Inference for Long-memory Time Series Models Based on Modified Empirical Likelihood

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Abstract

Empirical likelihood method has been applied to short-memory time series models by Monti (1997) through the Whittle's estimation method. Yau (2012) extended this idea to long-memory time series models. Asymptotic distributions of the empirical likelihood ratio statistic for short and long-memory time series have been derived to construct confidence regions for the corresponding model parameters. However, it experiences the undercoverage issue which causes the coverage probabilities of parameters lower than the given nominal levels, especially for small sample sizes. In this paper, we propose a modified empirical likelihood which combines the advantages of the adjusted empirical likelihood and the transformed empirical likelihood to modify the one proposed by Yau (2012) for autoregressive fractionally integrated moving average (ARFIMA) model for the purpose of improving coverage probabilities. Asymptotic null distribution of the test statistic has been established as the standard chi-square distribution with the degree of freedom one. Simulations have been conducted to investigate the performance of the proposed method as well as the comparisons of other existing methods to illustrate that the proposed method can provide better coverage probabilities especially for small sample sizes.

Keywords: modified empirical likelihood, long-memory time series, ARFIMA model, coverage probability .

1. Introduction

Owen (1988), Owen (1990), Owen (1991) introduced empirical likelihood (EL) method which is the data-driven method combining the advantages of parametric and nonparametric methods. The most appealing property of the EL method is that the associated empirical likelihood ratio statistics asymptotically follows standard chi-square distribution, which is same as the one used in parametric analysis. Since then, it has been widely used to make statistical inference of parameters and construct confidence regions. See Owen (2001) for more details. However, when the data is dependent, it becomes difficult to apply the empirical likelihood method as it is originally designed for independent observations. Using EL method to address dependent data problems has been studied by many researchers. Mykland (1995) established the connection between the dual likelihood and the empirical likelihood through the martingale estimating equations and applied it to time series model. Monti (1997) developed the idea of extending the EL method to short-memory stationary time series by using the Whittle's

(1953) method to obtain an M-estimator of the periodogram ordinates of time series models which are asymptotically independent. However, his method can not be applied directly to long-time memory time series model. Kitamura (1997) developed the blockwise empirical likelihood method for time series models. For long-memory or long-range dependence time series data, Hurvich and Beltrao (1993) showed that the normalized periodogram ordinates obtained from a Gaussian process are asymptotically neither independent identically distributed nor exponentially distributed. Nordman and Lahiri (2006) developed frequency domain empirical likelihood based on the spectral distribution through the fourier transformation to study short and long range dependence. Yau (2012) extended Monti's idea to autoregressive fractionally integrated moving average (ARFIMA) model by showing that the dependence in periodogram only applies to a small portion of the periodogram ordinates with fourier frequencies tending to zero. However, the profile empirical likelihood function computation which involves constrained maximization requires the convex hull of the estimating equation to have zero vector as an interior point. When the solution does not exist, Owen (2001) suggested assigning $-\infty$ to the log-EL statistic. Chen, Variyath, and Abraham (2008) pointed out the drawbacks in doing so and proposed an adjusted empirical likelihood (AEL) method by adding a pseudo term which always guarantees the existence of a solution. They further showed that the asymptotic results of the AEL are similar to that of the EL. Moreover, it achieves improved coverage probabilities without using Bartlett-corrections. Adapting their method, Gamage, Ning, and Gupta (2017) proposed an adjusted empirical likelihood procedure to modify the one proposed by Yau for the ARFIMA model and achieved better coverage probabilities of parameters. Jing, Tsao, and Zhou (2017), instead of considering the convex hull issue, developed the transformed empirical likelihood (TEL) to improve the coverage probabilities as well as preserving the properties of the original EL method with a simple form.

