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On the Ratio of Inverted Gamma Variates

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Abstract: In this paper the distribution and moments of the ratio of independent inverted gamma variates have been considered. Unbiased estimators of the parameter involved in the distribution have been proposed. As a particular case, the ratio of independent Levy variates have been studied.

Zusammenfassung: In diesem Aufsatz werden die Verteilung und Momente vom Quotient unabhängiger invertierter Gammavariablen betrachtet. Unverzerrte Schätzer für den Parameter in der Verteilung werden vorgeschlagen. Als ein spezieller Fall wird der Quotient unabhängiger Levy-Variablen untersucht.

Keywords: Moments, Unbiased Estimator, Maximum Likelihood Estimator, Levy Variables.

1 Introduction

The distribution of the ratio of random variables are of interest in problems in biological and physical sciences, econometrics, classification, and ranking and selection. Examples of the use of the ratio of random variables include Mendelian inheritance ratios in genetics, mass to energy ratios in nuclear physics, target to control precipitation in meteorology, and inventory ratios in economics. The distribution of ratio of random variables have been studied by several authors like Marsaglia (1965) and Korhonen and Narula (1989) for normal family, Press (1969) for student's t family, Basu and Lochner (1979) for Weibull family, Provost (1989) for gamma family, Pham-Gia (2000) for beta family, among others. The distribution of the ratio of independent gamma variates with shape parameters equal to 1 was studied by Bowman, Shenton, and Gailey (1998). Recently, Ali, Woo, and Pal (2006) obtained the distribution of the ratio of generalized uniform variates.

In this paper we derive the distribution of the ratio $V = X/(X+Y)$, where X and Y are independent inverted gamma variates, each with two parameters. An inverted gamma distribution IG(p,σ) is given by

$$f(x;p,\sigma) = \frac{\sigma^p}{\Gamma(p)} x^{-p-1} e^{-\sigma/x}\,, \qquad x>0\,,\ \sigma>0\,,\ p>0\,,$$

where p is the shape parameter and σ the scale parameter.

The moments of the distribution of the ratio have been obtained. As a particular case, the ratio of independent Levy variables has been considered. The Levy distribution is one of the few distributions that are stable and that have probability density functions that are analytically expressible. Moments of the Levy distribution do not exist. But the distribution is found to be very useful in analysis of stock prices and also in Physics for the study of dielectric susceptibility (see Jurlewicz and Weron, 1993).

2 Distribution of the Ratio of Inverted Gamma Variables

Let X and Y be independent random variables distributed as IG(p,σ_x) and IG(q,σ_y), respectively. Then $U=1/X$ and $W=1/Y$ are independently distributed as Gamma(p,σ_x) and Gamma(q,σ_y), respectively. We note that $V=X/(X+Y)=W/(U+W)$. Let $T=U+W$. Then V and T are jointly distributed with pdf

$$f_{V,T}(v,t)=\frac{\sigma_x^p\sigma_y^q}{\Gamma(p)\Gamma(q)}\,e^{-t\{\sigma_x+(\sigma_y-\sigma_x)v\}}t^{p+q-1}v^{q-1}(1-v)^{p-1}\,,\qquad 0<v<1\,,\ t>0\,.$$

Hence the marginal pdf of V is given by

$$f_V(v)=\frac{v^{q-1}(1-v)^{p-1}}{\rho^qB(q,p)}\left(1+\frac{1-\rho}{\rho}v\right)^{-p-q}\,,\quad 0<v<1\,,\ \rho=\frac{\sigma_x}{\sigma_y}>0\,. \qquad (1)$$

After some algebraic manipulation, and using formula 8.391 in Gradshteyn and Ryhzik (1965), the cumulative distribution function (cdf) of V is obtained as

$$F_V(v)=\frac{\{1+\rho(1-v)/v\}^{-q}}{qB(q,p)}{}_2F_1(q,1-p;q+1;\{1+\rho(1-v)/v\}^{-1})\,,\quad 0<v<1\,,$$

where

$${}_2F_1(a,b;c;x)=\sum_{i=0}^{\infty}\frac{(a)_i(b)_i}{(c)_i i!}x^i\,,\qquad (a)_i=a\cdot(a+1)\cdot\ldots\cdot(a+i-1)\,,\quad (a)_0=1\,,$$

is the Gauss hypergeometric series.

