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Kernel Estimator of the Conditional Hazard Function for Truncated Dependent Data with Functional Regressors

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Abstract

In this paper, we propose a new non-parametric kernel estimator of the conditional hazard function where the explanatory variable takes values in an infinite-dimensional space and the response variable is subject to random left truncation and satisfies the α-mixing property. Our main aim is to prove the strong uniform consistency rate for both functional and real arguments of the estimator. Furthermore, the performance of our estimator is evaluated via a simulation study and its practical relevance is illustrated by an application to real-world data.

Keywords: functional data, hazard function, kernel estimator, strong mixing, truncated data.

1. Introduction

Functional data analysis studies experiments whose results are generally curves. Under this assumption, the data are modeled as realizations of a random variable (rv) taking values in an infinite dimensional space. This area of modern statistics has received a great deal of attention over the last two decades and was popularized in the book of Ramsay and Silverman (2005). Non-parametric methods taking into account functional variables have been developed with very interesting practical motivations in the environmental field (Damon and Guillas 2002; Aneiros-Pérez, Cardot, Estévez-Pérez, and Vieu 2004), meteorological science (Besse, Cardot, and Stephenson 2000; Hall and Heckman 2002) and chemometrics and speech recognition problems (Ferraty and Vieu 2002, 2003). The monograph by Ferraty and Vieu (2006) offered a wide range of non-parametric estimation methods with functional data. Among other properties, the consistency of estimates of the conditional density, conditional distribution and regression was established in both the independent case and under dependence condition (strong mixing).

The hazard function, also known as the risk function, is a concept commonly used in survival analysis, medical research and reliability theory. It plays an essential role in statistics and is used in a variety of fields, including econometrics, epidemiology, environmental science and others. The estimation of the hazard function has attracted a great deal of interest in the statistical literature. The work of Ferraty, Rabhi, and Vieu (2008) is an important contribution to the conditional hazard rate for functional covariates in an infinite-dimensional space. They studied the almost complete convergence of a kernel estimator for the conditional hazard function and addressed different scenarios such as censored and/or dependent variables. Parallely, Quintela-del Río (2008) gave the asymptotic bias and variances, as well as the asymptotic normality of the three estimates (conditional density, distribution and hazard functions) for dependent data. Censored data are a type of data in which the values are incomplete or partially unknown. The hazard rate in the presence of a functional explanatory variable and censored data has been intensely explored in the literature. In addition to the work of Ferraty *et al.* (2008), Rabhi, Hammou, and Djebbouri (2015) investigated the maximum of the conditional hazard function under the strong mixing condition. Belabbaci, Rabhi, and Soltani (2015) considered the case where observations are linked to a single-index structure. More recently, Bellatrach, Bouabsa, and Attouch (2023) studied the case of spacial data using the k-nearest neighbor method.

Censorship is not the only type of incomplete data. Truncation is another type of incomplete data, which is entirely different from censorship. Particularly, Random Left Truncation (RLT) occurs when the variable of interest Y is observable only if it is greater than another rv T, called the truncation variable. For instance in insurance, when a policy excess is applied, claims with amounts lower than the excess are not declared to the insurer, this results in left truncation. This type of incomplete data arises in many fields such as astronomy and economics (see (Woodroofe 1985; Wang, Jewell, and Tsai 1986; Tsai, Jewell, and Wang 1987)) as well as epidemiology and biometry (see (He and Yang 1994)).

To the best of our knowledge, the problem of estimating the conditional hazard rate in the presence of functional explanatory variables, when observations are left-truncated, has not been addressed in the statistical literature. The aim of this paper is therefore to construct a family of such estimators for which a strong uniform consistency rate is established under the strong mixing condition. Thus, this work extends the results obtained by Ferraty *et al.* (2008) from complete or censored to truncated data. Furthermore, most existing work on non-parametric hazard functional estimation concerns only uniform consistency with respect to the real argument, whereas our main objective is to establish the strong uniform consistency rate for both functional and real arguments of the estimator. The remainder of the paper is organized as follows. Section 2 presents the model and introduces the necessary notations. In Section 3 we define the estimator. Section 4 states the assumptions and our main results. Section 5 reports simulation studies with numerical results and includes an application to real-world data. The final section provides the proofs and auxiliary results.

2. Model and notations

Let $\{(Y_i, T_i), i = 1, ..., N\}$ be a sequence of random vectors from (Y, T), where Y denotes the lifetime under study with continuous distribution function (df) F and T is the left truncation variable with continuous df G. In a RLT model, one observes (Y, T) only if $Y \geq T$. Let $\mu := \mathbb{P}(Y \geq T)$; it is clear that if $\mu = 0$ no data can be observed. Therefore, we assume that $\mu > 0$. Now, consider the presence of a covariate $\mathcal{X}$, valued in some functional space $\mathcal{S}$ equipped with a semi-metric $d(\cdot, \cdot)$. To ensure the identifiability of the model, we suppose that T is independent of $(\mathcal{X}, Y)$. Without possible confusion, we still denote by $\{(\mathcal{X}_i, Y_i, T_i), i = 1, ..., n\}$, with $n \leq N$, the vectors which one really observes (i.e., $Y_i \geq T_i$). As a consequence, our results will be formulated with respect to the actually observed n-sample. Thus, we introduce the conditional probability measure $\mathbf{P}(\cdot) := \mathbb{P}(\cdot | Y \geq T)$. Similarly, $\mathbb{E}$ and $\mathbf{E}$ denote the expectation operators related to $\mathbb{P}$ and $\mathbf{P}$, respectively. From now on, the star notation (*) will refer to any characteristic of the observed data.

Following Stute (1993) and Zhou (1996), the conditional joint df of an observed (Y, T) is given by

$$J^*(y,t) = \mathbf{P}(Y \le y, T \le t) = \mu^{-1} \int_{-\infty}^{y} G(t \wedge u) dF(u),$$

where $t \wedge u = \min(t,u)$. The marginal df's of Y and T are defined respectively by

$$F^*(y) = J^*(y,\infty) = \mu^{-1} \int_{-\infty}^{y} G(u)dF(u) \text{ and } G^*(t) = J^*(\infty,t) = \mu^{-1} \int_{-\infty}^{+\infty} G(t \wedge u)dF(u),$$

which are empirically estimated respectively by

$$F_n^*(y) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{Y_i \le y\}} \text{ and } G_n^*(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{T_i \le t\}},$$

where $\mathbb{1}_A$ denotes the indicator function of the set A.
Now define

$$C(y) = \mathbf{P}(T \le y \le Y).$$

It is easy to check that

$$C(y) = G^*(y) - F^*(y) = \mu^{-1} G(y) \left(1 - F(y)\right),$$

and its empirical estimate is

$$C_n(y) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{T_i \le y \le Y_i\}} = G_n^*(y) - F_n^*(y).$$

As the N-sample is unobservable due to truncation, Lynden-Bell (1971) proposed the non-parametric maximum likelihood estimates of F and G given, respectively, by

$$F_n(y) = 1 - \prod_{i:Y_i \le y} \frac{nC_n(Y_i) - 1}{nC_n(Y_i)} \text{ and } G_n(t) = \prod_{i:T_i > t} \frac{nC_n(T_i) - 1}{nC_n(T_i)}.$$

Let us denote, for any df L, the left and right endpoints of its support by $a_L := \inf\{t : L(t) > 0\}$ and $b_L := \sup\{t : L(t) < 1\}$, respectively. Woodroofe (1985) pointed out that F and G can be estimated only if

$$a_G \le a_F,\ b_G \le b_F \text{ and } \int_{a_F}^{+\infty} \frac{dF}{G} < \infty.$$

In addition, he established the following uniform consistency results

$$\sup_{y \ge a_F} |F_n(y) - F(y)| \xrightarrow{\mathbf{P}-a.s.} 0 \text{ and } \sup_{t \ge a_G} |G_n(t) - G(t)| \xrightarrow{\mathbf{P}-a.s.} 0.$$

Since N is unknown, μ can not be classically estimated by $\frac{n}{N}$. He and Yang (1998) showed that μ can be estimated by

$$\mu_n = \frac{G_n(y)\left(1 - F_n(y)\right)}{C_n(y)},$$

and proved that μ_n does not depend on y, so its value can be calculated for any y such that $C_n(y) \ne 0$. Furthermore, they showed its $\mathbf{P} - a.s.$ consistency in the independent case (see their Corollary 2.5).

In the sequel, $\{(\mathcal{X}_i, Y_i, T_i), i = 1, ..., n\}$ is assumed to be a stationary α-mixing sequence of random vectors. Recall that a sequence $\{Z_i, i = 1, ..., n\}$ is said to be α-mixing (strongly mixing) if the mixing coefficient

$$\alpha(n) := \sup_{k \ge 1} \sup \left\{ |\mathbf{P}(A \cap B) - \mathbf{P}(A)\mathbf{P}(B)|; A \in \mathcal{F}_1^k(Z), B \in \mathcal{F}_{k+n}^{+\infty}(Z) \right\}$$

converges to zero as n tends to infinity, where $\mathcal{F}_i^k(Z)$ denotes the σ-algebra generated by $\{Z_j, j = i, ..., k\}$. The α-mixing condition has several applications and is known to be fulfilled for many stochastic processes such as ARMA sequences, ARCH or GARCH models (for more details we refer to Bradley (2007)).

3. Definition of the estimator

For $\chi \in \mathcal{S}$, the conditional hazard function of Y given $\mathcal{X} = \chi$ is defined for any real y by

$$\lambda(y|\chi) = \lim_{\Delta y \to 0} \frac{\mathbb{P}(Y \leq y + \Delta y \mid Y \geq y, \mathcal{X} = \chi)}{\Delta y} = \frac{f(y|\chi)}{1 - F(y|\chi)} \mathbb{1}_{\{F(y|\chi) \neq 1\}},$$

where $F(y|\chi) = \mathbb{P}(Y \leq y|\mathcal{X} = \chi)$ is the conditional distribution function (cond-df) of Y given $\mathcal{X}$ and $f(y|\chi) = \dfrac{\partial F(y|\chi)}{\partial y}$ is the conditional probability density function (cond-pdf) of Y given $\mathcal{X}$.

In this section, our main purpose is to define an estimate for the conditional hazard function $\lambda(\cdot|\chi)$ under random left truncated data. We should note that in the complete data case, Ferraty and Vieu (2006) introduced an estimator for the cond-df using the familiar Nadaraya-Watson weights, defined by

$$F_N(y|\chi) := \frac{\sum_{i=1}^{N} K\left(\frac{d\left(\chi, \mathcal{X}_i\right)}{h_K}\right) H\left(\frac{y - Y_i}{h_H}\right)}{\sum_{i=1}^{N} K\left(\frac{d\left(\chi, \mathcal{X}_i\right)}{h_K}\right)},$$

where K is a real-valued kernel function, H is a df and the bandwidths h_K and h_H are sequences of positive real numbers that tend to 0 as n tends to infinity.