In this paper, we extend Yau's EL method for ARFIMA model by proposing a transformed adjusted empirical likelihood method (TAEL). The rest of the article is organized as follows. In Section 2, we briefly go over the AEL procedure proposed by Gamage *et al.* (2017) for a stationary ARFIMA model. In section 3, we proposed a transformed adjusted empirical likelihood (TAEL) based on the AEL by Chen *et al.* (2008) and the TEL by Jing *et al.* (2017) for a stationary ARFIMA model. Corresponding asymptotic results are also discussed. Simulations are carried out in section 4 to compare the coverage probabilities of the proposed TAEL method to other methods including original EL, TEL, and AEL for different distributions for the white noise term. In addition, coverage probabilities are calculated to illustrate the effectiveness of AEL method as compared to EL method with and without Bartlett-correction for different values of the parameters, different sample sizes and different distributions for the white noise term. Section 5 provides some discussion.

2. Adjusted empirical likelihood for ARFIMA models

With the definition of $p_j = P(X = x_j)$, the empirical likelihood ratio for any value $\beta \in \mathbb{B}$ is defined by

$$\hat{\lambda}(\beta) = \sup \prod_{j=1}^{n} p_j \bigg/ \prod_{j=1}^{n} \frac{1}{n} = \sup \prod_{j=1}^{n} n p_j,$$

subject to the constraints:(i) $\sum_{j=1}^{n} \psi(x_j, \beta)p_j = 0$, (ii) $\sum_{j=1}^{n} p_j = 1$, and (iii) $p_j \geq 0$, $(j = 1, 2, ...n)$. The maximization under the Lagrange multiplier method gives

$$p_j = [n\{1 + \xi(\beta)'\psi(x_j, \beta)\}]^{-1} \qquad (j = 1, 2, ..., n),$$

where $\xi(\beta)$ is the Lagrangian multiplier satisfying constraint (i). The empirical likelihood ratio statistic is thus defined

$$\hat{W}(\beta) = -2 \ln \hat{\lambda}(\beta) = 2 \sum_{j=1}^{n} \ln\{1 + \xi(\beta)'\psi(x_j, \beta)\} \tag{1}$$

Owen (1988) showed that $\hat{W}(\beta)$ is asymptotically distributed as χ^2. For time series data, the M-estimator, β_0, which is used to estimate the parameter β is based on the periodogram using the Whittle's method. It is the solution of

$$\sum_{i=1}^{n} \psi(x_i, \beta) = 0,$$

where $\psi(x_i, \beta)$ is an estimating equation which is defined by $E\psi(X_i, \beta) = 0$. Under some moment conditions on estimating equations (Owen, 2001), the convex hull of the estimating equation $\psi(x_i, \beta)$, $\{\sum p_i \psi(x_i, \beta) | p_i \geq 0, \sum p_i = 1, i = 1, 2, \cdots, n\}$ contains 0 as its interior point with probability 1 as $n \to \infty$. When the parameter β is not close to the true value β_0, or when n is small, there is a good chance that the solution to constraint doesn't exist which raises some computational issues as mentioned by Chen *et al.* (2008). When the solution does not exist, Owen (2001) suggested assigning $-\infty$ to the empirical log-likelihood ratio statistic. To overcome this difficulty, Chen *et al.* (2008) proposed an adjusted empirical likelihood (AEL) ratio function by adding a pseudo term, ψ_{n+1}-th term, to guarantee the zero to be an interior point of the convex hull of estimating equations, therefore, the required numerical maximization always has the solution. By doing so, they modified Owen's method and applied to independent observations with the establishment of the Wilks' theorem for AEL statistic same as the one Owen obtained for EL statistic.