Using formula 3.197(3) in Gradshteyn and Ryhzik (1965), formulas 15.3.3 and 15.3.5 in Abramowitz and Stegtun (1970), and the density (1), we obtain the moments of the ratio $V=X/(X+Y)$ as

$$\mathrm{E}(V^k)=\begin{cases}\dfrac{B(q+k,p)}{B(q,p)}\,{}_2F_1(k,p;p+q+k;(\rho-1)/\rho)\,, & \text{if } \rho>1\,,\\ \rho^k\dfrac{B(q+k,p)}{B(q,p)}\,{}_2F_1(k,q+k;p+q+k;1-\rho)\,, & \text{if } 0<\rho<1\,.\end{cases} \qquad (2)$$

In order to estimate ρ, we make use of the following lemma.

Lemma 2.1: Let $R=V/(1-V)$. Then

(a) R is distributed as the ratio of two independent random variables with distributions Gamma(q,σ_y) and Gamma(p,σ_x).

(b) $\mathrm{E}(R^k)=\rho^kB(q+k,p-k)/B(q,p)$, provided $p>k$.

Proof: We have $R=V/(1-V)=X/Y=Y^{-1}/X^{-1}$. Since X and Y are independently distributed as IG(p,σ_x) and IG(q,σ_y), respectively, (a) easily follows.

The distribution of R is therefore defined by the pdf

$$f_R(r)=\frac{1}{\rho^qB(q,p)}\frac{r^{q-1}}{(1+r/\rho)^{p+q}}\,,\qquad r>0\,.$$

Hence, the k-th moment of R comes out to be

$$\mathrm{E}(R^k) = \frac{B(q+k, p-k)}{B(q,p)}\rho^k , \qquad \text{provided } p > k .$$

From the lemma, for $p > 1$, we have

$$\mathrm{E}(R) = \mathrm{E}\left(\frac{V}{1-V}\right) = \frac{q}{p-1}\rho \tag{3}$$

so that $\mathrm{E}(R) \cdot (p-1)/q = \rho$. Thus, for a random sample $V_1, \ldots, V_n$ of size n from the distribution of V, an unbiased estimator of ρ will be given by

$$\hat{\rho} = \frac{p-1}{nq}\sum_{i=1}^{n}\frac{V_i}{1-V_i} , \qquad \text{if } p > 1 .$$

The variance of this estimator is

$$\mathrm{var}(\hat{\rho}) = \frac{p+q-1}{nq(p-2)}\rho^2 , \qquad \text{for } p > 2 .$$

On the basis of independent random samples $X_1, \ldots, X_{n_1}$ and $Y_1, \ldots, Y_{n_2}$ drawn from the distributions of X and Y, respectively, the maximum likelihood estimator (MLE) of ρ is $\tilde{\rho} = \tilde{\sigma}_x/\tilde{\sigma}_y$, where $\tilde{\sigma}_x$ and $\tilde{\sigma}_y$ are the MLEs of σ_x and σ_y given by

$$\tilde{\sigma}_x = \frac{n_1 p}{\sum_{i=1}^{n_1}(1/X_i)} , \qquad \tilde{\sigma}_y = \frac{n_2 q}{\sum_{i=1}^{n_2}(1/Y_i)} .$$

Noting that $U = \sum_{i=1}^{n_1}(1/X_i)$ and $W = \sum_{i=1}^{n_2}(1/Y_i)$ are independently distributed as $\mathrm{Gamma}(n_1 p, \sigma_x)$ and $\mathrm{Gamma}(n_2 q, \sigma_y)$, respectively, $Z = (1/U)/(1/U + 1/W)$ is distributed with pdf given by (1) where p and q are replaced by $n_1 p$ and $n_2 q$, respectively. Also,

$$\tilde{\rho} = \frac{n_1 p}{n_2 q}\frac{Z}{1-Z} ,$$

such that, from (3),

$$\mathrm{E}(\tilde{\rho}) = \frac{n_1 p}{n_1 p - 1}\rho .$$

Hence,

$$\tilde{\tilde{\rho}} = \frac{n_1 p - 1}{n_1 p}\tilde{\rho} \tag{4}$$

is an unbiased estimator of ρ with

$$\mathrm{var}(\tilde{\tilde{\rho}}) = \frac{n_1 p + n_2 q - 1}{n_2 q(n_1 p - 2)}\rho^2 . \tag{5}$$

It can be easily seen that for $n_1 = n_2 = n$ we get $\mathrm{var}(\hat{\rho}) > \mathrm{var}(\tilde{\tilde{\rho}})$.

