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Decomposition of Measure for Marginal Homogeneity in Square Contingency Tables with Ordered Categories

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Abstract: For the analysis of square contingency tables with ordered categories, Tomizawa et al. (2003) considered a measure to represent the degree of departure from marginal homogeneity (MH). Tomizawa (1993) considered an extended marginal homogeneity (EMH) model. This paper (i) proposes a measure to represent the degree of departure from EMH, (ii) proposes a measure from equality of marginal means (E), and (iii) gives a theorem that the value of measure for MH is equal to the sum of the value of measure for EMH and that for E.

Zusammenfassung: Für die Analyse von quadratischen Kontingenztafeln mit geordneten Kategorien betrachtete Tomizawa et al. (2003) ein Maß, um den Grad der Abweichung von der marginalen Homogenität (MH) zu beschreiben. Tomizawa (1993) betrachtete ein erweitertes marginales Homogeniätsmodel (EMH). Dieser Artikel (i) schlägt ein Maß vor, um den Grad der Abweichung von EMH zu repräsentieren, (ii) schlägt eine Distanz von der Gleichheit von marginalen Mitteln (E) vor, und (iii) liefert einen Satz über die Gleichheit des Wertes von MH und der Summe der Werte von EMH und von E.

Keywords: Extended Marginal Homogeneity, Kullback-Leibler Information, Marginal Mean.

1 Introduction

Consider an $r \times r$ square contingency table with the same row and column classifications. Let p_{ij} denote the probability that an observation will fall in the ith row and jth column of the table ($i = 1, \dots, r$, $j = 1, \dots, r$).

Consider the marginal homogeneity (MH) model defined by

$$p_{i\bullet} = p_{\bullet i} \qquad \text{for} \quad i = 1, \dots, r \,,$$

where

$$p_{i\bullet} = \sum_{k=1}^{r} p_{ik} \,, \qquad p_{\bullet i} = \sum_{k=1}^{r} p_{ki} \,;$$

see, e.g., Stuart (1955) and Bishop, Fienberg, and Holland (1975, p.282). Let

$$G_{1(i)} = \sum_{s=1}^{i} \sum_{t=i+1}^{r} p_{st} \qquad \text{and} \qquad G_{2(i)} = \sum_{s=i+1}^{r} \sum_{t=1}^{i} p_{st}$$

for $i = 1, \dots, r-1$. By considering the difference between the cumulative marginal probabilities, the MH model may be expressed as

$$G_{1(i)} = G_{2(i)} \qquad \text{for} \quad i = 1, \dots, r-1 \,.$$

This states that the cumulative probability that an observation will fall in row category i or below and column category $i+1$ or above is equal to the cumulative probability that the observation falls in column category i or below and row category $i+1$ or above.

Consider the extended marginal homogeneity (EMH) model (Tomizawa, 1993) defined by

$$G_{1(i)} = \delta G_{2(i)} \qquad \text{for} \quad i = 1, \ldots, r-1 .$$

Let X and Y denote the row and column variables, respectively. Consider the model of equality of marginal means (E) defined by

$$\sum_{i=1}^{r} i p_{i\bullet} = \sum_{i=1}^{r} i p_{\bullet i} \quad [\text{i.e., } \mathrm{E}(X) = \mathrm{E}(Y)] .$$

Tomizawa (1991) pointed out that the MH model holds if and only if both the EMH and the E models hold.

Tomizawa (1995) and Tomizawa and Makii (2001) considered the measures to represent the degree of departure from MH for the data on a *nominal* scale, and Tomizawa et al. (2003) considered them for the data on an *ordinal* scale; see Appendix for the Kullback-Leibler (KL) information type measure Γ_{MH} proposed in Tomizawa et al. (2003).

When we want to see the degree of departure from EMH, we cannot use the measure Γ_{MH} because Γ_{MH} can measure the degree of departure from MH, however it cannot measure it from EMH. Therefore, for the data on an *ordinal* scale, we are interested in a measure to represent what degree the departure from EMH is.