The EL-based method proposed by Monti (1997) which used the asymptotically independent periodogram ordinates, cannot be directly used in the long-memory scenario. However, Yau (2012) showed that the periodogram ordinates obtained by the long-memory time series can be considered asymptotically independent, hence the EL method could be extended to the long-memory time series models. Consider a stationary ARFIMA(p,d,q) process Z_t given by

$$\Phi(B)(1 - B)^d Z_t = \Theta(B) a_t, \tag{2}$$

for some $-\frac{1}{2} < d < \frac{1}{2}$ where B is the backward shift operator ($BZ_t = Z_{t-1}$), with $\Theta(B) = 1 + \theta_1 B + \theta_2 B^2 + ... + \theta_q B^q$ and $\Phi(B) = 1 - \phi_1 B - \phi_2 B^2 - ... - \phi_p B^p$. The absolute values of the roots of these two polynomials are all greater than 1 to guarantee the stationarity and invertibility of the model. We also assume that $\Theta(B)$ and $\Phi(B)$ have no common factors to avoid the redundancy of the parameters. Similar to Gamage *et al.* (2017), we consider the values of d to be between 0 and 0.5 since this is the only condition under which the dependence structure of periodogram ordinates has been established (Yau 2012).

Let $\beta = (\beta_{(1)}, \sigma^2)$, where $\beta_{(1)} = (\phi_1, ..., \phi_p, \theta_1, ..., \theta_q, d)$ is the parameter of interest. Denote $\psi_j = \psi_j(\beta) = \psi(x_j, \beta)$ and $\bar{\psi}_n = \bar{\psi}_n(\beta) = \frac{1}{n} \sum_{j=1}^{n} \psi_j$. For some positive constant a_n, define

$$\psi_{n+1} = \psi_{n+1}(\beta) = -\frac{a_n}{n} \sum_{j=1}^{n} \psi_j = -a_n \bar{\psi}_n.$$

Here we choose $a_n = \max(1, log(n)/2)$ coupled with a trimmed version of $\bar{\psi}_n$ when appropriate suggested by Chen *et al.* (2008).

Gamage *et al.* (2017) extended the EL approach for the long-memory time series models and proposed an adjusted empirical likelihood (AEL) ratio statistic as follows.

$$\hat{W}^*(\beta_{(1)}) = 4 \sum_{j=1}^{n+1} \ln \left[1 + \xi(\beta_{(1)})' \tilde{\psi}_j(I(\omega_j), \beta_{(1)}) \right], \tag{3}$$

where $\tilde{\psi}$-function defined as

$$\tilde{\psi}_j \{ I(\omega_j), \beta_{(1)} \} = \frac{I(\omega_j)}{g_j^1(\beta_{(1)})} \frac{\partial \ln \{ g_j^1(\beta_{(1)}) \}}{\partial \beta_{(1)}}, \tag{4}$$

with $\tilde{\psi}_{n+1}$ as defined above, and $\xi(\beta_{(1)})$ satisfies,

$$\sum_{j=1}^{n+1} [1 + \xi(\beta_{(1)})' \tilde{\psi}_j \{I(\omega_j), \beta_{(1)}\}]^{-1} \tilde{\psi}_j \{I(\omega_j), \beta\} = 0.$$

Here g_j^1 is the profile spectral density function of the model, and $I(\omega_j)$ is the periodogram ordinate evaluated at Fourier frequency ω_j. Same discuss as Chen *et al.* (2008), because $\bar{\tilde{\psi}}_n = \frac{1}{n}\sum_{i=1}^{n} \tilde{\psi}_i$ and $\tilde{\psi}_{n+1}$ are on the opposite of 0, therefore, the convex hull of $\tilde{\psi}_i, i = 1, 2, \cdots, n+1$, that is, $\{\sum p_i \tilde{\psi}_i | p_i \geq 0, \sum p_i = 1\}$, contains 0 as an interior point for any given $\beta_{(1)}$. Consequently, $W^*(\beta_{(1)})$ is well defined for any given $\beta_{(1)}$. Moreover, $\hat{W}^*(\beta_{(1)})$, defined in (3) has an asymptotic χ^2 with $k = p + q + 1$ degrees of freedom (Gamage *et al.* 2017). As mentioned by Yau (2012), this result also applies to ARFIMA process with $d \in [\delta, 0.5 - \delta]$ for any fixed $\delta > 0$, since β belongs to a compact space.