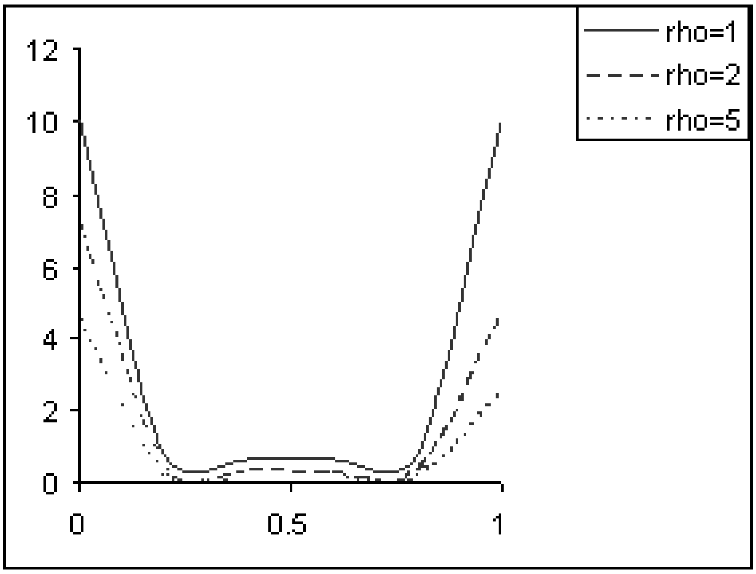

Figure 1: Plots of the pdf (6) for $\rho = 1, 2, 5$.

3 Distribution of the Ratio of Levy Variables

For $p = q = 1/2$, X and Y are two independent Levy variables with scale parameters σ_x and σ_y, respectively. The pdf and cdf of V then reduces to

$$\begin{aligned} f_V(v) &= \frac{1}{\pi\sqrt{\rho}} v^{-1/2}(1-v)^{-1/2}\left(1+\frac{1-\rho}{\rho}v\right)^{-1}, \qquad (6) \\ &= \frac{\sqrt{\rho}}{\pi} v^{-3/2}(1-v)^{-1/2}\left(1+\rho\frac{1-v}{v}\right)^{-1}, \qquad 0<v<1\,,\ \rho=\frac{\sigma_x}{\sigma_y}>0 \end{aligned}$$

and

$$\begin{aligned} F_V(v) &= \frac{2}{\pi}\left(1+\rho\frac{1-v}{v}\right)^{-1/2} {}_2F_1(1/2,1/2;3/2;(1+\rho(1-v)/v)^{-1}) \\ &= \frac{2\sqrt{\rho}}{\pi}\sqrt{\frac{1-v}{v}}\left(1+\rho\frac{1-v}{v}\right)^{-1} {}_2F_1(1,1;3/2;(1+\rho(1-v)/v)^{-1}) \qquad (7) \\ &= \frac{2}{\pi}\sin^{-1}\frac{1}{\sqrt{1+\rho\frac{1-v}{v}}}\,, \qquad 0<v<1\,. \qquad (8) \end{aligned}$$

The expression (7) of the cdf is obtained using formulas 3.381(3) and 6.455(1) in Gradshteyn and Ryhzik (1965), and expression (8) follows from formula 15.1.6 in Abramowitz and Stegtun (1970).

From (2) the moments of the distribution are

$$\mathrm{E}(V^k) = \begin{cases} \dfrac{B(k+1/2,1/2)}{\pi} {}_2F_1(k,1/2;1+k;(\rho-1)/\rho) & \text{if } \rho>1 \\ \dfrac{\rho^k B(k+1/2,1/2)}{\pi} {}_2F_1(k,k+1/2;1+k;1-\rho) & \text{if } 0<\rho<1\,. \end{cases}$$

From Abramowitz and Stegtun (1970), we have the following useful relations for the hypergeometric function ${}_2F_1(a, b; c; z)$:

Lemma 3.1:

(i) (Formula 15.2.2)

$$\frac{\partial^n}{\partial z^n} {}_2F_1(a, b; c; z) = \frac{(a)_n (b)_n}{(c)_n} {}_2F_1(a+n, b+n; c+n; z)$$

