional variance and log returns. Peaks in conditional variance during 2009, 2011, 2013, and 2016 correspond to similar peaks in the log return graph.

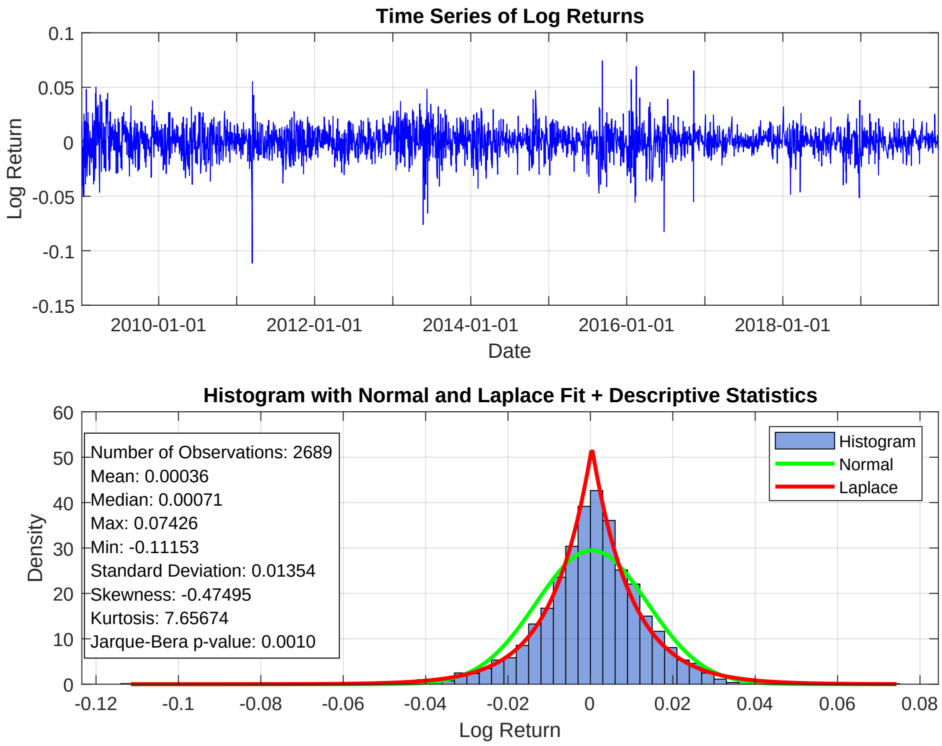

Figure 6: Trajectory of log-returns for the daily closing values of the N225 index (January 2009–December 2019), with histogram of data

Table 6: ARCH test

Test Statistic	p-Value	Critical Value
68.529	1.1102×10^{-16}	3.8415

3.3. Compare with classical GARCH models

The Akaike's information criterion (AIC) and the Bayesian information criterion (BIC) formulas for these tv-GARCH(p,q) processes are given by:

$$\mathrm{AIC} = -2\ell(\hat{\boldsymbol{\theta}}(u)) + 2(1+p+q)$$
$$\mathrm{BIC} = -2\ell(\hat{\boldsymbol{\theta}}(u)) + (1+p+q)\log(n),$$

where $\ell(\hat{\boldsymbol{\theta}}(u))$ represents the log-likelihood evaluated at $\hat{\boldsymbol{\theta}}(u)$. We use this method to compare the models from the previous section with the classical GARCH model on the same dataset. Lower AIC and BIC values indicate a better model fit. Tables 7 and 8 below show the comparison. The uniform kernel and Gaussian kernel are used for estimating tvGARCH(p,q) models.

Conclusion. The tables show that tvGARCH models outperform classical GARCH models, as indicated by lower AIC/BIC values. The Epanechnikov kernel slightly outperforms the uniform kernel and the Gaussian kernel.