3. Transformed adjusted empirical likelihood for ARFIMA models

The undercoverage problem in the EL method where the coverage probability tends to be lower than the nominal level, is partly due to the convex hull constraint which defines the confidence region inside a bounded region in the parmeter space (Tsao 2013). This could be serious especially in small sample and in multidimensional scenarios. A transformed version of the EL (TEL) has been proposed by Jing *et al.* (2017) as a solution to the undercoverage problem in EL as follows. Define

$$g_T(l(\theta); \gamma) = l(\theta) \times max\{1 - l(\theta)/n, 1 - \gamma\}, \tag{5}$$

for a constant $\gamma \in [0, 1]$, where $l(\theta)$ is the original empirical log-likelihood ratio (Owen 2001). Thus $g_T(l(\theta); \gamma)$ is a truncated quadratic transformation of $l(\theta)$. By default, Jing *et al.* (2017) sets $\gamma = 1/2$. Hence, the transformed empirical log-likelihood ratio (TEL), $l_T(\theta)$ is,

$$l_T(\theta) = g_T(l(\theta); \gamma = 1/2) = l(\theta) \times max\{1 - l(\theta)/n, 1/2\}, \tag{6}$$

which is equivalent to,

$$\ell_T(\theta) = \begin{cases} \ell(\theta)[1 - \ell(\theta)/n] & \ell(\theta) \leq n/2 \\ \ell(\theta)/2 & \ell(\theta) > n/2. \end{cases} \tag{7}$$

Jing *et al.* (2017) showed that the confidence region based on $g_T(l(\theta), \gamma = 1/2)$ is the expansion of the original empirical likelihood confidence region, that is, $\{\theta : l(\theta) \leq c\} \subseteq \{\theta : g_T(l(\theta), \gamma = 1/2) \leq c\}$ where c is the $(1-\alpha)-$ th quantile of χ_d^2. Consequently, TEL improves the coverage probabilities of EL. Moreover, Jing *et al.* (2017) also showed that TEL shares the same asymptotic properties with the EL. See Jing *et al.* (2017) for more details.

Consider the ARFIMA (p,d,q) model in (2) where the AEL ratio test statistic is given by (3) which has an asymptotic χ^2 distribution with $p + q + 1$ degrees of freedom. If we apply the transformation to this test statistic, then the resulting transformed AEL (TAEL) ratio statistic, $W_T(\beta_{(1)})$, satisfies the following criterions:

a. for any $\beta_{(1)}$, $0 \leq W_T(\beta_{(1)}) \leq \hat{W}^*(\beta_{(1)})$,

b. $W_T(\beta_1) \leq W_T(\beta_2)$ iff $\hat{W}^*(\beta_1) \leq \hat{W}^*(\beta_2)$ for any $\beta_1, \beta_2 \in \Theta$,

c. limiting distribution of $W_T(\beta_0)$ is same as that of $\hat{W}^*(\beta_0)$, and

d. $W_T(\beta)$ contours have data driven shape and they are centered around $\tilde{\beta}$.

By incorporating the suggestion on transformation for the EL by Jing *et al.* (2017) to the AEL ratio statistic, we consider the following transformation:

$$\hat{W}_T(\beta_{(1)}) = \hat{W}^*(\beta_{(1)}) \times \max\left\{1 - \frac{\hat{W}^*(\beta_{(1)})}{n+1}, 1 - \gamma\right\}, \tag{8}$$

where $\gamma \in [0, 1]$. As follows we check if the transformed adjusted empirical likelihood (TAEL) ratio satisfies the four conditions given above.