Under RLT data, following Helal and Ould Saïd (2016), the previous estimator becomes

$$F_n(y|\chi) := \frac{\sum_{i=1}^{n} G_n^{-1}(Y_i) K\left(\frac{d\left(\chi, \mathcal{X}_i\right)}{h_K}\right) H\left(\frac{y - Y_i}{h_H}\right)}{\sum_{i=1}^{n} G_n^{-1}(Y_i) K\left(\frac{d\left(\chi, \mathcal{X}_i\right)}{h_K}\right)},$$

for the i's such that $G_n(Y_i) \neq 0$. Then, the cond-pdf estimator is given by

$$f_n(y|\chi) := \frac{\partial F_n(y|\chi)}{\partial y} = \frac{\frac{1}{h_H} \sum_{i=1}^{n} G_n^{-1}(Y_i) K\left(\frac{d\left(\chi, \mathcal{X}_i\right)}{h_K}\right) H'\left(\frac{y - Y_i}{h_H}\right)}{\sum_{i=1}^{n} G_n^{-1}(Y_i) K\left(\frac{d\left(\chi, \mathcal{X}_i\right)}{h_K}\right)},$$

where H' is the first derivative of H.

Consequently, an estimator for the conditional hazard function is given, in a natural way, by

$$\lambda_n(y|\chi) := \frac{f_n(y|\chi)}{1 - F_n(y|\chi)} \mathbb{1}_{\{F_n(y|\chi) \neq 1\}}.$$

It is well known that the estimation behavior in functional non-parametric data problems is controlled by the concentration properties of the functional variable $\mathcal{X}$, defined by

$$\phi_\chi\left(h_K\right) := \mathbb{P}(\mathcal{X} \in \mathcal{B}(\chi, h_K)) = \mathbb{P}\left(\mathcal{X} \in \{\chi' \in \mathcal{S}, d\left(\chi, \chi'\right) < h_K\}\right),$$

where $\mathcal{B}\left(\chi, h_K\right)$ denotes the ball of center χ and radius h_K.

4. Main results

Throughout this paper, C_1 and C_2 define any finite positive constant which is allowed to change from line to line. Furthermore, we assume that $a_G < a_F$ and $b_G \leq b_F$ and we consider two real numbers a and b such that $a_F < a < b < b_F$. Denote by Ξ a compact subset of the semi-metric space $\mathcal{S}$. Let us introduce the following assumptions.

4.1. Assumptions

A1 (i) For all $h_K > 0$, $\mathbb{P}(\mathcal{X} \in \mathcal{B}(\chi, h_K)) = \phi_\chi(h_K) > 0$ and $\lim_{h_K \to 0} \phi_\chi(h_K) = 0$.

(ii) There exist two functions ζ and ϕ, where ϕ is supposed to be increasing and differentiable, such that

$$\lim_{h_K \to 0} \sup_{\chi \in \Xi} \left| \frac{\phi_\chi(h_K)}{\phi(h_K)} - \zeta(\chi) \right| = 0,$$

with $m_\zeta := \inf_{\chi \in \Xi} \zeta(\chi) > 0$ and $M_\zeta := \sup_{\chi \in \Xi} \zeta(\chi) < \infty$.

(iii) The joint density of (Y_i, Y_j) given $(\mathcal{X}_i, \mathcal{X}_j)$ exists, is bounded and satisfies

$$\sup_{i \neq j} \mathbf{P}\left((\mathcal{X}_i, \mathcal{X}_j) \in \mathcal{B}(\chi, h_K) \times \mathcal{B}(\chi, h_K)\right) =: \Psi_\chi(h_K) = O\left(\phi_\chi^2\left(h_K\right)\right).$$

A2 The functions $F(\cdot|\chi)$ and $f(\cdot|\chi)$ satisfy a Hölder condition with respect to each variable. That is, there exist a generic finite constant C_χ, depending on χ, constants $b_1, b_2 > 0$, such that for any $(\chi_1, y_1) \in \mathcal{V}(\chi) \times [a, b]$ and $(\chi_2, y_2) \in \mathcal{V}(\chi) \times [a, b]$, we have

(i) $|F(y_1|\chi_1) - F(y_2|\chi_2)| \leq C_\chi \left(d\left(\chi_1, \chi_2\right)^{b_1} + |y_1 - y_2|^{b_2}\right)$,

(ii) $|f(y_1|\chi_1) - f(y_2|\chi_2)| \leq C_\chi \left(d\left(\chi_1, \chi_2\right)^{b_1} + |y_1 - y_2|^{b_2}\right)$,

where $\mathcal{V}(\chi)$ is a neighborhood of χ.

A3 The kernel K is

(i) compactly supported such that $0 < C_1 \leq K(\cdot) \leq C_2 < \infty$,

(ii) continuously differentiable with bounded derivative $K'(\cdot)$.

A4 The df H is such that

(i) $\int_{\mathbb{R}} |z|^{b_2} H'(z) dz < \infty$, $\forall z \in \mathbb{R}$,

(ii) H' is bounded and Lipschitzian.

A5 $\{(\mathcal{X}_i, Y_i, T_i), i = 1, ..., n\}$ is a strictly stationary α-mixing sequence with coefficient $\alpha(n) = O(n^{-\nu})$, for some $\nu > 5$.

A6 $h_K h_H^{\frac{\nu+5}{2}} \phi^{\frac{\nu+3}{2}}(h_K) = O\left(n^{-\delta}\right)$, with $0 < \delta < \frac{\nu-5}{2}$ and $\nu > 5$.

A7 The bandwidths h_K and h_H satisfy

(i) $h_K = O\left(n^{-\frac{\delta}{2}}\right)$ and $h_H = O\left(n^{-\frac{2\delta}{3(\nu+5)}}\right)$,

(ii) $\lim_{n \to \infty} \frac{\log n}{n h_H \phi(h_K)} = 0$,

(iii) $\lim_{n \to \infty} \frac{\phi^{\nu-2}(h_K) \log n}{h_H} = 0$.

Remark 1. *Assumption **A1**(i) concerns the small ball probabilities and represents the most minimal condition that can be made. Assumption **A1**(ii) plays an important role in the non-parametric functional estimation (see Ferraty and Vieu (2006)), as it overcomes the issue of the non-existence of the probability density function. Assumption **A1**(iii) permits to treat the covariance term. Assumption **A3** concerns the kernel and states conditions that are standard in non-parametric functional estimation. Assumptions **A2** and **A4** are technical conditions used to study the bias term. Assumption **A5** defines the type of dependence, while Assumptions **A6** and **A7** address the suitable choice of bandwidth related to the small ball probability.*

4.2. Results

Proposition 1. *Under Assumptions **A1-A7**, for n large enough we have*

$$\sup_{\chi\in\Xi}\sup_{y\in[a,b]}|f_n(y|\chi)-f(y|\chi)| = O\left(h_K^{b_1}+h_H^{b_2}\right)+O\left(\sqrt{\frac{\log n}{nh_H\phi(h_K)}}+\sqrt{\frac{\phi^{\nu-2}(h_K)\log n}{h_H}}\right)\mathbf{P}-a.s.$$

Remark 2. *The result in Proposition 1 extends the result of Ferraty, Laksaci, Tadj, and Vieu (2010) (Theorem 5) from the independent to the dependent truncated case.*

Proposition 2. *Under Assumptions **A1-A7**, for n large enough we have*

$$\sup_{\chi\in\Xi}\sup_{y\in[a,b]}|F_n(y|\chi)-F(y|\chi)| = O\left(h_K^{b_1}+h_H^{b_2}\right)+O\left(\sqrt{\frac{\log n}{n\phi(h_K)}}+\sqrt{\phi^{\nu-2}(h_K)\log n}\right)\mathbf{P}-a.s.$$

Remark 3. *Note that the rate of convergence in Proposition 2 quantifies the dependence effect which reduces, for $\nu\to\infty$, to the same as that obtained by Helal and Ould Saïd (2016) (Theorem 4.1) in the independent case.*

Theorem 1. *Under Assumptions **A1-A7**, for n large enough we have*

$$\sup_{\chi\in\Xi}\sup_{y\in[a,b]}|\lambda_n(y|\chi)-\lambda(y|\chi)| = O\left(h_K^{b_1}+h_H^{b_2}\right)+O\left(\sqrt{\frac{\log n}{nh_H\phi(h_K)}}+\sqrt{\frac{\phi^{\nu-2}(h_K)\log n}{h_H}}\right)\mathbf{P}-a.s.$$

Remark 4. *Theorem 1 ensures uniform consistency for both functional and real arguments. Thus, on one hand, our result generalizes the result obtained by Ferraty et al. (2010) (Theorem 6) from the case of complete independent data to the case of truncated dependent data. On the other hand, in the absence of truncation ($T=-\infty$), our results extend those of Ferraty et al. (2008) (Theorem 3.2) and Quintela-del Río (2008) (Theorem 4), who established the uniform consistency only with respect to the real argument y.*

5. Simulation and application

5.1. Simulation study

In this section, we aim to evaluate the efficiency of our estimator with regard to truncation rates and sample sizes. Specifically, we proceed according to the following algorithm. The sample size n is fixed and assumed to be known.

step 1 The functional covariates $\mathcal{X}_i(t)$ are generated as

$$\mathcal{X}_i(t) = B_iZ_{1i}(t)+(1-B_i)Z_{2i}(t) \text{ with } t\in[0,1]\,,\, i\geq 1,$$

where $B_i\sim\mathcal{B}(0.5)$ (Bernoulli distribution with success probability 0.5), Z_{i1} and Z_{i2} are two diffusion processes such that

$$Z_{i1}(t) = 2-\cos(\pi tW_i) \text{ and } Z_{i2}(t)=\cos(\pi tW_i),$$

where $W_i\sim\mathcal{U}_{[0,1]}$ (uniform distribution on the interval $[0,1]$).

step 2 The variables of interest Y_i are generated according to the model

$$Y_i = \int_0^1 \left(\mathcal{X}_i'(t)\right)^2 dt + \varepsilon_i,\ i \geq 1,$$

where $\mathcal{X}_i'(\cdot)$ is the first derivative of $\mathcal{X}_i(\cdot)$. The ε_i's are independent of $\mathcal{X}_i$ and generated according to the following autoregressive process to ensure α-mixing dependency

$$\varepsilon_i = \frac{1}{\sqrt{2}}\left(\varepsilon_{i-1} + \eta_i\right),\ i \geq 1,$$

with η_i independent random variables from standard normal distribution.

step 3 The truncating rv's T_i are generated from the normal distribution $\mathcal{N}(\theta, 1)$, where θ is adapted in order to get different values of truncation rate (TR).

step 4 The triplets $(\mathcal{X}_i(t), Y_i, T_i)$ were drawn until n of them satisfy the condition $Y \geq T$. In this way, we obtained the truncated sample $\{(\mathcal{X}_i, Y_i, T_i), i = 1, ..., n\}$.