The purpose of this paper is (i) to propose a measure which represents the degree of departure from EMH (denoted by Γ_{EMH}), (ii) to propose that from E (denoted by Γ_{E}), and (iii) to give the theorem that the value of Γ_{MH} is equal to the sum of the value of Γ_{EMH} and the value of Γ_{E}. We emphasize that the measure Γ_{EMH} proposed in this paper is entirely different from the measures which represent the degree of departure from MH in Tomizawa (1995), Tomizawa and Makii (2001), and Tomizawa et al. (2003).

2 Measures

2.1 Measure for Extended Marginal Homogeneity

We shall consider the measure to represent the degree of departure from EMH. Let

$$\Delta = \sum_{i=1}^{r-1} \left(G_{1(i)} + G_{2(i)} \right) \quad (> 0) , \qquad G_{1(i)}^{*} = \frac{G_{1(i)}}{\Delta} , \qquad G_{2(i)}^{*} = \frac{G_{2(i)}}{\Delta} ,$$

$$\Delta_U^{*} = \sum_{i=1}^{r-1} G_{1(i)}^{*} , \qquad \Delta_L^{*} = \sum_{i=1}^{r-1} G_{2(i)}^{*} .$$

Assuming that $\Delta_U^{*} > 0$, $\Delta_L^{*} > 0$, and $\{G_{1(i)} + G_{2(i)} > 0\}$, consider a measure defined by

$$\Gamma_{\mathrm{EMH}} = \frac{1}{\log 2} \, I \left(\left\{ G_{1(i)}^{*}, G_{2(i)}^{*} \right\} \; ; \; \left\{ G_{1(i)}^{\mathrm{EMH}}, G_{2(i)}^{\mathrm{EMH}} \right\} \right) , \tag{1}$$

where

$$I(\cdot;\cdot) = \sum_{i=1}^{r-1} \left\{ G^*_{1(i)} \log \left(\frac{G^*_{1(i)}}{G^{\mathrm{EMH}}_{1(i)}} \right) + G^*_{2(i)} \log \left(\frac{G^*_{2(i)}}{G^{\mathrm{EMH}}_{2(i)}} \right) \right\} ,$$

$$G^{\mathrm{EMH}}_{1(i)} = \Delta^*_U \left(G^*_{1(i)} + G^*_{2(i)} \right) , \qquad G^{\mathrm{EMH}}_{2(i)} = \Delta^*_L \left(G^*_{1(i)} + G^*_{2(i)} \right) .$$

Note that $I(\cdot;\cdot)$ is the KL information between $\{G^*_{1(i)}, G^*_{2(i)}\}$ and $\{G^{\mathrm{EMH}}_{1(i)}, G^{\mathrm{EMH}}_{2(i)}\}$.

We see that (i) $0 \leq \Gamma_{\mathrm{EMH}} \leq 1$, (ii) $\Gamma_{\mathrm{EMH}} = 0$ if and only if the EMH model holds, and (iii) $\Gamma_{\mathrm{EMH}} = 1$ if and only if the degree of departure from EMH is the largest in a sense that $G_{1(i)} = 0$ (then $G_{2(i)} > 0$) or $G_{2(i)} = 0$ (then $G_{1(i)} > 0$), $i = 1, \ldots, r-1$ and $\Delta^*_U = \Delta^*_L = 1/2$.

According to the KL information, Γ_{EMH} represents the degree of departure from EMH, and the degree increases as the value of Γ_{EMH} increases.

2.2 Measure for the Equality of Marginal Means

We shall consider the measure to represent the degree of departure from the E model. We note that $\mathrm{E}(X) = \mathrm{E}(Y)$ is equivalent to $\sum_{i=1}^{r-1} G_{1(i)} = \sum_{i=1}^{r-1} G_{2(i)}$ (i.e., $\Delta^*_U = \Delta^*_L$), although the details are omitted here. Assuming that $\Delta^*_U \geq 0$ and $\Delta^*_L \geq 0$, consider a measure defined by

$$\Gamma_{\mathrm{E}} = \frac{1}{\log 2} I \left(\{\Delta^*_U, \Delta^*_L\} ; \left\{ \frac{1}{2}, \frac{1}{2} \right\} \right) , \tag{2}$$

where

$$I(\cdot;\cdot) = \Delta^*_U \log \left(\frac{\Delta^*_U}{1/2} \right) + \Delta^*_L \log \left(\frac{\Delta^*_L}{1/2} \right) .$$

This may be expressed as

$$\Gamma_{\mathrm{E}} = 1 - \frac{1}{\log 2} H \left(\{\Delta^*_U, \Delta^*_L\} \right) ,$$

where

$$H(\cdot) = -\Delta^*_U \log \Delta^*_U - \Delta^*_L \log \Delta^*_L , \quad [0 \log 0 = 0] .$$

Thus, essentially, Γ_{E} represents the Shannon entropy $H(\{\Delta^*_U, \Delta^*_L\})$.