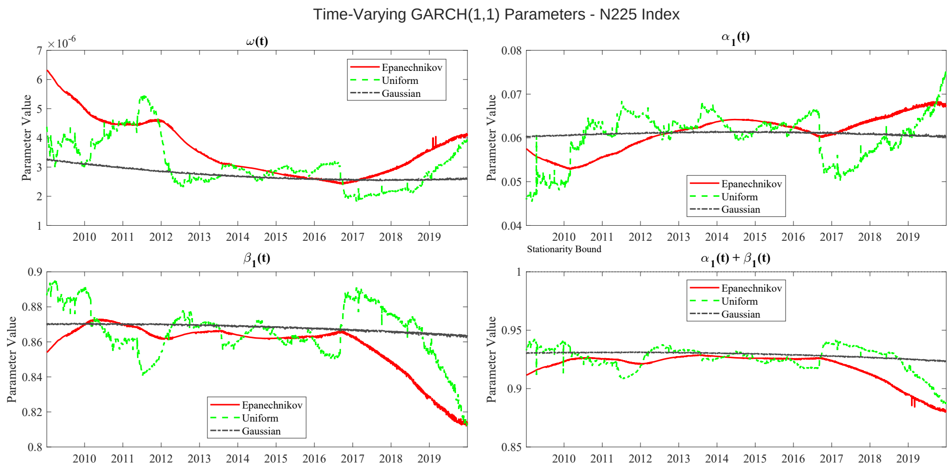

Figure 7: Estimations of $\omega(t), \alpha_1(t),$ and $\beta_1(t)$ for the log-returns of the N225 index (January 2009–December 2019)

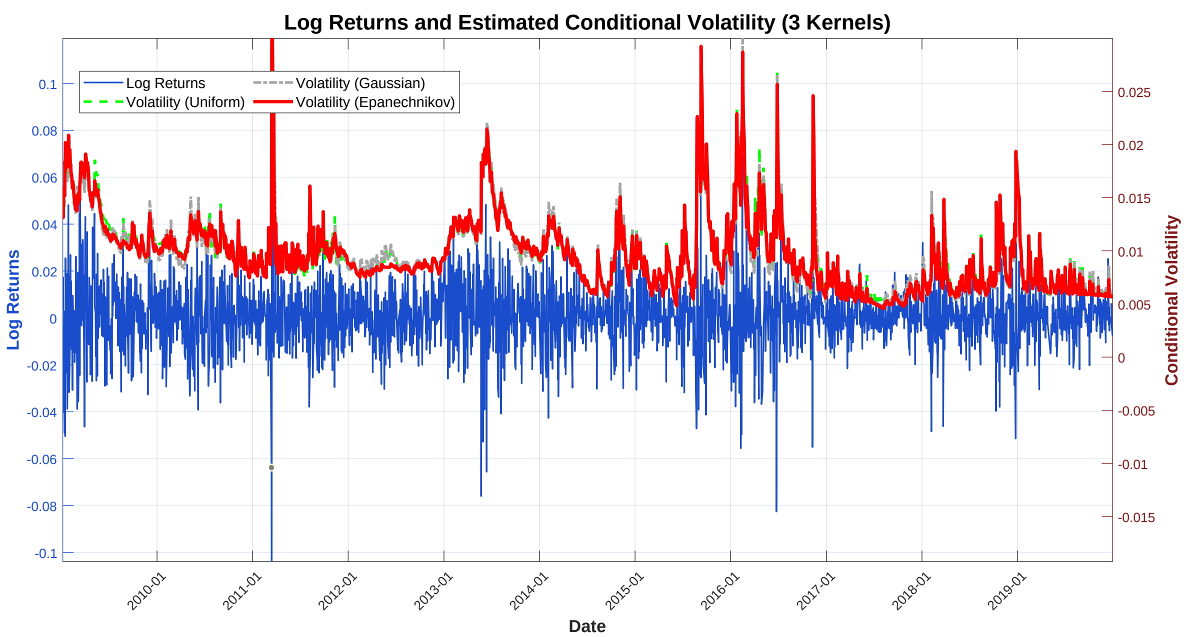

Figure 8: Estimated conditional variance for log-returns path for the N225 index (January 2009–December 2019)