For this simulation study, we use a quadratic kernel defined by

$$K(x) = \frac{3}{2}(1 - x^2)\mathbb{1}_{[0,1[}(x) \text{ with } K(1) > 0$$

and the distribution function H given by

$$H(z) = \int_{-\infty}^{z} \frac{3}{4}(1 - t^2)\mathbb{1}_{[-1,1]}(t)dt.$$

In addition, it is necessary to select a suitable semi-metric $d(\cdot,\cdot)$ adapted to our data. Several types of semi-metrics can be considered (see Ferraty and Vieu (2006)). Here we choose the $\mathcal{L}_1$-distance between the first derivatives of the curves defined by

$$d(\chi_i, \chi_j) = \sqrt{\int_0^{\pi/2} \left(\chi_i'(t) - \chi_j'(t)\right)^2 dt}.$$

Finally, we split our data into two subsets: a training sample and a test sample of size n_I. From the training sample, we calculate the predicted values of our estimator $\lambda_n(\cdot|\chi)$ corresponding to the test sample curves.

Prediction efficiency is assessed using the Global Mean Square Error (GMSE), calculated along $B = 200$ training samples and evaluated for the test sample, defined as

$$GMSE = \frac{1}{B \times n_I} \sum_{b=1}^{B} \sum_{i=1}^{n_I} (\lambda_{n,b}(y_i|\chi_i) - \lambda(y_i|\chi_i))^2,$$

where $\lambda_{n,b}(y_i|\chi_i)$ is the value of $\lambda_n(y_i|\chi_i)$ computed at iteration b. Without possible confusion, we denote by n the size of the training sample. Note that, in this simulation study, we choose the optimal bandwidths that minimize the GMSE over a grid of bandwidths for h_K and h_H.

Figure 1 below shows 200 curves $\mathcal{X}_i(t)$. The time is discretised into 100 points.

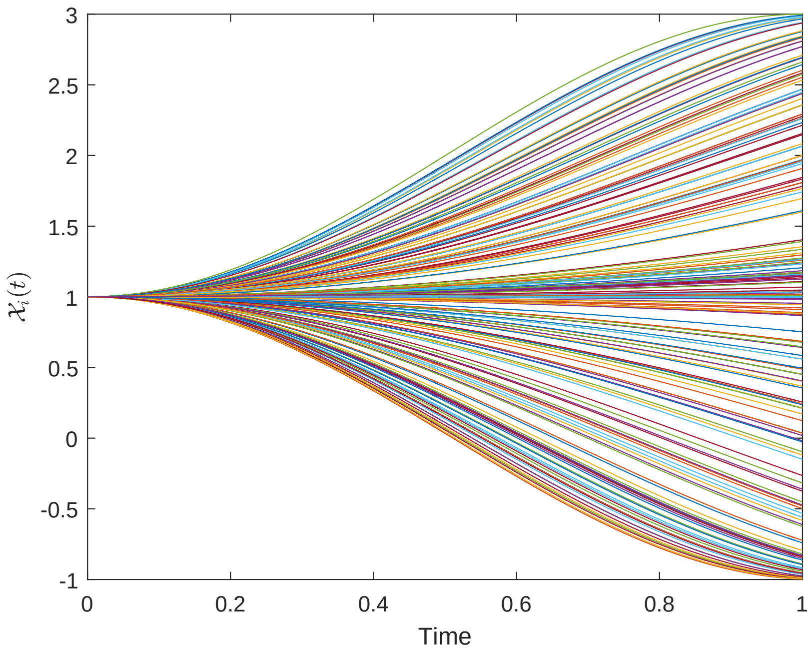

Figure 1: Curves $\mathcal{X}_i(t)$, for $i = 1, ..., 200$ and $t \in [0, 1]$

Sample size effects

First, we fix the truncation rate $TR = 40\%$ and vary the training sample size $n = 50$, 200 and 500 (with the test sample size fixed at 30). We plot the points $\{(i; \lambda(y_i|\chi_i)), (i; \lambda_n(y_i|\chi_i))\}$. The true values are represented in blue and the predicted values are represented in red.

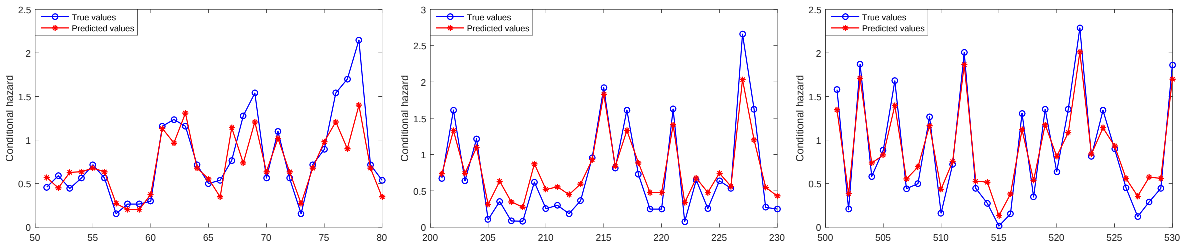

Figure 2: $\lambda_n(y_i|\chi_i)$ versus $\lambda(y_i|\chi_i)$ with $TR = 40\%$ for $n = 50$, 200 and 500, respectively

Figure 2 shows sufficiently good performance even for small sample sizes n and it is clear that the estimation quality improves as the sample size increases.

Truncation rate effects

We investigated whether prediction quality is influenced by the rate of truncation. To this end, we fixed the sample size at $n = 200$ and choose different truncation rates $TR = 10\%$, 40% and 60%. As shown in Figure 4 below, the quality of prediction appears to be slightly affected by the truncation rate.

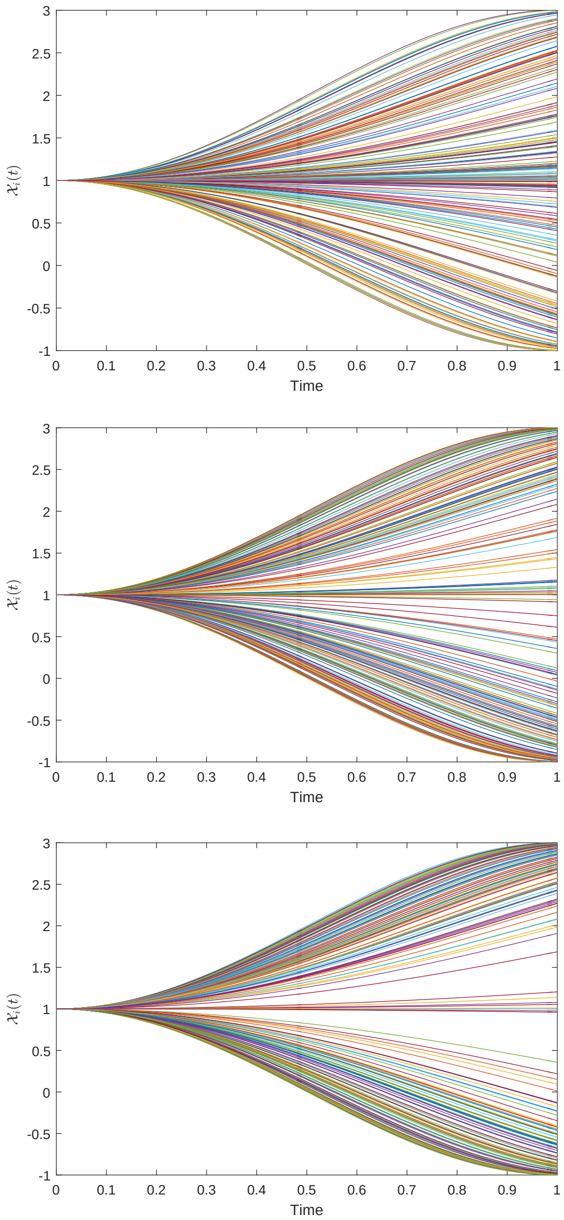

Figure 3: Observed curves with $n = 200$ for $TR = 10\%$, 40% and 60%, respectively

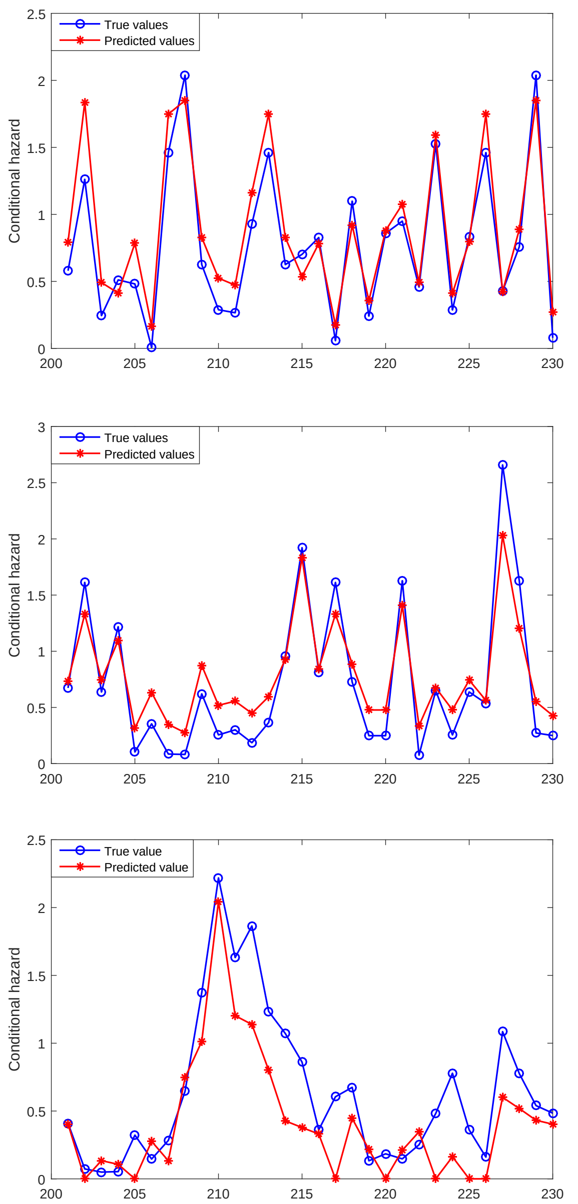

Figure 4: $\lambda_n(y_i|\chi_i)$ vs. $\lambda(y_i|\chi_i)$ with $n = 200$ for $TR = 10\%$, 40% and 60%, respectively

These conclusions are also confirmed by the numerical results of GMSE reported in Table 1.