We see that $H(\{\Delta^*_U, \Delta^*_L\})$ must lie between 0 (when $\Delta^*_U = 0$ then $\Delta^*_L = 1$, or when $\Delta^*_L = 0$ then $\Delta^*_U = 1$) and $\log 2$ (when $\Delta^*_U = \Delta^*_L = 1/2$), and therefore Γ_{E} must lie between 0 and 1. We also see that (i) $\Gamma_{\mathrm{E}} = 0$ if and only if there is a structure of E in the $r \times r$ table, and (ii) $\Gamma_{\mathrm{E}} = 1$ if and only if the degree of departure from E is the largest in a sense that $\Delta^*_U = 0$ (then $\Delta^*_L = 1$) or $\Delta^*_L = 0$ (then $\Delta^*_U = 1$). Namely, (i) $\Gamma_{\mathrm{E}} = 0$ if and only if $\mathrm{E}(X) = \mathrm{E}(Y)$, i.e., $\sum_{i=1}^{r-1} G_{1(i)} = \sum_{i=1}^{r-1} G_{2(i)}$, and (ii) $\Gamma_{\mathrm{E}} = 1$ if and only if $\sum\sum_{i<j} p_{ij} = 0$ ($\sum\sum_{i>j} p_{ij} \neq 0$) or $\sum\sum_{i>j} p_{ij} = 0$ ($\sum\sum_{i<j} p_{ij} \neq 0$) (i.e., $\Pr(X < Y) = 0$, $\Pr(X > Y) \neq 0$, or $\Pr(X > Y) = 0$, $\Pr(X < Y) \neq 0$).

According to the KL information or the Shannon entropy, Γ_{E} represents the degree of departure from E model, and the degree increases as the value of Γ_{E} increases.

2.3 Relationships between the Measures

Assume that $\Delta_U^* > 0$, $\Delta_L^* > 0$, and $\{G_{1(i)} + G_{2(i)} > 0\}$. Then we obtain the following theorem.

Theorem 1. *The value of* Γ_{MH} *equals the sum of the value of* Γ_{EMH} *and the value of* Γ_{E}.
Proof. It is easily seen that the right term of equation (1) plus the right term of equation (2) equals the right term of equation (3) in the Appendix. Thus, the proof is completed.

From Theorem 1, Γ_{EMH} is expressed as $\Gamma_{\mathrm{EMH}} = \Gamma_{\mathrm{MH}} - \Gamma_{\mathrm{E}}$. Therefore, the measure Γ_{EMH} also would indicate the degree of departure from MH excluding the influence of degree of departure from E.

From (1) we see that $\Gamma_{\mathrm{EMH}} \geq 0$. Thus we obtain the next theorem.

Theorem 2. *The value of* Γ_{MH} *is greater than or equal to the value of* Γ_{E}. *The equality holds if and only if there is a structure of* EMH *in the* $r \times r$ *table.*

From $0 \leq \Gamma_{\mathrm{MH}} \leq 1$, $0 \leq \Gamma_{\mathrm{E}} < 1$ (note that $\Gamma_{\mathrm{E}} \neq 1$ because $\Delta_U^* > 0$ and $\Delta_L^* > 0$ being the assumption), and Theorems 1 and 2, we see that $0 \leq \Gamma_{\mathrm{EMH}} \leq 1$. From $\Gamma_{\mathrm{EMH}} = \Gamma_{\mathrm{MH}} - \Gamma_{\mathrm{E}}$, we see that (i) $\Gamma_{\mathrm{EMH}} = 0$ if and only if $\Gamma_{\mathrm{MH}} = \Gamma_{\mathrm{E}}$; namely, the degree of departure from the equality of $G_{1(i)}$ and $G_{2(i)}$ for $i = 1, \dots, r-1$, is equal to the degree of departure from the equality of $\sum_{i=1}^{r-1} G_{1(i)}$ and $\sum_{i=1}^{r-1} G_{2(i)}$. This seems natural when the EMH model holds (i.e., $\Gamma_{\mathrm{EMH}} = 0$).