Table 7: AIC values for different values of $p \in \{1,2\}$ and $q \in \{0,1\}$

	Uniform Kernel		Epanechnikov Kernel		Gaussian Kernel		Classical GARCH(p,q)	
$p\backslash q$	0	1	0	1	0	1	0	1
1	7757.17	9045.30	**7751.78**	**9041.62**	7761.96	9047.74	8125.57	9404.73
2	–	9232.30	–	**9227.46**	–	9227.50	–	9615.23

Table 8: BIC values for different values of $p \in \{1,2\}$ and $q \in \{0,1\}$

	Uniform Kernel		Epanechnikov Kernel		Gaussian Kernel		Classical GARCH(p,q)	
$p\backslash q$	0	1	0	1	0	1	0	1
1	7768.38	9062.10	**7762.98**	**9058.42**	7773.16	9064.54	8136.78	9421.53
2	–	9254.70	–	**9249.87**	–	9249.91	–	9643.24

4. Proof of main results

Proof of Lemma 2.1. We aim to show that $\widetilde{\mathcal{L}}_N(u_0, \boldsymbol{\theta})$ converges to $\mathcal{L}(u_0, \boldsymbol{\theta})$.
i) By definition, we have

$$\widetilde{\mathcal{L}}_N(u_0, \boldsymbol{\theta}) = \frac{1}{N} \sum_{t=1}^{N} \frac{1}{b} K\left(\frac{u_0 - \frac{t}{N}}{b}\right) \widetilde{q}_{t,N}(\boldsymbol{\theta}),$$

where

$$\widetilde{q}_{t,N}(\boldsymbol{\theta}) = \log \widetilde{\sigma}_t(u_0) + \frac{\left|\tilde{X}_t(u_0)\right|}{\widetilde{\sigma}_t(u_0)}.$$

ii) Consider the difference

$$\widetilde{\mathcal{L}}_N(u_0, \boldsymbol{\theta}) - \mathcal{L}(u_0, \boldsymbol{\theta}) = \frac{1}{N} \sum_{t=1}^{N} \frac{1}{b} K\left(\frac{u_0 - \frac{t}{N}}{b}\right) \widetilde{q}_{t,N}(\boldsymbol{\theta}) - \mathbb{E}(\widetilde{q}_0(u_0, \boldsymbol{\theta})).$$

Define the deviation term

$$Z_{t,N} = Z_t(\boldsymbol{\theta}, u) = \widetilde{q}_{t,N}(\boldsymbol{\theta}) - \mathbb{E}(\widetilde{q}_0(u, \boldsymbol{\theta})).$$

Applying the law of large numbers, we obtain

$$\frac{1}{N} \sum_{t=1}^{N} Z_{t,N} \overset{N\to\infty}{\to} \mathbb{E}(\widetilde{q}_0(u, \boldsymbol{\theta})).$$

Thus, it follows that

$$\widetilde{\mathcal{L}}_N(u_0, \boldsymbol{\theta}) \to \mathcal{L}(u_0, \boldsymbol{\theta}).$$

This completes the proof of the first part of the lemma. Now we establish the second part, we have

$$\mathcal{B}_{u_0,N}(\boldsymbol{\theta}) = \mathcal{L}_N(u_0, \boldsymbol{\theta}) - \tilde{\mathcal{L}}_N(u_0, \boldsymbol{\theta}),$$

the difference $|q_{t,N}(u_0, \boldsymbol{\theta}) - \tilde{q}_{t,N}(\boldsymbol{\theta})|$ is uniformly small. Specifically, there exists a sequence $\epsilon_N \to 0$ such that:

$$\sup_{1\le t\le N} \sup_{\boldsymbol{\theta}\in\Theta} |q_{t,N}(u_0, \boldsymbol{\theta}) - \tilde{q}_{t,N}(\boldsymbol{\theta})| \le \epsilon_N.$$