Table 1: GMSE and corresponding optimal bandwidth pairs

	$TR \approx 10\%$			$TR \approx 40\%$			$TR \approx 60\%$		
	h_H	h_K	GMSE	h_H	h_K	GMSE	h_H	h_K	GMSE
$n = 50$	1.41	0.91	0.0464	1.81	0.96	0.0800	1.70	1.32	0.2816
$n = 200$	1.30	0.45	0.0308	1.71	0.84	0.0685	1.66	1.10	0.1312
$n = 500$	1.28	0.33	0.0076	1.26	0.61	0.0613	1.34	0.59	0.1180

5.2. Application to real data

In the energy sector, anticipating peak consumption based on temperature can be valuable for assessing the risk of grid overload due to excessive demand, whether from heating or cooling. This application aims to predict the instantaneous risk that the daily peak consumption exceeds a given threshold, given that the temperature is fixed at a specific value. For this purpose, we rely on data including hourly electricity consumption (measured in MWh) and temperatures (measured in degrees Celsius) covering the period from 1 January 2016 to 31 December 2016. The data were sourced from a hypermarket's smart meter, see Pîrjan, Oprea, Căruțașu, Petroșanu, Bâra, and Coculescu (2017) and Fetitah, Attouch, Khardani, and Righi (2021) for a detailed description of this dataset.

The collected sample consists of 366 data points, where each data point is a pair of the daily peak electricity consumption $Y_i = \max_{j=1,\ldots,24} (Y_i(t_j))$ and the corresponding daily temperature curve $\mathcal{X}_i(t_j)_{j=1,\ldots,24}$ for a fixed day i (see Figure 5).

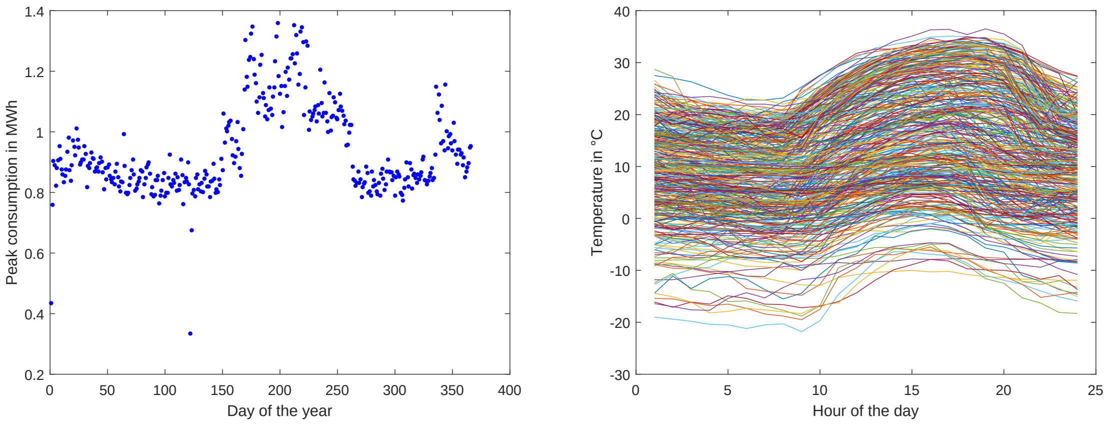

Figure 5: Daily peak electricity consumption and corresponding temperature curves (2016)

Due to a technical failure, peak consumptions below 0.865 MWh were not recorded. As a result, only 219 peak consumption values were observed. This restriction corresponds to a left-truncation with a proportion of approximately 40% (see Figure 6).

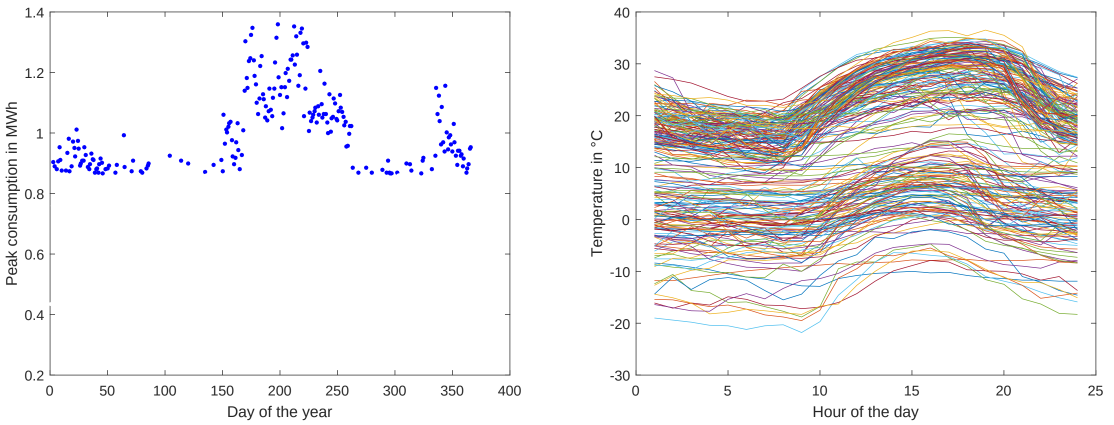

Figure 6: Observed daily peak electricity consumption and corresponding temperature curves (2016)

Now, we divide our sample into a learning sample containing the first 200 days and a testing sample comprising the last 19 days. The non-parametric estimates used are those defined in Section 3. The kernel K, the df H and the semi-metric are the same as those used for generated data. To select the optimal bandwidth, we apply a leave-one-out cross-validation (LOOCV) approach, see Estévez-Pérez, Quintela-del Río, and Vieu (2002).

In Figure 7, we plot the resulting $\lambda_n(y_i|\chi_i)$ for $i = 201, ..., 219$.

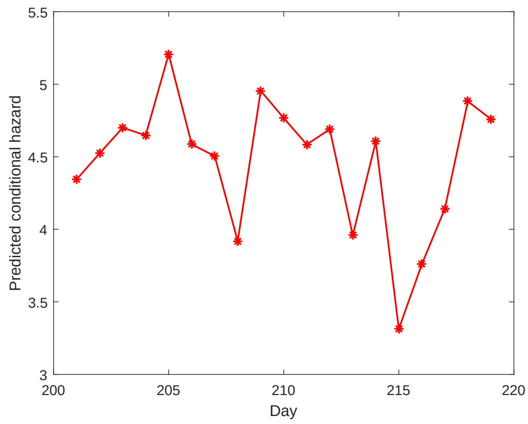

Figure 7: $\lambda_n\,(y_i|\chi_i)$ for $i = 201, ..., 219$

Since Figure 7 displays only pointwise estimates, it does not allow for a detailed interpretation of the model behavior. To address this limitation, we include Figure 8 where we plot the resulting estimates conditioned on the 201st functional variable $\lambda_n(y|\chi_{201})$.

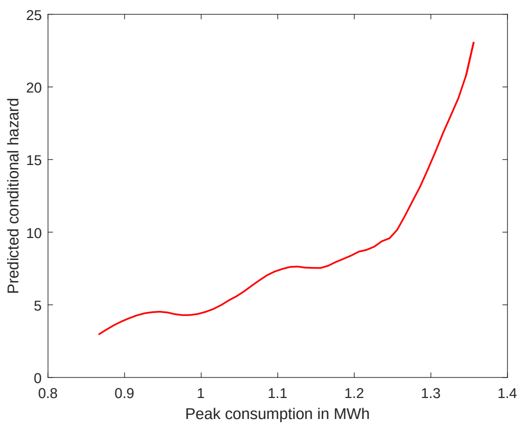

Figure 8: $\lambda_n(y|\chi_{201})$

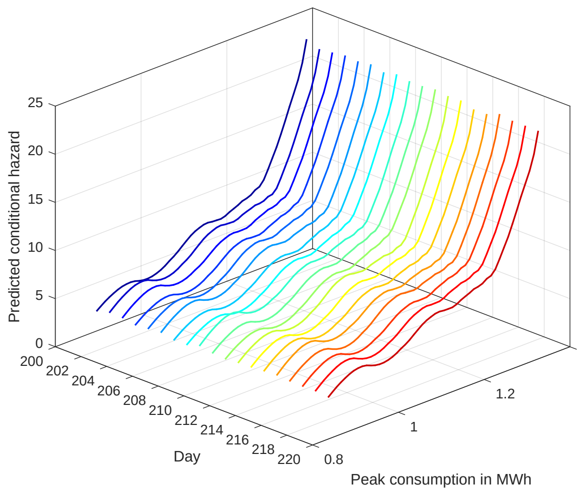

Figure 9: $\lambda_n(y|\chi_i)$, for $i = 201, ..., 219$

Figure 8 indicates a greater instantaneous risk of peak occurring as consumption rises. Notably, the shape of the curve reflects nonlinearity, with a more pronounced increase beyond a threshold of approximately 1.25 MWh. In Figure 9, we plot the conditional hazard function given all covariates $\chi_{201}, ..., \chi_{219}$ of the test sample.

Figure 9 reveals a consistent pattern over the days, with conditional hazard functions exhibiting similar shapes but varying slightly in magnitude and curvature. Although the baseline hazard increases with peak consumption, its level and steepness may be modulated by day-specific covariates, potentially capturing seasonal or behavioral variations in electricity usage.