We also see that (ii) $\Gamma_{\mathrm{EMH}} = 1$ if and only if $\Gamma_{\mathrm{MH}} = 1$ and $\Gamma_{\mathrm{E}} = 0$; namely, $G_{1(i)} = 0$ (then $G_{2(i)} > 0$) or $G_{2(i)} = 0$ (then $G_{1(i)} > 0$) for $i = 1, \dots, r-1$, and $\mathrm{E}(X) = \mathrm{E}(Y)$. Namely $\Gamma_{\mathrm{EMH}} = 1$ indicates that $G_{1(i)}/G_{2(i)} = \infty$ for some i and $G_{1(i)}/G_{2(i)} = 0$ for the other i, and $\mathrm{E}(X) = \mathrm{E}(Y)$ (i.e., $\sum_{i=1}^{r-1} G_{1(i)} = \sum_{i=1}^{r-1} G_{2(i)}$). It seems appropriate to consider that then the degree of departure from EMH is largest.

3 Approximate Confidence Intervals for the Measures

Let n_{ij} denote the observed frequency in the ith row and jth column of the $r \times r$ square table ($i = 1, \dots, r$, $j = 1, \dots, r$). Assuming that the $\{n_{ij}\}$ result from a full multinomial sampling, we consider the approximate standard errors and large-sample confidence intervals for Γ_{EMH} and Γ_{E} using the delta method as described by Bishop et al. (1975, Sec.14.6) and Agresti (1990, Sec.12.1). The sample version of Γ_{EMH}, i.e., $\hat{\Gamma}_{\mathrm{EMH}}$, is given by Γ_{EMH} with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$, where $\hat{p}_{ij} = n_{ij}/n$ and $n = \sum\sum n_{ij}$. Similarly, $\hat{\Gamma}_{\mathrm{E}}$ is given. Using the delta method, $\sqrt{n}(\hat{\Gamma}_{\mathrm{EMH}} - \Gamma_{\mathrm{EMH}})$ has asymptotically (as $n \to \infty$) a normal distribution with mean zero and variance,

$$\mathrm{var}[\Gamma_{\mathrm{EMH}}] = \frac{1}{\Delta^2} \sum_{k=1}^{r-1} \sum_{l=k+1}^{r} \left(p_{kl} w_{kl}^2 + p_{lk} v_{lk}^2\right),$$

where

$$w_{kl} = \frac{1}{\log 2} \left\{ \sum_{i=k}^{l-1} \log\left(\frac{G_{1(i)}^*}{G_{1(i)}^{\mathrm{EMH}}}\right) - (l-k)\Gamma_{\mathrm{EMH}} \log 2 \right\},$$

$$v_{lk} = \frac{1}{\log 2} \left\{ \sum_{i=k}^{l-1} \log\left(\frac{G_{2(i)}^*}{G_{2(i)}^{\mathrm{EMH}}}\right) - (l-k)\Gamma_{\mathrm{EMH}} \log 2 \right\}.$$

Table 1: Occupational status for Japanese father-son pairs

(a) examined in 1955

	Son's status								
Father's status	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	Total
(1)	36	4	14	7	8	2	3	8	82
(2)	20	20	27	24	11	11	2	11	126
(3)	9	6	23	12	9	5	3	16	83
(4)	15	14	39	81	17	16	11	15	208
(5)	6	7	22	13	72	20	6	13	159
(6)	3	2	5	12	18	19	9	7	75
(7)	5	3	10	11	21	15	38	25	128
(8)	39	30	76	80	69	52	45	614	1005
Total	133	86	216	240	225	140	117	709	1866

(b) examined in 1975

	Son's status								
Father's status	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	Total
(1)	44	18	28	8	6	8	1	5	118
(2)	15	50	45	20	18	17	4	7	176
(3)	18	25	47	30	24	18	5	7	174
(4)	16	27	53	77	40	29	9	6	257
(5)	18	25	42	31	122	43	17	13	311
(6)	12	15	21	15	36	33	3	8	143
(7)	3	5	8	7	26	21	9	3	82
(8)	44	65	114	92	184	195	58	325	1077
Total	170	230	358	280	456	364	106	374	2338