Then,

$$|\mathcal{B}_{u_0,N}(\boldsymbol{\theta})| \le \frac{1}{bN} \sum_{t=1}^{N} \left| K\left(\frac{u_0 - t/N}{b}\right)\right| \cdot |q_{t,N}(u_0, \boldsymbol{\theta}) - \tilde{q}_{t,N}(\boldsymbol{\theta})| \le \epsilon_N \cdot \frac{1}{bN} \sum_{t=1}^{N} \left| K\left(\frac{u_0 - t/N}{b}\right)\right|.$$

The term $\frac{1}{bN}\sum_{t=1}^{N}|K((u_0-t/N)/b)|$ converges to $\int|K(u)|\,du<\infty$ by the Riemann sum approximation. Hence,

$$\sup_{\boldsymbol{\theta}\in\Theta}|\mathcal{B}_{u_0,N}(\boldsymbol{\theta})|\leq\epsilon_N\cdot O_p(1)\xrightarrow{\mathbb{P}}0.$$

This completes the proof of the second part.

□

Proof of Theorem 2.3. Using (5), we have pointwise convergence:

$$\mathcal{L}_N(u_0,\boldsymbol{\theta})\xrightarrow{\mathbb{P}}\mathcal{L}(u_0,\boldsymbol{\theta}),$$

since $\widehat{\boldsymbol{\theta}}(u)=\operatorname{argmin}_{\boldsymbol{\theta}}\mathcal{L}(u_0,\boldsymbol{\theta})$. It follows that:

$$\mathcal{L}_N(u_0,\widehat{\boldsymbol{\theta}}(u_0))\leq\mathcal{L}_N(u_0,\boldsymbol{\theta}_0(u_0))\xrightarrow{\mathbb{P}}\mathcal{L}(u_0,\boldsymbol{\theta}_0(u_0))\leq\mathcal{L}(u_0,\widehat{\boldsymbol{\theta}}(u_0)).$$

Using (5) again, we obtain:

$$\mathcal{L}_N(u_0,\widehat{\boldsymbol{\theta}}(u_0))\xrightarrow{\mathbb{P}}\mathcal{L}(u_0,\boldsymbol{\theta}_0(u_0)).$$

From the continuity of $\mathcal{L}(u_0,\cdot)$ and the compactness of the parameter space, we conclude:

$$\widehat{\boldsymbol{\theta}}(u_0)\xrightarrow{\mathbb{P}}\boldsymbol{\theta}_0(u_0),$$

provided that $\mathcal{L}(u_0,\cdot)$ has a unique minimizer. Note that $\mathcal{L}(u_0,\cdot)$ corresponds to the stationary case considered in Boularouk (2023). □

Proof of Theorem 2.4. The asymptotic normality of the estimator is established through a Taylor expansion of the quasi-likelihood function around $\boldsymbol{\theta}_{\mathbf{0}}(u)$.

The derivative of the likelihood function $\frac{\partial\mathcal{L}_N(u,\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}$ generates δ-functions. These functions are differentiable on $\boldsymbol{\theta}$ except for a subspace with Lebesgue measure zero. In the case of the Laplace distribution, the likelihood function involves the term:

$$q_{t,N}(\boldsymbol{\theta})=\log\sigma_t+\frac{|X_{t,N}|}{\sigma_t},$$

which behaves similarly to the Least Absolute Deviation $|X_{t,N}|$. This function is differentiable except at $X_t=0$. Since we assume $P(X_t=0)=0$, the derivative can still be computed in a generalized sense. The absolute value function introduces a discontinuity at $x=0$, where its derivative is given by $\operatorname{sign}(x)$. However, the sign function $\operatorname{sign}(x)$ is not continuous at $x=0$, making it non-differentiable at this point in the usual sense. To handle this, we use the fact that differentiating $\operatorname{sign}(x)$ results in a Dirac delta function:

$$\frac{\partial}{\partial x}\operatorname{sign}(x)=2\delta(x).$$

Thus, at $X_t=0$, the derivative generates a Dirac delta function, which concentrates probability mass at that point. Therefore, the derivative of the likelihood function $\mathcal{L}_{t,N}(u,\boldsymbol{\theta})$ is written as:

$$\frac{\partial\mathcal{L}_N(u,\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}=\frac{1}{bN}\sum_{t=1}^{N}K(\frac{u_0-\frac{t}{N}}{b})\frac{\partial q_{t,N}(u,\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}.$$

This formulation allows us to incorporate the non-differentiability of the Laplace likelihood while maintaining a valid derivative expression in a generalized sense. For more details, see Boularouk (2023).

We have $\dfrac{\partial \widetilde{q}_{t,N}(u_0,\boldsymbol{\theta}_0)}{\partial \boldsymbol{\theta}} = \dfrac{\partial}{\partial \boldsymbol{\theta}}\left(\log \widetilde{\sigma}_t(u_0) + \frac{|\tilde{X}_t(u)|}{\widetilde{\sigma}_t(u_0)}\right)$ that satisfies a multidimensional Central Limit Theorem. We notice that $\dfrac{\partial \mathbb{E}\left(\widetilde{q}_{t,N}(u_0,\boldsymbol{\theta}_0)|\mathcal{F}_0\right)}{\partial \boldsymbol{\theta}} = 0$ as $(\boldsymbol{\theta_0})$ is the unique minimiser of $\mathbb{E}\left(\widetilde{q}_{t,N}(u_0,\boldsymbol{\theta}_0)|\mathcal{F}_0\right)$ over the open set $\overset{\circ}{\Theta}$.

The function $\boldsymbol{\theta} \in \boldsymbol{\theta} \to \mathbb{E}\left(\tilde{q}_{t,N}(\boldsymbol{\theta}_0)|\mathcal{F}_0\right)$ is differentiable under the condition $\left|\dfrac{\partial \widetilde{q}_{t,N}(u_0,\boldsymbol{\theta}_0)}{\partial \boldsymbol{\theta}}\right| \in \mathrm{Lip}_p(\boldsymbol{\theta})$. Thus $\dfrac{\partial \widetilde{q}_{t,N}(u_0,\boldsymbol{\theta}_0)}{\partial \boldsymbol{\theta}}$ constitutes a differences of martingale sequence. We also have $\mathbb{E}\left[\left\|\dfrac{\partial \widetilde{q}_{t,N}(u_0,\boldsymbol{\theta}_0)}{\partial \boldsymbol{\theta}}\right\|^2\right] < \infty$ and we can apply the Central Limit Theorem for differences of martingale sequences:

$$\frac{1}{\sqrt{Nb}}\sum_{t=1}^{N}\frac{\partial \widetilde{q}_{t,N}(u,\boldsymbol{\theta}_0)}{\partial \boldsymbol{\theta}} K\left(\frac{\frac{t}{N}-u}{b}\right) \underset{N\to+\infty}{\xrightarrow{\mathcal{L}}} \mathcal{N}(0,\Sigma(\boldsymbol{\theta}_0(u)))$$

with

$$\begin{aligned}\Sigma(\boldsymbol{\theta}_0(u)) = &\int_{\mathbb{R}} K^2(x) \\ &\times \sum_{t\in\mathbb{Z}}\left(\mathrm{Cov}\left[\frac{\partial \widetilde{q}_{t,N}(u,\boldsymbol{\theta}_0)}{\partial \boldsymbol{\theta}_i}, \frac{\partial \widetilde{q}_{t,N}(u,\boldsymbol{\theta}_0)}{\partial \boldsymbol{\theta}_j}\right]\right)_{1\le i,j\le d}.\end{aligned}$$