6. Auxiliary results and proofs

We introduce the following notations, for $i = 1, ..., n$

$$K_i(\chi) := K\left(\frac{d(\chi, \mathcal{X}_i)}{h_K}\right) \text{ and } H_i(y) := H\left(\frac{y - Y_i}{h_H}\right).$$

Clearly we have

$$F_n(y|\chi) =: \frac{F_{1,n}(\chi, y)}{g_n(\chi)} \text{ and } f_n(y|\chi) =: \frac{f_{1,n}(\chi, y)}{g_n(\chi)},$$

where

$$F_{1,n}(\chi, y) := \frac{\mu_n}{n\mathbb{E}(K_1(\chi))}\sum_{i=1}^{n} G_n^{-1}(Y_i)K_i(\chi)H_i(y),$$

$$g_n(\chi) := \frac{\mu_n}{n\mathbb{E}(K_1(\chi))}\sum_{i=1}^{n} G_n^{-1}(Y_i)K_i(\chi),$$

and

$$f_{1,n}(\chi, y) := \frac{\mu_n}{nh_H\mathbb{E}(K_1(\chi))}\sum_{i=1}^{n} G_n^{-1}(Y_i)K_i(\chi)H_i'(y). \tag{1}$$

Note that, Assumptions **A3**(i) and **A1**(i) ensure $\inf_{\chi\in\Xi}\mathbb{E}(K_1(\chi)) > 0$, since

$$0 < C_1\phi_\chi(h_K) < \mathbb{E}(K_1(\chi)) < C_2\phi_\chi(h_K). \tag{2}$$

We also define $\widetilde{F}_{1,n}(\chi, y)$, $\widetilde{g}_n(\chi)$ and $\widetilde{f}_{1,n}(\chi, y)$ the pseudo-estimators according to $F_{1,n}(\chi, y)$, $g_n(\chi)$ and $f_{1,n}(\chi, y)$, respectively, as follows

$$\widetilde{F}_{1,n}(\chi, y) := \frac{\mu}{n\mathbb{E}(K_1(\chi))}\sum_{i=1}^{n} G^{-1}(Y_i)K_i(\chi)H_i(y), \tag{3}$$

$$\widetilde{g}_n(\chi) := \frac{\mu}{n\mathbb{E}(K_1(\chi))}\sum_{i=1}^{n} G^{-1}(Y_i)K_i(\chi), \tag{4}$$

and

$$\widetilde{f}_{1,n}(\chi, y) := \frac{\mu}{nh_H\mathbb{E}(K_1(\chi))}\sum_{i=1}^{n} G^{-1}(Y_i)K_i(\chi)H_i'(y). \tag{5}$$

Following the proofs of Horrigue and Ould Saïd (2011), under Assumption **A1**(ii) we have

$$1 \le \frac{M_\zeta}{m_\zeta} =: \Gamma < \infty$$

and for $\epsilon = \frac{m_\zeta}{3} > 0$, there exists n_ϵ such that, uniformly on $\chi \in \Xi$, for $n > n_\epsilon$ we have

$$\zeta(\chi) - \epsilon \le \frac{\phi_\chi(h_K)}{\phi(h_K)} \le \zeta(\chi) + \epsilon,$$

which implies that for all $\chi \in \Xi$

$$\frac{2}{3}m_\zeta \le \frac{\phi_\chi(h_K)}{\phi(h_K)} \le \frac{4}{3}M_\zeta, \tag{6}$$

and for all χ_1, χ_2 in Ξ

$$0 < \frac{1}{2\Gamma} \le \frac{\phi_{\chi_1}(h_K)}{\phi_{\chi_2}(h_K)} \le 2\Gamma < \infty. \tag{7}$$

Proof of Proposition 1. For $y \in [a,b]$ and $\chi \in \Xi$, set

$$\begin{aligned} f_n(y|\chi) - f(y|\chi) &= \frac{1}{g_n(\chi)}\left(f_{1,n}(\chi,y) - \widetilde{f}_{1,n}(\chi,y)\right) + \frac{1}{g_n(\chi)}\left(\widetilde{f}_{1,n}(\chi,y) - \mathbf{E}\left(\widetilde{f}_{1,n}(\chi,y)\right)\right) \\ &+ \frac{1}{g_n(\chi)}\left(\mathbf{E}\left(\widetilde{f}_{1,n}(\chi,y)\right) - f(y|\chi)\right) + \frac{f(y|\chi)}{g_n(\chi)}\left(g_n(\chi) - \widetilde{g}_n(\chi)\right) \\ &+ \frac{f(y|\chi)}{g_n(\chi)}\left(\mathbf{E}\left(\widetilde{g}_n(\chi)\right) - \widetilde{g}_n(\chi) - \mathbf{E}\left(\widetilde{g}_n(\chi)\right) + 1\right). \end{aligned} \tag{8}$$

Therefore, Proposition 1 is a consequence of the following lemmata. □

Lemma 1. *Under Assumptions **A2**(ii) and **A4**(i), we have*

i. $\sup_{\chi\in\Xi}\sup_{y\in[a,b]}\left|\mathbf{E}\left(\widetilde{f}_{1,n}(\chi,y)\right) - f(y|\chi)\right| = O\left(h_K^{b_1} + h_H^{b_2}\right)$ *as* $n\to\infty$,

ii. $\mathbf{E}\left(\widetilde{g}_n(\chi)\right) = 1.$

Proof. • i) In what follows, $d\mathbf{P}_\chi(\cdot)$ represents the distribution of $\mathcal{X}$ under left truncation. From (5) and under the truncation effect we have

$$\begin{aligned} \mathbf{E}\left(\widetilde{f}_{1,n}(\chi,y)\right) &= \frac{\mu}{h_H\mathbb{E}\left(K_1(\chi)\right)}\int\int G^{-1}(y)\,K_1(\chi)\,H_1'(y)\,f^*(y|u)\,dy\,d\mathbf{P}_\chi(u) \\ &= \frac{1}{h_H\mathbb{E}\left(K_1(\chi)\right)}\int\int K_1(\chi)\,H_1'(y)\,f(y|u)\,dy\,d\mathbb{P}_\chi(u) \\ &= \frac{1}{h_H\mathbb{E}\left(K_1(\chi)\right)}\mathbb{E}\left[K_1(\chi)\,H_1'(y)\right] \\ &= \frac{1}{h_H\mathbb{E}\left(K_1(\chi)\right)}\mathbb{E}\left[K_1(\chi)\,\mathbb{E}\left(H_1'(y)\,|\mathcal{X}_1\right)\right]. \end{aligned} \tag{9}$$

Moreover, by changing variables, we have

$$\begin{aligned} \mathbb{E}\left(H_1'(y)\,|\mathcal{X}_1\right) &= \int H'\left(\frac{y-u}{h_H}\right) f(u|\mathcal{X}_1)\,du \\ &= h_H\int H'(z)\,f(y - zh_H|\mathcal{X}_1)\,dz \\ &= h_H\int H'(z)\left[f(y - zh_H|\mathcal{X}_1) - f(y|\chi)\right]dz \\ &+ h_H\int H'(z)\,f(y|\chi)\,dz. \end{aligned} \tag{10}$$

Substituting (10) in (9) we obtain

$$\begin{aligned} \mathbf{E}\left(\widetilde{f}_{1,n}(\chi,y)\right) &= \frac{1}{\mathbb{E}\left(K_1(\chi)\right)}\mathbb{E}\left(K_1(\chi)\int H'(z)\left[f(y - zh_H|\mathcal{X}_1) - f(y|\chi)\right]dz\right) \\ &+ \frac{1}{\mathbb{E}\left(K_1(\chi)\right)}\mathbb{E}\left(K_1(\chi)\,f(y|\chi)\right) \\ &=: \mathcal{I}_1 + \mathcal{I}_2. \end{aligned}$$

It is clear that $\mathcal{I}_2 = f(y|\chi)$. For $\mathcal{I}_1$, under Assumptions **A2**(ii) and **A4**, we get

$$\begin{aligned} \int H'(z)\left|f(y - zh_H|\mathcal{X}_1) - f(y|\chi)\right|dz &\le C_\chi\int H'(z)\left(d(\chi,\mathcal{X}_1)^{b_1} + |zh_H|^{b_2}\right)dz \\ &\le C_\chi\int H'(z)\left(h_K^{b_1} + |z|^{b_2}h_H^{b_2}\right)dz \\ &= O\left(h_K^{b_1} + h_H^{b_2}\right), \end{aligned}$$

hence $\mathcal{I}_1 = O\left(h_K^{b_1} + h_H^{b_2}\right)$. This ends the proof.

• ii) The proof is done similarly to (9) by using formula (4). □

Lemma 2. *Under Assumptions **A1**, **A3**, **A4**(ii) and **A5-A7**, for n large enough we have*

$$\sup_{\chi\in\Xi}\sup_{y\in[a,b]}\left|\widetilde{f}_{1,n}(\chi,y)-\mathbf{E}\left(\widetilde{f}_{1,n}(\chi,y)\right)\right|=O\left(\sqrt{\frac{\log n}{nh_H\phi(h_K)}}+\sqrt{\frac{\phi^{\nu-2}\left(h_K\right)\log n}{h_H}}\right)\mathbf{P}-a.s.$$

Proof. As Ξ and $[a,b]$ are compact sets, then they can be covered by a finite number l_n of balls $\mathcal{B}_k:=\mathcal{B}_k(\chi_k,r_n)$ of radius r_n ($r_n\to 0$), centered at $\chi_1,...,\chi_{l_n}$ and a finite number d_n of intervals $\mathcal{I}_t:=[y_t-u_n,y_t+u_n]$ of half length u_n ($u_n\to 0$) centered at $y_1,...,y_{d_n}$, respectively. Therefore, it follows that for all $(\chi,y)\in\Xi\times[a,b]$, there exist a ball $\mathcal{B}_k$ and an interval $\mathcal{I}_t$ such that $d\left(\chi,\chi_k\right)\leq r_n$ and $|y-y_t|\leq u_n$. Moreover, there exist two positive constants C_1 and C_2 such that $l_nr_n\leq C_1$ and $d_nu_n\leq C_2$.