(ii) (Formula 15.1.13)

$${}_2F_1(a, a+1/2; 1+2a; z) = \frac{2^{2a}}{(1+\sqrt{1-z})^{2a}}$$

(iii) (Formula 15.1.14)

$${}_2F_1(a, a+1/2; 2a; z) = \frac{2^{2a-1}}{\sqrt{1-z}(1+\sqrt{1-z})^{2a-1}}$$

(iv) (Formula 15.3.5)

$${}_2F_1(a, b; c; z) = (1-z)^{-b} {}_2F_1(b, c-a; c; z/(z-1)) \, .$$

From Lemma 3.1 we get the following lemmas:

Lemma 3.2:

(a)

$${}_2F_1(1, 3/2; 2; z) = \frac{2}{\sqrt{1-z}(1+\sqrt{(1-z)})}$$

(b)

$${}_2F_1(2, 5/2, 3; z) = \frac{4}{3} \frac{1+2\sqrt{1-z}}{(1-z)^{3/2}(1+\sqrt{1-z})^2} \, .$$

Proof: (a) follows from Lemma 3.1(iii) by substituting $a = 1$. By Lemma 3.1(i) we get

$${}_2F_1(2, 5/2; 3; z) = \frac{4}{3} \frac{\partial}{\partial z} {}_2F_1(1, 3/2; 2; z) \, .$$

Hence, using (a), we have (b).

Lemma 3.3:

(a)

$${}_2F_1(1/2, 1; 2; (\rho-1)/\rho) = \frac{2\sqrt{\rho}}{1+\sqrt{\rho}}$$

(b)

$$ {}_2F_1(1/2, 2; 3; (\rho-1)/\rho) = \rho^2 {}_2F_1(2, 5/2; 3; 1-\rho) = \frac{4}{3}\frac{\sqrt{\rho}(1+2\sqrt{\rho})}{(1+\sqrt{\rho})^2}\,. $$

Proof: (a) follows from Lemma 3.1(ii) by substituting $a = 1/2$, $z = (\rho-1)/\rho$, and (b) follows from Lemma 3.1(iv) and Lemma 3.2(b) by taking $a = 1/2$, $b = 2$, $c = 3$, and $z = (\rho-1)/\rho$.

Using Lemmas 3.2 and 3.3 we have

$$ \mathrm{E}(V) = \frac{\sqrt{\rho}}{1+\sqrt{\rho}}\,, \qquad \mathrm{E}(V^2) = \frac{\sqrt{\rho}(1+2\sqrt{\rho})}{2(1+\sqrt{\rho})^2}\,, \tag{9} $$

such that

$$ \mathrm{var}(V) = \frac{\sqrt{\rho}}{2(1+\sqrt{\rho})^2}\,. \tag{10} $$

If $\tilde{\rho}$ denotes the MLE of ρ based on independent random samples $X_1,\dots,X_{n_1}$ and $Y_1,\dots,Y_{n_2}$ from the distributions of X and Y, then, from (4) and (5), an unbiased estimator of ρ and its variance are given by

$$ \tilde{\tilde{\rho}} = \frac{n_1-2}{n_1}\tilde{\rho} $$

$$ \mathrm{var}(\tilde{\tilde{\rho}}) = \left(\frac{n_1-2}{n_1}\right)^2 \frac{2(n_1+n_2-2)}{n_2(n_1-4)}\rho^2\,, \qquad \text{for } n_1 > 4\,. $$

Since $\rho^* = \sqrt{\rho}/(1+\sqrt{\rho})$ is a monotone increasing and bounded function of ρ, inference on ρ^* will be equivalent to inference on ρ. Hence, from any estimator of ρ^* we can obtain an estimator of ρ by a one-to-one transformation.

From (9) and (10), for a random sample $V_1,\dots,V_n$ of size n from the distribution (6), an unbiased estimator of ρ^* is $\bar{V} = n^{-1}\sum_{i=1}^n V_i$ with variance $\rho^*(1-\rho^*)/2n$. The corresponding estimator of ρ is $\hat{\hat{\rho}} = (\bar{V}/(1-\bar{V}))^2$ with asymptotic variance $2\rho^{3/2}(1+\sqrt{\rho})^2/n$.

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