Now we use a Taylor-Lagrange expansion for establishing

$$\begin{aligned}\frac{1}{\sqrt{Nb}}\sum_{t=1}^{N}\frac{\partial \widetilde{q}_{t,N}(u,\boldsymbol{\theta}_0)K\left(\frac{u-\frac{t}{N}}{b}\right)}{\partial \boldsymbol{\theta}} &= \frac{1}{\sqrt{Nb}}\sum_{t=1}^{N}\frac{\partial \widetilde{q}_{t,N}(u,\boldsymbol{\theta}_0)K\left(\frac{u-\frac{t}{N}}{b}\right)}{\partial \boldsymbol{\theta}} \\ &+ \sqrt{Nb}\times\frac{1}{Nb}\sum_{t=1}^{N}\frac{\partial^2 \widetilde{q}_{t,N}(u,\bar{\boldsymbol{\theta}})K\left(\frac{u-\frac{t}{N}}{b}\right)}{\partial \boldsymbol{\theta}^2}\left(\widehat{\boldsymbol{\theta}}(u)-\boldsymbol{\theta}_0(u)\right),\end{aligned}$$

with $\bar{\boldsymbol{\theta}}$ in the segment bounded by $\widehat{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}_0$. This leads to:

$$\begin{aligned}\left(\widehat{\boldsymbol{\theta}}(u)-\boldsymbol{\theta}_0(u)\right) &= \frac{\sum_{t=1}^{N}\dfrac{\partial \widetilde{q}_{t,N}(u,\widehat{\boldsymbol{\theta}})K\left(\frac{u-\frac{t}{N}}{b}\right)}{\partial \boldsymbol{\theta}} - \sum_{t=1}^{N}\dfrac{\partial \widetilde{q}_{t,N}(u,\boldsymbol{\theta_0})K\left(\frac{u-\frac{t}{N}}{b}\right)}{\partial \boldsymbol{\theta}}}{\sum_{t=1}^{N}\dfrac{\partial^2 \widetilde{q}_{t,N}(u,\bar{\boldsymbol{\theta}})K\left(\frac{u-\frac{t}{N}}{b}\right)}{\partial \boldsymbol{\theta}^2}} \\ &= \frac{-\sum_{t=1}^{N}\dfrac{\partial \widetilde{q}_{t,N}(u,\boldsymbol{\theta_0})K\left(\frac{u-\frac{t}{N}}{b}\right)}{\partial \boldsymbol{\theta}}}{\sum_{t=1}^{N}\dfrac{\partial^2 \widetilde{q}_{t,N}(u,\bar{\boldsymbol{\theta}})K\left(\frac{u-\frac{t}{N}}{b}\right)}{\partial \boldsymbol{\theta}^2}}.\end{aligned}$$

Multiplying by $\sqrt{Nb}$ results in

$$\sqrt{Nb}\left(\widehat{\boldsymbol{\theta}}(u)-\boldsymbol{\theta}_0(u)\right) = -\sqrt{Nb}\frac{\sum_{t=1}^{N}\dfrac{\partial \widetilde{q}_{t,N}(u,\boldsymbol{\theta_0})K\left(\frac{u-\frac{t}{N}}{b}\right)}{\partial \boldsymbol{\theta}}}{\sum_{t=1}^{N}\dfrac{\partial^2 \widetilde{q}_{t,N}(u,\bar{\boldsymbol{\theta}})K\left(\frac{u-\frac{t}{N}}{b}\right)}{\partial \boldsymbol{\theta}^2}}. \qquad (6)$$

So by law of large numbers, the denominator of (6) deliver the following information

$$\frac{\partial^2 \tilde{\mathcal{L}}_N(u,\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^2} = \frac{1}{Nb}\sum_{t=1}^{N} \frac{\partial^2 \widetilde{q}_{t,N}(u,\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^2} K\left(\frac{u-\frac{t}{N}}{b}\right) \xrightarrow[N\to+\infty]{\mathbb{P}} \mathbb{E}\left[\frac{\partial^2 \widetilde{q}_{t,N}(u,\boldsymbol{\theta_0})}{\partial \boldsymbol{\theta}^2}\right].$$