Then, we consider the following decomposition

$$\begin{aligned}\sup_{\chi\in\Xi}\sup_{y\in[a,b]}\left|\widetilde{f}_{1,n}(\chi,y)-\mathbf{E}\left(\widetilde{f}_{1,n}(\chi,y)\right)\right| &\leq \max_{1\leq k\leq l_n}\sup_{\chi\in\mathcal{B}_k}\sup_{y\in[a,b]}\left|\widetilde{f}_{1,n}(\chi,y)-\widetilde{f}_{1,n}(\chi_k,y)\right|\\ &+ \max_{1\leq k\leq l_n}\max_{1\leq t\leq d_n}\sup_{y\in\mathcal{I}_t}\left|\widetilde{f}_{1,n}(\chi_k,y)-\widetilde{f}_{1,n}(\chi_k,y_t)\right|\\ &+ \max_{1\leq k\leq l_n}\max_{1\leq t\leq d_n}\left|\widetilde{f}_{1,n}(\chi_k,y_t)-\mathbf{E}\left(\widetilde{f}_{1,n}(\chi_k,y_t)\right)\right|\\ &+ \max_{1\leq k\leq l_n}\max_{1\leq t\leq d_n}\sup_{y\in\mathcal{I}_t}\left|\mathbf{E}\left(\widetilde{f}_{1,n}(\chi_k,y_t)\right)-\mathbf{E}\left(\widetilde{f}_{1,n}(\chi_k,y)\right)\right|\\ &+ \max_{1\leq k\leq l_n}\sup_{\chi\in\mathcal{B}_k}\sup_{y\in[a,b]}\left|\mathbf{E}\left(\widetilde{f}_{1,n}(\chi_k,y)\right)-\mathbf{E}\left(\widetilde{f}_{1,n}(\chi,y)\right)\right|\\ &=: \sum_{i=1}^{5}\mathcal{J}_{i,n}.\end{aligned}$$

Using Jensen's inequality, $\mathcal{J}_{5,n}\leq\mathbf{E}\left(\mathcal{J}_{1,n}\right)$, thus we deal only with $\mathcal{J}_{1,n}$. Under Assumption **A4**(ii), we have

$$\begin{aligned}\sup_{y\in[a,b]}\left|\widetilde{f}_{1,n}(\chi,y)-\widetilde{f}_{1,n}(\chi_k,y)\right| &\leq \frac{1}{nh_H}G^{-1}(a)\sum_{i=1}^{n}H_i'\left(y\right)\left|\frac{K_i(\chi)}{\mathbb{E}\left(K_1\left(\chi\right)\right)}-\frac{K_i(\chi_k)}{\mathbb{E}\left(K_1\left(\chi_k\right)\right)}\right|\\ &\leq C_1G^{-1}(a)\frac{1}{nh_H}\sum_{i=1}^{n}\frac{|K_i(\chi)-K_i(\chi_k)|}{\mathbb{E}\left(K_1\left(\chi\right)\right)}\\ &+ C_1G^{-1}(a)\frac{1}{nh_H}\sum_{i=1}^{n}\left|\frac{K_i(\chi_k)}{\mathbb{E}\left(K_1\left(\chi\right)\right)}-\frac{K_i(\chi_k)}{\mathbb{E}\left(K_1\left(\chi_k\right)\right)}\right|\\ &=: C_1G^{-1}(a)\left(\mathcal{J}_{11,n}\left(\chi,\chi_k\right)+\mathcal{J}_{12,n}\left(\chi,\chi_k\right)\right).\end{aligned}$$

Under Assumption **A3**, we have

$$\sup_{\chi\in\mathcal{B}_k}\mathcal{J}_{11,n}\left(\chi,\chi_k\right)\leq\frac{\|K'\|_\infty}{h_H\mathbb{E}\left(K_1\left(\chi\right)\right)}\sup_{\chi\in\mathcal{B}_k}\frac{d\left(\chi,\chi_k\right)}{h_K},$$

and

$$\sup_{\chi\in\mathcal{B}_k}\mathcal{J}_{12,n}\left(\chi,\chi_k\right)\leq\frac{\|K\|_\infty\mathbb{E}\left|K_1'\left(\chi^*\right)\right|}{h_H\mathbb{E}\left(K_1\left(\chi\right)\right)\mathbb{E}\left(K_1\left(\chi_k\right)\right)}\sup_{\chi\in\mathcal{B}_k}\frac{d\left(\chi,\chi_k\right)}{h_K},$$

where $\chi^*\in\mathcal{B}_k(\chi_k,r_n)$. From the fact that

$$\mathbb{E}\left|K_1'\left(\chi^*\right)\right|\leq\|K'\|_\infty\phi_{\chi^*}(h_K),$$

and making use of the results in (6) and (7), we get

$$\sup_{\chi\in\mathcal{B}_k}\left(\mathcal{J}_{11,n}\left(\chi,\chi_k\right)+\mathcal{J}_{12,n}\left(\chi,\chi_k\right)\right)\leq\frac{3}{2m_\xi}\frac{\|K'\|_\infty}{C_1}\frac{r_n}{h_Hh_K\phi(h_K)}\left(1+\frac{2\Gamma\|K\|_\infty}{C_1}\right).\qquad(11)$$

Choosing $r_n = h_K\sqrt{\frac{h_H\phi(h_K)}{n}}$, combining with (11), we obtain for n large enough

$$\mathcal{J}_{1,n} = O\left(\frac{1}{\sqrt{nh_H\phi(h_K)}}\right)\mathbf{P} - a.s., \tag{12}$$

and consequently,

$$\mathcal{J}_{5,n} = O\left(\frac{1}{\sqrt{nh_H\phi(h_K)}}\right)\mathbf{P} - a.s. \tag{13}$$

As before, $\mathcal{J}_{4,n} \le \mathbf{E}\left(\mathcal{J}_{2,n}\right)$, thus we deal only with $\mathcal{J}_{2,n}$. Under Assumptions **A3**(i) and **A4**(ii) we have

$$\begin{aligned}
\sup_{y\in\mathcal{I}_t}\left|\widetilde{f}_{1,n}(\chi_k,y) - \widetilde{f}_{1,n}(\chi_k,y_t)\right| &\le \frac{C_1}{h_H\mathbb{E}\left(K_1\left(\chi_k\right)\right)}G^{-1}(a)\|K\|_\infty\sup_{y\in\mathcal{I}_t}\frac{|y-y_t|}{h_H}\\
&\le \frac{C_1}{\phi(h_K)}\sup_{y\in\mathcal{I}_t}\frac{|y-y_t|}{h_H^2}\\
&\le \frac{C_1}{\phi(h_K)}\frac{u_n}{h_H^2}.
\end{aligned}$$

By choosing $u_n = \sqrt{\frac{h_H^3\phi(h_K)}{n}}$, we get for n large enough

$$\mathcal{J}_{2,n} = O\left(\frac{1}{\sqrt{nh_H\phi(h_K)}}\right)\mathbf{P} - a.s., \tag{14}$$

then

$$\mathcal{J}_{4,n} = O\left(\frac{1}{\sqrt{nh_H\phi(h_K)}}\right)\mathbf{P} - a.s. \tag{15}$$

Now we focus on upper bounding the principal term $\mathcal{J}_{3,n}$. Set

$$\Delta_i(\chi_k,y_t) := \mu\left\{G^{-1}\left(Y_i\right)K_i\left(\chi_k\right)H_i'\left(y_t\right) - \mathbf{E}\left(G^{-1}\left(Y_i\right)K_i\left(\chi_k\right)H_i'\left(y_t\right)\right)\right\},$$

for all $1 \le i \le n$, $1 \le k \le l_n$ and $1 \le t \le d_n$. Obviously we have

$$\widetilde{f}_{1,n}(\chi_k,y_t) - \mathbf{E}\left(\widetilde{f}_{1,n}(\chi_k,y_t)\right) = \frac{1}{nh_H\mathbb{E}\left(K_1\left(\chi_k\right)\right)}\sum_{i=1}^{n}\Delta_i(\chi_k,y_t).$$

In order to apply the Fuk-Nagaev's exponential inequality (see Rio (2000) formula 6.19b) slightly modified in Ferraty and Vieu (2006) (proposition A.11-ii), note that by Assumptions **A3**(i) and **A4**(ii) we have

$$|\Delta_i(\chi_k,y_t)| \le \frac{2C_1\mu\|K\|_\infty}{G(a)} < \infty$$

and the covariance term

$$\begin{aligned}
S_n^2 &:= \sum_{i=1}^{n}\sum_{j=1}^{n}\left|\mathrm{Cov}\left(\Delta_i(\chi_k,y_t),\Delta_j(\chi_k,y_t)\right)\right|\\
&= \sum_{\substack{i=1\\ i\ne j}}^{n}\sum_{j=1}^{n}\left|\mathrm{Cov}\left(\Delta_i(\chi_k,y_t),\Delta_j(\chi_k,y_t)\right)\right| + n\mathrm{Var}\left(\Delta_1(\chi_k,y_t)\right)\\
&=: S_{1n}^2 + S_{2n}^2.
\end{aligned} \tag{16}$$

To evaluate the asymptotic behavior of S_{1n}^2, we use a technique developed in Masry (1986). Define the sub-sets E_1 and E_2 as

$$E_1 = \left\{(i,j); 1 \le |i-j| \le m_n\right\} \text{ and } E_2 = \left\{(i,j); m_n+1 \le |i-j| \le n-1\right\},$$

where $m_n = o(n)$ at a rate specified later. Then,

$$S_{1n}^2 = \sum\sum_{E_1} |\text{Cov}(\Delta_i(\chi_k, y_t), \Delta_j(\chi_k, y_t))| + \sum\sum_{E_2} |\text{Cov}(\Delta_i(\chi_k, y_t), \Delta_j(\chi_k, y_t))|. \qquad (17)$$

On the one hand, to evaluate the sum over E_1, we write

$$\begin{aligned} |\text{Cov}(\Delta_i(\chi_k, y_t), \Delta_j(\chi_k, y_t))| &= |\mathbf{E}[\Delta_i(\chi_k, y_t)\Delta_j(\chi_k, y_t)]| \\ &\leq \mu^2 G^{-2}(a)\,\mathbf{E}\left(K_i(\chi_k) K_j(\chi_k) H_i'(y_t) H_j'(y_t)\right) \\ &+ \mu^2 G^{-2}(a)\,\mathbf{E}\left(K_i(\chi_k) H_i'(y_t)\right)\mathbf{E}\left(K_j(\chi_k) H_j'(y_t)\right). \end{aligned}$$

Using the fact that under Assumption **A1**(iii) we have

$$\mathbf{E}\left(H_i'(y_t) H_j'(y_t) \,|\, (\mathcal{X}_i, \mathcal{X}_j)\right) = O\left(h_H^2\right)$$

and

$$\mathbf{E}\left(H_i'(y_t) \,|\mathcal{X}_i\right) = O(h_H),$$

we get

$$\mathbf{E}\left(K_i(\chi_k) K_j(\chi_k) H_i'(y_t) H_j'(y_t)\right) \leq C_1 h_H^2 \mathbf{P}\left((\mathcal{X}_i, \mathcal{X}_j) \in \mathcal{B}(\chi, h_K) \times \mathcal{B}(\chi, h_K)\right)$$

and

$$\mathbf{E}\left(K_i(\chi_k) H_i'(y_t)\right) \leq C_1 h_H \mathbf{P}\left(\mathcal{X}_i \in \mathcal{B}(\chi, h_K)\right).$$

Again, under Assumption **A1**(iii) we obtain

$$|\text{Cov}(\Delta_i(\chi_k, y_t), \Delta_j(\chi_k, y_t))| = O\left(h_H^2 \phi^2(h_K)\right). \qquad (18)$$