Condition a: We know that $\hat{W}^*(\beta_{(1)}) > 0$. Therefore, if $\hat{W}^*(\beta_{(1)}) < \gamma(n+1)$, then $\max\left\{1 - \frac{\hat{W}^*(\beta_{(1)})}{n+1}, 1 - \gamma\right\} = 1 - \frac{\hat{W}^*(\beta_{(1)})}{n+1}$, and if $\hat{W}^*(\beta_{(1)}) \geq \gamma(n+1)$, then $\max\left\{1 - \frac{\hat{W}^*(\beta_{(1)})}{n+1}, 1 - \gamma\right\} = 1 - \gamma$. Thus, $0 < \max\left\{1 - \frac{\hat{W}^*(\beta_{(1)})}{n+1}, 1 - \gamma\right\} \leq 1$. Furthermore,

$$\hat{W}_T(\beta_{(1)}) = \begin{cases} \hat{W}^*(\beta_{(1)}) \times \left[1 - \frac{\hat{W}^*(\beta_{(1)})}{n+1}\right] & ,\text{if } \hat{W}^*(\beta_{(1)}) < \gamma(n+1) \\ \hat{W}^*(\beta_{(1)})(1 - \gamma) & ,\text{if } \hat{W}^*(\beta_{(1)}) \geq \gamma(n+1) \end{cases}$$

Condition b: We need to prove that the transformed function of AEL ratio, $\hat{W}_t(\beta_{(1)})$ is a monotonically increasing function of $\hat{W}^*(\beta_{(1)})$. We consider two cases.

i. When $\hat{W}^*(\beta_{(1)}) \in [0, (\gamma(n+1)]$, then $\hat{W}_T(\beta_{(1)}) = \hat{W}^*(\beta_{(1)}) \times \left[1 - \frac{\hat{W}^*(\beta_{(1)})}{n+1}\right]$. So, $\hat{W}_T(\beta_{(1)}) \geq 0$ for all $\hat{W}^*(\beta_{(1)})$, which implies that $\hat{W}_T(\beta_{(1)})$ is monotonically increasing function of $\hat{W}^*(\beta_{(1)})$.

ii. When $\hat{W}^*(\beta_{(1)}) > \gamma(n+1)$, then $\hat{W}_T(\beta_{(1)}) = \hat{W}^*(\beta_{(1)}) \times (1 - \gamma) > 0$, which implies that $\hat{W}_T(\beta_{(1)})$ is monotonically increasing function.

Since $\hat{W}_T(\beta_{(1)})$ is continuous at $\hat{W}^*(\beta_{(1)}) = \gamma(n+1)$, we can conclude that $\hat{W}_T(\beta_{(1)})$ is a monotonically increasing function of $\hat{W}^*(\beta_{(1)})$.

Condition c: Gamage *et al.* (2017) showed $\hat{W}^*(\beta_{(1)}) = o_p(1)$ in the equation (A.10). Thus with probability tending to unity, we have $\hat{W}^*(\beta_{(1)}) \leq (1 - \gamma)(n+1)$. Combining them together we obtain,

$$\hat{W}_T(\beta_{(1)}) = \hat{W}^*(\beta_{(1)}) + o_p(1).$$

Condition d: Let $\tau_1 \in [0, \infty)$. Then the τ_1-level contour of $\hat{W}_T(\beta_{(1)})$ is $\{\beta_{(1)} : \hat{W}_T(\beta_{(1)}) = \tau_1\}$. Now, if we consider the τ_2-level contour of $\hat{W}^*(\beta_{(1)})$, $\{\beta_{(1)} : \hat{W}^*(\beta_{(1)}) = \tau_2\}$, to have the same shape as τ_1-level contour, then we have $\{\beta_{(1)} : g_T(\hat{W}^*(\beta_{(1)})) = \tau_1\} = \{\beta_{(1)} : \hat{W}^*(\beta_{(1)}) = \tau_2\}$, where $g_T(\hat{W}^*(\beta_{(1)})) = \hat{W}_T(\beta_{(1)})$. Then, we get $\hat{W}^*(\beta_{(1)}) = g_T^{-1}(\tau_1) = \tau_2$. This holds, since $\hat{W}_T(\beta_{(1)})$ is a monotonically increasing function of $\hat{W}^*(\beta_{(1)})$. This proves that the data driven shape is retained in the transformed statistic. Further, suppose at $\tilde{\beta}_{(1)}$, $\hat{W}^*(\beta_{(1)})$ has a unique minimum, and also $\hat{W}_T(\beta_{(1)})$ is monotonically increasing. Then, $\hat{W}_T(\tilde{\beta}_{(1)}) < \hat{W}_T(\beta_{(1)})$ for $\beta_{(1)} \neq \tilde{\beta}_{(1)}$.