Status: professional (1) managers (2) clerical (3) sales (4) skilled manual (5) semiskilled manual (6) unskilled manual (7) farmers (8)

Similarly, $\sqrt{n}(\hat{\Gamma}_{\mathrm{E}} - \Gamma_{\mathrm{E}})$ has asymptotically a normal distribution with mean zero and variance

$$\mathrm{var}(\Gamma_{\mathrm{E}}) = \frac{1}{\Delta^2} \sum_{k=1}^{r-1} \sum_{l=k+1}^{r} \left(p_{kl}\alpha_{kl}^2 + p_{lk}\beta_{lk}^2 \right) ,$$

where

$$\alpha_{kl} = \frac{l-k}{\log 2} \{\log(2\Delta_U^*) - \Gamma_{\mathrm{E}} \log 2\} , \qquad \beta_{lk} = \frac{l-k}{\log 2} \{\log(2\Delta_L^*) - \Gamma_{\mathrm{E}} \log 2\} .$$

Let $\widehat{\mathrm{var}}(\Gamma)$ denote $\mathrm{var}(\Gamma)$ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$. Then $\widehat{\mathrm{var}}^{1/2}[\Gamma]/\sqrt{n}$ is the estimated approximate standard error for $\hat{\Gamma}$, giving a confidence interval for Γ.

4 Examples

Example 1. The data in Table 1 taken from Tominaga (1979, p.131) describe the cross-classification of father's and son's occupational status categories in Japan which were examined in 1955 and 1975.

Table 2: Estimates of Γ_{EMH}, its approximate standard error, and 95% confidence interval

Applied data	Estimated measure	Standard error	Confidence interval
Table 1a	0.023	0.005	(0.013, 0.034)
Table 1b	0.055	0.007	(0.042, 0.068)

Since the confidence interval for Γ_{EMH} applied to each of Tables 1a and 1b does not include zero (see Table 2), this would indicate that there is not a structure of EMH in each table. Let G^2 denote the likelihood ratio chi-squared statistic for testing goodness-of-fit of the model. The values of G^2 for the EMH model are 116.76 for Table 1a and 280.73 for Table 1b with $r - 2 = 6$ degrees of freedom (df). Therefore the EMH model fits each of these data poorly.

We compare the degree of departure from EMH between these tables using $\hat{\Gamma}_{\mathrm{EMH}}$. We can see from $\hat{\Gamma}_{\mathrm{EMH}}$ that (i) for Table 1a, the degree of departure from EMH is estimated to be 0.023 times the maximum degree of departure from EMH, and (ii) for Table 1b, it is estimated to be 0.055 times the maximum degree of departure from EMH.

When the degrees of departure from EMH in Tables 1a and 1b are compared using the confidence interval for Γ_{EMH}, it would be greater in Table 1b than in Table 1a.

Table 3: Cross-classification of Merino ewes according to number of lambs born in consecutive years

	Number of Lambs 1952			
Number of Lambs 1953	0	1	2	Total
0	58	52	1	111
1	26	58	3	87
2	8	12	9	29
Total	92	122	13	227

Table 4: Estimates of Γ_{EMH}, Γ_{E}, and Γ_{MH}, their approximate standard errors and 95% confidence intervals, applied to Table 3

Measures	Estimated measures	Standard errors	Confidence intervals
Γ_{EMH}	0.102	0.043	(0.018, 0.187)
Γ_{E}	0.001	0.004	(-0.007, 0.008)
Γ_{MH}	0.103	0.043	(0.018, 0.188)

Example 2. The data in Table 3 taken from Tallis (1962) describe the cross-classification of Merino ewes according to the number of lambs born in consecutive years, 1952 and 1953 (also see Bishop et al., 1975, p.288).

Since the confidence intervals for Γ_{MH} and Γ_{EMH} do not include zero (see Table 4), these would indicate that there is not each structure of MH and EMH in the table. However, since the confidence interval for Γ_{E} includes zero (see Table 4), this would indicate that there is a structure of E in the table. Also, the degree of departure from EMH is larger than the degree of departure from E. Therefore we can state from Theorem 1 that the lack of structure of the MH model is caused by the lack of structure of the EMH model rather than that of the E model.