Since $\bar{\theta}$ in the segment bounded by $\hat{\boldsymbol{\theta}}$ and $\boldsymbol{\theta_0}$ and by consistency result of previous section, we have $\widehat{\boldsymbol{\theta}}(u) \xrightarrow[N\to+\infty]{\mathbb{P}} \boldsymbol{\theta_0}(u)$ and

$$\frac{1}{\sqrt{Nb}}\sum_{t=1}^{N}\left(\frac{\partial^2 \widetilde{q}_{t,N}(u,\bar{\boldsymbol{\theta}})}{\partial \boldsymbol{\theta}^2}\right) K\left(\frac{u-\frac{t}{N}}{b}\right) \xrightarrow[N\to+\infty]{\mathbb{P}} \Gamma\left(\boldsymbol{\theta}_0(u)\right),$$

with $\Gamma\left(\boldsymbol{\theta}_0(u)\right) = \mathbb{E}\left[\dfrac{\partial^2 \widetilde{q}_{t,N}(u,\boldsymbol{\theta_0})}{\partial \boldsymbol{\theta}^2}\right]$.

Application of CLT on the numerator of (6) gives

$$\frac{1}{\sqrt{Nb}}\sum_{t=1}^{N}\left(\frac{\partial \widetilde{q}_{t,N}(u,\bar{\boldsymbol{\theta}})}{\partial \boldsymbol{\theta}}\right) K\left(\frac{u-\frac{t}{N}}{b}\right) \xrightarrow[N\to+\infty]{\mathcal{L}} \mathcal{N}\left(0,\Sigma\left(\boldsymbol{\theta}_0(u)\right)\right).$$

Combining these results, by using Slutsky Lemma we obtain

$$-\sqrt{Nb}\frac{\sum_{t=1}^{N}\dfrac{\partial \widetilde{q}_{t,N}(u,\boldsymbol{\theta_0}) K\left(\frac{u-\frac{t}{N}}{b}\right)}{\partial \boldsymbol{\theta}}}{\sum_{t=1}^{N}\dfrac{\partial^2 \widetilde{q}_{t,N}(u,\bar{\boldsymbol{\theta}}) K\left(\frac{u-\frac{t}{N}}{b}\right)}{\partial \boldsymbol{\theta}^2}}\Gamma\left(\boldsymbol{\theta}_0(u)\right) \xrightarrow[N\to+\infty]{\mathcal{L}} \mathcal{N}\left(0,\Sigma\left(\boldsymbol{\theta}_0(u)\right)\right), \tag{7}$$

which means from (7) that

$$\sqrt{Nb}\left(\hat{\boldsymbol{\theta}}(u)-\boldsymbol{\theta}_0(u)\right) \underset{N\to+\infty}{\mathcal{L}} \mathcal{N}\left(0,\Gamma^{-1}\left(\boldsymbol{\theta}_0(u)\right)\Sigma\left(\boldsymbol{\theta}_0(u)\right)\Gamma^{-1}\left(\boldsymbol{\theta}_0(u)\right)\right).$$

□

5. Conclusion

In this study, we proposed a kernel-based quasi-maximum likelihood estimator (QMLE) for time-varying GARCH processes with Laplace-distributed innovations. Theoretical analysis established the consistency and asymptotic normality of the estimator under general regularity conditions, extending the framework of local likelihood methods to heavy-tailed, locally stationary settings. The convergence properties were derived using kernel-based arguments and asymptotic expansions adapted to the Laplace likelihood. Numerical experiments supported the theoretical results, illustrating the robustness and accuracy of the proposed estimator in capturing local volatility dynamics under non-Gaussian environments. We believe that this work provides a useful foundation for further investigations of Laplacian locally stationary processes and contributes to the broader development of nonparametric inference methods for time-varying models.

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