On the other hand, to evaluate the sum under E_2, we use Davydov-Rio's covariance inequality (Rio (2000), formula 1.12a) for mixing processes. For all $i \neq j$,

$$|\text{Cov}(\Delta_i(\chi_k, y_t), \Delta_j(\chi_k, y_t))| \leq C\alpha(|i-j|).$$

By the fact that $\sum_{\kappa \geq m_n+1} \kappa^{-\nu} \leq \int_{m_n}^{\infty} \theta^{-\nu} d\theta = \frac{m_n^{1-\nu}}{\nu - 1}$ and under Assumption **A5**, we get

$$\sum\sum_{E_2} |\text{Cov}(\Delta_i(\chi_k, y_t), \Delta_j(\chi_k, y_t))| \leq \frac{n m_n^{1-\nu}}{\nu - 1}. \qquad (19)$$

Combining (18) and (19) with (17), we write

$$S_{1n}^2 \leq C_1 n m_n h_H^2 \phi^2(h_K) + C_2 n^2 m_n^{-\nu}.$$

Choosing $m_n = \lfloor\left((h_H \phi(h_K))^{-1}\right)\rfloor$ (where $\lfloor\cdot\rfloor$ denotes the greatest integer less than or equal to the argument), we find

$$S_{1n}^2 = O(n h_H \phi(h_K)) + O\left(n^2 (h_H \phi(h_K))^{\nu}\right). \qquad (20)$$

Concerning the variance term, under Assumptions **A3**(i) and **A4**(ii) we have

$$\begin{aligned} S_{2n}^2 &= n\mathbf{E}\left[\Delta_1^2(\chi_k, y_t)\right] \\ &\leq n\mu^2 G^{-2}(a)\,\mathbf{E}\left[K_1^2(\chi_k) H_1'^2(y_t)\right] \\ &\leq Cn\|K_1\|_\infty \mathbf{E}\left[K_1(\chi_k)\,\mathbf{E}\left(H_1'^2(y_t) \,|\chi_k\right)\right], \end{aligned}$$

with $\mathbf{E}\left(H_1'^2(y_t)\,|\chi_k\right) = O(h_H)$, which yields

$$S_{2n}^2 = O\left(nh_H\phi(h_K)\right). \tag{21}$$

From (20) and (21) combining with (16), we get

$$S_n^2 = O\left(nh_H\phi(h_K)\right) + O\left(n^2\left(h_H\phi\left(h_K\right)\right)^{\nu}\right). \tag{22}$$

Hence, according to the Fuk-Nagaev's exponential inequality, for all $\varepsilon > 0$, $q \geq 1$, $\nu \geq 1$, for some $C_1 < \infty$, combining with (22), we have

$$\begin{aligned}
\mathbf{P}\left(\left|\widetilde{f}_{1,n}(\chi_k, y_t) - \mathbf{E}\left(\widetilde{f}_{1,n}(\chi_k, y_t)\right)\right| > \varepsilon\right) &= \mathbf{P}\left(\left|\sum_{i=1}^{n}\Delta_i(\chi_k, y_t)\right| > nh_H\mathbb{E}\left(K_1\left(\chi_k\right)\right)\varepsilon\right) \\
&\leq C_1\left(1 + \frac{\left(nh_H\mathbb{E}\left(K_1\left(\chi_k\right)\right)\varepsilon\right)^2}{q\left(nh_H\phi(h_K) + n^2h_H^{\nu}\phi^{\nu}\left(h_K\right)\right)}\right)^{-q/2} \\
&+ C_1\frac{n}{q}\left(\frac{q}{nh_H\mathbb{E}\left(K_1\left(\chi_k\right)\right)\varepsilon}\right)^{\nu+1} \\
&=: C_1\left(A_{1n} + A_{2n}\right).
\end{aligned}$$

Set $\varepsilon = \varepsilon_0\sqrt{\dfrac{\log n}{nh_H\phi(h_K)} + \dfrac{\phi^{\nu-2}(h_K)\log n}{h_H}}$ with $\varepsilon_0 > 0$, using the results in (2) and (6) we get

$$A_{1n} \leq \left(1 + \frac{\varepsilon_0^2\log n}{q}\right)^{-q/2}.$$

By choosing $q = C_1(\log n)^2$ and using a Taylor expansion of $\log(1+x)$ we have

$$A_{1n} \leq C_1 n^{-\frac{\varepsilon_0^2}{2}}$$

and

$$A_{2n} \leq C_1 n^{\frac{1-\nu}{2}}\left(\log n\right)^{\frac{3\nu-1}{2}}\left(h_H\phi\left(h_K\right)\right)^{-\frac{\nu+1}{2}}.$$

Then, we can write

$$\begin{aligned}
\mathbf{P}\left(J_{3,n} > \varepsilon\right) &\leq \sum_{k=1}^{l_n}\sum_{t=1}^{d_n}\left(A_{1n} + A_{2n}\right) \\
&\leq C_1\left(r_n u_n\right)^{-1}\left(A_{1n} + A_{2n}\right) \\
&\leq C_1\frac{n}{\phi\left(h_K\right)}\left(h_H^2 h_K\right)^{-1}\left(A_1 + A_2\right) \\
&=: \widetilde{A}_{1n} + \widetilde{A}_{2n}.
\end{aligned}$$

Assumption **A6** permits to get

$$\widetilde{A}_{2n} \leq C_1 n^{\frac{3+2\delta-\nu}{2}}\left(\log n\right)^{\frac{3\nu-1}{2}}$$

and ensures that $\frac{3+2\delta-\nu}{2} < -1$. Hence $\widetilde{A}_{2n}$ is bounded by a general term of a convergent series.

In the same way, Assumption **A6** gives

$$\widetilde{A}_{1n} \leq C_1 n^{\frac{2-\varepsilon_0^2+\delta}{2} + \frac{2\delta}{\nu+5} + \frac{\delta}{3(\nu+3)}}.$$

Note that Assumption **A7**(i) together with Assumption **A6** ensure that $\phi\left(h_K\right) \to 0$ as $n \to \infty$. Hence, for a suitable choice of ε_0, the upper bound of $\widetilde{A}_{1n}$ becomes a general term of a

convergent series.
Consequently $\sum_{n\geq 1}\left(\widetilde{A}_{1n}+\widetilde{A}_{2n}\right)<\infty$, and therefore Borel-Cantelli's Lemma permits to get

$$\mathcal{J}_{3,n}=O\left(\sqrt{\frac{\log n}{nh_H\phi\left(h_K\right)}}+\sqrt{\frac{\phi^{\nu-2}(h_K)\log n}{h_H}}\right). \tag{23}$$

Finally, from (12)-(15) and (23), we end the proof. □

Lemma 3. *Under Assumptions **A1**, **A3** and **A5-A7**, we have*

$$\sup_{\chi\in\Xi}\left|\widetilde{g}_n(\chi)-\mathbf{E}\left(\widetilde{g}_n(\chi)\right)\right|=O\left(\sqrt{\frac{\log n}{n\phi(h_K)}}+\sqrt{\phi^{\nu-2}(h_K)\log n}\right)\mathbf{P}-a.s. \text{ as } n\to\infty.$$

Proof. Following the same steps as in the proof of Lemma 2, we use only the covering for the compact Ξ and we write

$$\begin{aligned}\sup_{\chi\in\Xi}\left|\widetilde{g}_n(\chi)-\mathbf{E}\left(\widetilde{g}_n(\chi)\right)\right| &\leq \max_{1\leq k\leq l_n}\sup_{\chi\in\mathcal{B}_k}\left|\widetilde{g}_n(\chi)-\widetilde{g}_n(\chi_k)\right|\\ &+ \max_{1\leq k\leq l_n}\left|\widetilde{g}_n(\chi_k)-\mathbf{E}\left(\widetilde{g}_n(\chi_k)\right)\right|\\ &+ \max_{1\leq k\leq l_n}\sup_{\chi\in\mathcal{B}_k}\left|\mathbf{E}\left(\widetilde{g}_n(\chi_k)\right)-\mathbf{E}\left(\widetilde{g}_n(\chi)\right)\right|\\ &=: \mathcal{J'}_{1,n}+\mathcal{J'}_{2,n}+\mathcal{J'}_{3,n}.\end{aligned}$$

Choosing $r_n=\dfrac{\phi(h_K)}{\sqrt{n}}$ we find

$$\mathcal{J'}_{1,n}=O\left(\frac{1}{h_K\sqrt{n}}\right)\mathbf{P}-a.s. \text{ and } \mathcal{J'}_{3,n}=O\left(\frac{1}{h_K\sqrt{n}}\right).$$

For $\mathcal{J'}_{2,n}$, we apply Fuk Nagaev's exponential inequality by setting

$$\Delta'_i(\chi_k):=\mu\left\{G^{-1}\left(Y_i\right)K_i\left(\chi_k\right)-\mathbf{E}\left(G^{-1}\left(Y_i\right)K_i\left(\chi_k\right)\right)\right\}$$

and choosing $\varepsilon=\varepsilon_0\sqrt{\dfrac{\log n}{n\phi(h_K)}+\phi^{\nu-2}\left(h_K\right)\log n}$ with $\varepsilon_0>0$, we get

$$\mathcal{J'}_{2,n}=O\left(\sqrt{\frac{\log n}{n\phi\left(h_K\right)}}+\sqrt{\phi^{\nu-2}(h_K)\log n}\right)\mathbf{P}-a.s.$$

□

Lemma 4. *Under the same assumptions as Lemma 2, we have*

i. $\displaystyle\sup_{\chi\in\Xi}\sup_{y\in[a,b]}\left|f_{1,n}(\chi,y)-\widetilde{f}_{1,n}(\chi,y)\right|=O\left(\sqrt{\frac{\log\log n}{n}}\right)\mathbf{P}-a.s. \text{ as } n\to\infty,$

ii. $\displaystyle\sup_{\chi\in\Xi}\left|g_n(\chi)-\widetilde{g}_n(\chi)\right|=O\left(\sqrt{\frac{\log\log n}{n}}\right)\mathbf{P}-a.s. \text{ as } n\to\infty.$