Table 5: Average doses of conjugated oestrogen used by cases and matched controls: Los Angeles endometrial cancer study

Average dose	Average dose for control (mg/day)				
for case	0	0.1-0.299	0.3-0.625	0.626+	Total
(mg/day)	(1)	(2)	(3)	(4)	
0 (1)	6	2	3	1	12
0.1-0.299 (2)	9	4	2	1	16
0.3-0.625 (3)	9	2	3	1	15
0.626+ (4)	12	1	2	1	16
Total	36	9	10	4	59

Table 6: Estimates of Γ_{EMH}, Γ_{E}, and Γ_{MH}, their approximate standard errors and 95% confidence intervals, applied to Table 5

Measures	Estimated measures	Standard errors	Confidence intervals
Γ_{EMH}	0.004	0.006	(-0.009, 0.016)
Γ_{E}	0.302	0.125	(0.057, 0.548)
Γ_{MH}	0.306	0.124	(0.062, 0.550)

Example 3. Table 5 taken directly from Breslow and Day (1980, p.185) is the data from the Los Angeles study of endometrial cancer. These data are obtained from the 59 matched pairs using four dose levels of conjugated oestrogen, (1) none, (2) 0.1–0.299mg, (3) 0.3–0.625mg, and (4) 0.626+mg. Since the confidence intervals for Γ_{MH} and Γ_{E} do not include zero (see Table 6), these would indicate that there is not each structure of MH and E in the table. While, since the confidence interval for Γ_{EMH} includes zero (see Table 6), this would indicate that there is a structure of EMH in the table. In addition, the degree of departure from E is larger than the degree of departure from EMH. Therefore we can state from Theorem 1 that the lack of structure of the MH model is caused by the lack of structure of the E model rather than that of the EMH model. This is contrast to the case of Example 2.

5 Concluding Remarks

The measures $\hat{\Gamma}_{\mathrm{EMH}}$ and $\hat{\Gamma}_{\mathrm{E}}$ always range between 0 and 1 independent of the dimension r and sample size n. So, $\hat{\Gamma}_{\mathrm{EMH}}$ and $\hat{\Gamma}_{\mathrm{E}}$ may be useful for *comparing* the degree of departure from EMH and E, respectively, in several tables.

Table 7: Artificial data with $\hat{G}_{1(i)}/\hat{G}_{2(i)}$ $(i = 1, 2, 3)$, $\hat{\Gamma}_{\text{EMH}}$, and G^2 for the EMH model

(a) $n = 2611$ (sample size)

511	216	9	45
27	304	63	90
18	198	497	90
54	36	90	363

(b) $n = 2106$

604	81	18	8
9	326	45	4
13	120	455	3
6	4	2	408

(c) $n = 2419$

604	162	36	16
18	326	90	8
26	240	455	6
12	8	4	408

(d) $\{\hat{G}_{1(i)}/\hat{G}_{2(i)}\}$, $\hat{\Gamma}_{\text{EMH}}$, and G^2 for EMH

	$i = 1$	2	3	$\hat{\Gamma}_{\text{EMH}}$	G^2
Table 7a	2.73	0.68	1.25	0.054	154.17
Table 7b	3.82	0.52	1.25	0.134	99.75
Table 7c	3.82	0.52	1.25	0.134	199.50

Note: $\hat{G}_{1(i)}/\hat{G}_{2(i)}$ indicates $G_{1(i)}/G_{2(i)}$ with $\{p_{st}\}$ replaced by $\{\hat{p}_{st} = n_{st}/n\}$.

Consider the artificial data in Table 7. From the values of G^2 (with 2 df) for the EMH model (see Table 7d), we see that the EMH model fits the data in Table 7a worse than the data in Table 7b. In contrast, the value of $\hat{\Gamma}_{\text{EMH}}$ is less for Table 7a than for Table 7b (see Table 7d). In terms of $\{\hat{G}_{1(i)}/\hat{G}_{2(i)}\}$, $i = 1, 2, 3$ (see Table 7d), it seems natural to conclude that the degree of departure from EMH is less for Table 7a than for Table 7b. Therefore $\hat{\Gamma}_{\text{EMH}}$ may be preferable to G^2 for *comparing* the degree of departure from EMH in several tables. (By the similar reason, $\hat{\Gamma}_{\text{EMH}}$ may also be preferable to the P-values for *comparing* them.)

It may seem, to many readers, that G^2/n is also a reasonable measure for representing the degree of departure from EMH. However, it does not seem to us that G^2/n is a reasonable measure. For example, consider the artificial data in Tables 7b and 7c. The values of G^2/n are 0.05 for Table 7b, and 0.08 for Table 7c. Therefore the value of G^2/n is less for Table 7b than for Table 7c. However, the value of $\hat{\Gamma}_{\text{EMH}}$ for Table 7b is theoretically identical to that for Table 7c because $\{\hat{G}_{1(i)}/\hat{G}_{2(i)}\}$, $i = 1, 2, 3$, for Table 7b is identical to those for Table 7c (see Table 7d). It seems natural to conclude that the degree of departure from EMH for Table 7b is equal to that for Table 7c. Therefore $\hat{\Gamma}_{\text{EMH}}$ may also be preferable to G^2/n for *comparing* the degree of departure from EMH in several tables.

The $\hat{\Gamma}_{\text{EMH}}$ would be useful when we want to see what degree the departure from EMH is toward the *maximum* departure from EMH. Note that we cannot use the G^2 and $\hat{\Gamma}_{\text{MH}}$ when we want to see it.

We observe that (i) the measure Γ_{EMH} should be applied to contingency tables with *ordered* categories, (ii) the asymptotic normal distribution of $\sqrt{n}(\hat{\Gamma}_{\text{EMH}} - \Gamma_{\text{EMH}})$ may be not applicable when $\Gamma_{\text{EMH}} = 0$ and $\Gamma_{\text{EMH}} = 1$ because then $\text{var}[\Gamma_{\text{EMH}}] = 0$, and (iii) $\hat{\Gamma}_{\text{EMH}}$ cannot be used for testing goodness-of-fit of EMH (although G^2 can be used for it).

Table 8: Artificial data

$n = 1169$ (sample size)			
100	54	50	46
8	120	250	104
8	20	150	50
4	13	12	180

6 Discussion

Consider the artificial data in Table 8. The values of G^2 for the MH and EMH models are 389.17 with 3 df and 4.91 with 2 df, respectively. Therefore, these would indicate that for these data, there is not the structure of MH, however, there is the structure of EMH. Then the estimated value of measure Γ_{MH} is 0.517 and the estimated value of measure Γ_{EMH} is 0.002. Thus, when we want to see the degree of departure from EMH, the measure Γ_{EMH} would be useful although the measure Γ_{MH} is not useful.

We emphasize that the measure Γ_{EMH} proposed in this paper is entirely different from the measures which represent the degree of departure from MH as the measure Γ_{MH} although the form may be similar.

Finally we note that Tomizawa et al. (2003) also gave the power-divergence type measure to represent the degree of departure from MH; however, using the power-divergence, we cannot obtain the similar result to Theorem 1.

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Appendix

The KL information type measure Γ_{MH}, which represents the degree of departure from MH for the data on an *ordinal* scale, proposed in Tomizawa et al. (2003), is given as follows: assuming that $\{G_{1(i)} + G_{2(i)} > 0\}$,

$$\Gamma_{\text{MH}} = \frac{1}{\log 2} I\left(\{G^*_{1(i)}, G^*_{2(i)}\}\,;\,\{Q^*_i, Q^*_i\}\right)\,, \tag{3}$$

where

$$I(\cdot;\cdot) = \sum_{i=1}^{r-1}\left\{G^*_{1(i)}\log\left(\frac{G^*_{1(i)}}{Q^*_i}\right) + G^*_{2(i)}\log\left(\frac{G^*_{2(i)}}{Q^*_i}\right)\right\}\,,$$

$$Q^*_i = \frac{1}{2}(G^*_{1(i)} + G^*_{2(i)})\,.$$

Note that $0 \le \Gamma_{\text{MH}} \le 1$, and $\Gamma_{\text{MH}} = 0$ if and only if the MH model holds.

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