Proof. • i) From (1) and (5) we write

$$\begin{aligned}\left|f_{1,n}(\chi,y)-\widetilde{f}_{1,n}(\chi,y)\right| &= \frac{1}{nh_H\mathbb{E}\left(K_1\left(\chi\right)\right)}\sum_{i=1}^{n}\left|\frac{\mu_n}{G_n(Y_i)}-\frac{\mu}{G(Y_i)}\right|K_i\left(\chi\right)H_i'\left(y\right)\\ &= \frac{1}{nh_H\mathbb{E}\left(K_1\left(\chi\right)\right)}\sum_{i=1}^{n}\left|\frac{\mu_n-\mu}{G_n(Y_i)}-\frac{\mu\left(G_n(Y_i)-G(Y_i)\right)}{G_n(Y_i)G(Y_i)}\right|\\ &\times\ K_i\left(\chi\right)H_i'\left(y\right)\\ &\leq v_n(\chi,y)\left(\frac{|\mu_n-\mu|}{G_n(a)}+\frac{\mu}{G_n(a)G(a)}\sup_{y\in[a,b]}|G_n(y)-G(y)|\right),\end{aligned}$$

with $v_n(\chi,y):=\dfrac{1}{nh_H\mathbb{E}\left(K_1\left(\chi\right)\right)}\sum_{i=1}^{n}K_i\left(\chi\right)H_i'\left(y\right).$

We prove easily that $\sup_{\chi\in\Xi}|v_n(\chi,y)|=O\left(1\right)\mathbf{P}-a.s.$ by writing

$$v_n(\chi,y)\leq|v_n(\chi,y)-\mathbf{E}\left(v_n(\chi,y)\right)|+|\mathbf{E}\left(v_n(\chi,y)\right)|. \tag{24}$$

The first term of the right hand side of (24) is a particular case of Lemma 2 which gives $|v_n(\chi,y)-\mathbf{E}\left(v_n(\chi,y)\right)|=O\left(\sqrt{\frac{\log n}{nh_H\phi(h_K)}}+\sqrt{\frac{\phi^{\nu-2}\left(h_K\right)\log n}{h_H}}\right)$, wheras $\mathbf{E}\left(v_n(\chi,y)\right)=O\left(1\right)$. On the other hand, from Ould-Saïd and Tatachak (2009) (Lemma 5.2), under Assumption **A5**, we have

$$|\mu_n-\mu|=O\left(\sqrt{\frac{\log\log n}{n}}\right)\mathbf{P}-a.s.$$

and

$$\sup_{y\in[a,b]}|G_n(y)-G(y)|=O\left(\sqrt{\frac{\log\log n}{n}}\right)\mathbf{P}-a.s.$$

Moreover, $G_n\left(a\right)\overset{\mathbf{P}-a.s.}{\longrightarrow}G\left(a\right)>0$ which concludes the proof.

• ii) The proof can be done by following the same arguments as those used in the proof of i). □

Lemma 5. *Under Assumptions **A1** and **A3**, there exists* $\rho>0$ *such that*

$$\sum_{n\geq1}\mathbf{P}\left(\inf_{\chi\in\Xi}\widetilde{g}_n(\chi)\leq\rho\right)<\infty.$$

Proof. From the following inequality

$$\inf_{\chi\in\Xi}\widetilde{g}_n(\chi)\geq\inf_{\chi\in\Xi}\mathbf{E}\left[\widetilde{g}_n(\chi)\right]-\sup_{\chi\in\Xi}|\widetilde{g}_n(\chi)-\mathbf{E}\left[\widetilde{g}_n(\chi)\right]|,$$

making use of Lemma 3, we get the result. Note that the result of Lemma 5 combining with Lemma 4 (ii) permit to get

$$\sum_{n\geq1}\mathbf{P}\left(\inf_{\chi\in\Xi}g_n(\chi)\leq\rho\right)<\infty.$$

□

Proof of Proposition 2. Similarly to (8), we have

$$\begin{aligned}F_n(y|\chi)-F(y|\chi) &= \frac{1}{g_n(\chi)}\left(F_{1,n}(\chi,y)-\widetilde{F}_{1,n}(\chi,y)\right)+\frac{1}{g_n(\chi)}\left(\widetilde{F}_{1,n}(\chi,y)-\mathbf{E}\left(\widetilde{F}_{1,n}(\chi,y)\right)\right)\\ &+\frac{1}{g_n(\chi)}\left(\mathbf{E}\left(\widetilde{F}_{1,n}(\chi,y)\right)-F(y|\chi)\right)+\frac{F(y|\chi)}{g_n(\chi)}\left(g_n(\chi)-\widetilde{g}_n(\chi)\right)\\ &+\frac{F(y|\chi)}{g_n(\chi)}\left(\mathbf{E}\left(\widetilde{g}_n(\chi)\right)-\widetilde{g}_n(\chi)-\mathbf{E}\left(\widetilde{g}_n(\chi)\right)+1\right).\end{aligned}$$

Therefore, Proposition 2 is a consequence of the following intermediate results. □

Lemma 6. *Under Assumptions **A2**(i) and **A4**(i), we have*

$$\sup_{\chi\in\Xi}\sup_{y\in[a,b]}\left|\mathbf{E}\left[\widetilde{F}_{1,n}(\chi,y)\right]-F(y|\chi)\right|=O\left(h_K^{b_1}+h_H^{b_2}\right)\ \text{as } n\to\infty.$$

Proof. From (3) and similarly to (9), we have

$$\mathbf{E}\left[\widetilde{F}_{1,n}(\chi,y)\right]=\frac{1}{\mathbb{E}\left(K_1\left(\chi\right)\right)}\mathbb{E}\left(K_1\left(\chi\right)\mathbb{E}\left[H_1\left(y\right)|\mathcal{X}_1\right]\right). \tag{25}$$

Otherwise, by integrating by parts and changing variables, we get

$$\begin{aligned}\mathbb{E}\left(H_1\left(y\right)|\mathcal{X}_1\right) &= \int H\left(\frac{y-u}{h_H}\right)f\left(u|\mathcal{X}_1\right)du\\ &= \int H'\left(z\right)F\left(y-zh_H|\mathcal{X}_1\right)dz\\ &= \int H'\left(z\right)\left[F\left(y-zh_H|\mathcal{X}_1\right)-F\left(y|\chi\right)\right]dz+\int H'\left(z\right)F\left(y|\chi\right)dz. \end{aligned}\tag{26}$$

Substituting (26) in (25) we get

$$\begin{aligned}\mathbf{E}\left[\widetilde{F}_{1,n}(\chi,y)\right] &= \frac{1}{\mathbb{E}\left(K_1\left(\chi\right)\right)}\mathbb{E}\left\{K_1\left(\chi\right)\int H'\left(z\right)\left[F\left(y-zh_H|\mathcal{X}_1\right)-F\left(y|\chi\right)\right]dz\right\}\\ &+ \frac{1}{\mathbb{E}\left(K_1\left(\chi\right)\right)}\mathbb{E}\left[K_1\left(\chi\right)F\left(y|\chi\right)\right].\end{aligned}$$

Following the same idea as in the proof of Lemma 1 and under Assumptions **A2**(i) and **A4**(i) we get the result. □

Lemma 7. *Under Assumptions **A1**, **A3**, **A4**(ii) and **A5**-**A7**, we have*

$$\sup_{\chi\in\Xi}\sup_{y\in[a,b]}\left|\widetilde{F}_{1,n}(\chi,y)-\mathbf{E}\left(\widetilde{F}_{1,n}(\chi,y)\right)\right|=O\left(\sqrt{\frac{\log n}{n\phi(h_K)}}+\sqrt{\phi^{\nu-2}(h_K)\log n}\right)\mathbf{P}-a.s.\ \text{as } n\to\infty.$$

Proof. The proof is analogous to that in Lemma 2, replacing $\Delta_i\left(\chi_k,y_t\right)$ by

$$\Delta_i''(\chi_k,y_t):=\mu\left\{G^{-1}\left(Y_i\right)K_i\left(\chi_k\right)H_i\left(y_t\right)-\mathbf{E}\left(G^{-1}\left(Y_i\right)K_i\left(\chi_k\right)H_i\left(y_t\right)\right)\right\}$$

and choosing $\varepsilon=\varepsilon_0\sqrt{\frac{\log n}{n\phi(h_K)}+\phi^{\nu-2}\left(h_K\right)\log n}$ with $\varepsilon_0>0$, we get the result. □

Lemma 8. *Under the same assumptions as Lemma 7, we have*

$$\sup_{\chi\in\Xi}\sup_{y\in[a,b]}\left|F_{1,n}(\chi,y)-\widetilde{F}_{1,n}(\chi,y)\right|=O\left(\sqrt{\frac{\log\log n}{n}}\right)\mathbf{P}-a.s.\ \text{as } n\to\infty.$$

Proof. As we have treated the Lemma 4, observe that

$$\left|F_{1,n}(\chi,y)-\widetilde{F}_{1,n}(\chi,y)\right|\leq|w_n(\chi)|\left\{\frac{|\mu_n-\mu|}{G_n(a)}+\frac{\mu}{G_n(a)G(a)}\sup_{y\in[a,b]}|G_n(y)-G(y)|\right\},$$

where $w_n(\chi):=\frac{1}{n\mathbb{E}\left(K_1\left(\chi\right)\right)}\sum_{i=1}^{n}K_i\left(\chi\right)$.

Similarly to (24), we write

$$w_n(\chi)\leq|w_n(\chi)-\mathbf{E}\left(w_n(\chi)\right)|+|\mathbf{E}\left(w_n(\chi)\right)|. \tag{27}$$

The first term of the right hand side of (27) is a particular case of Lemma 7 and $\mathbf{E}\left(w_n(\chi)\right)=O\left(1\right)$. □

Proof of Theorem 1. The proof is based on the following decomposition

$$\begin{aligned}\lambda_n(y|\chi) - \lambda(y|\chi) &= \frac{f_n(y|\chi) - f(y|\chi) + f(y|\chi)\left[F_n(y|\chi) - F(y|\chi)\right]}{(1 - F_n(y|\chi))\,(1 - F(y|\chi))} \\ &- \frac{F(y|\chi)\left[f_n(y|\chi) - f(y|\chi)\right]}{(1 - F_n(y|\chi))\,(1 - F(y|\chi))},\end{aligned}$$

which implies

$$\sup_{\chi\in\Xi}\sup_{y\in[a,b]} |\lambda_n(y|\chi) - \lambda(y|\chi)| \leq C_1 \frac{\sup\limits_{\chi\in\Xi}\sup\limits_{y\in[a,b]} |f_n(y|\chi) - f(y|\chi)| + |F_n(y|\chi) - F(y|\chi)|}{\inf\limits_{\chi\in\Xi}\inf\limits_{y\in[a,b]} |1 - F_n(y|\chi))\,|}. \tag{28}$$

The result is a direct consequence of Propositions 1 and 2 combining with (28). □

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