Since $\hat{W}_T(\beta_{(1)})$ satistifies the conditions $(a) - (d)$ and $\hat{W}^*(\beta_{(1)}) \sim \chi^2_{p+q+1}$ (Gamage *et al.* 2017), $\hat{W}_T(\beta_{(1)}) \sim \chi^2_{p+q+1}$ as $n \to \infty$. We can similarly show that the transformed versions of the empirical likelihood with theoretical Bartlett correction (TB) and the empirical likelihood with estimated Bartlett correction (EB) satisfies the conditions (a)-(d), thus, both asymptotic distributions are χ^2_{p+q+1} as $n \to \infty$ for ARFIMA (p,d,q) model. We omit the details here.

4. Simulations

In this section, we compare the coverage probabilities based on the transformed EL and AEL methods to the (untransformed) EL and AEL method as well as the EL with estimated Bartlett correction (EB) for models with different sample sizes and different distributions of error terms.

4.1. Coverage probabilities

ARFIMA (0,d,0) model

Monte Carlo simulations are conducted to compute the coverage probabilities to examine the efficiency and accuracy of the transformed EL and AEL methods for ARFIMA model. We compare the results to the coverage probabilities of the untransformed (or original) EL and AEL methods and EB. We consider the simplest model, ARFIMA(0,d,0) given below:

$$Z_t = (1 - B)^{-d} a_t$$

Simulations are carried out for different sample sizes, values of d ($d = (0.1, 0.2, 0.3, 0.4)$) and three different distributions of the white noise term, namely, $N(0,1)$, t_5, and χ_5^2 distributions. In t_5 and χ_5^2 distributions, the mean is subtracted from the white noise process in order to make the white noise process to have mean zero. Under each case, 1000 series of size $T = (50, 70, 100, 200, 1000)$ are drawn and the coverage probabilities are computed. We choose $a_n = \log(n)/2$ as in the definition of ψ_{n+1}. The transformed empirical likelihood coverage probabilities and the transformed adjusted empirical likelihood coverage probabilities are compared to the coverage probabilities of the (untransformed) empirical likelihood and adjusted empirical likelihood and EB respectively. Table 1 shows the results for the simulations with a nominal level of 95%. It can be seen that the coverage probabilities computed using the transformed EL and transformed AEL are higher than their untransformed counterparts. Comparing to EB, TAEL provides better coverage probabilities always while TEL performs better most times. Also when the sample size is small, the transformed AEL performs better and gives accurate results than the other methods.

ARFIMA (0,d,1) model

We consider ARFIMA (0,d,1) model:

$$(1 - B)^d Z_t = (1 + \theta B) a_t,$$

where a_t is the white noise process with mean zero and variance σ^2. The value of the model parameter, θ is set to be 0.5. We consider three different error distributions for the white noise process a_t: $N(0,1)$, t_5, and χ_5^2. The last two distributions are centered around zero by subtracting their corresponding mean values. Four different d values are considered: $d = (0.1, 0.2, 0.3, 0.4)$. Table 2 shows that the coverage probability for models under different white noise distributions are different as it changes to adapt the differences in error distributions depicting the non-parametric property. Further, when the sample size is small, the transformed EL and AEL methods give better coverage probability as compared to the untransformed ones. Similar to Table 1, TAEL performs better in all scenarios while TEL performs better most times comparing to EB.

Table 1: Coverage probabilities for the parameter of ARFIMA(0,d,0) model

n	Method	$d = 0.1$	$d = 0.2$	$d = 0.3$	$d = 0.4$
		Model: $a_t \sim N(0,1)$			
T=50	EL	0.670	0.668	0.675	0.69
	EL.T	0.692	0.693	0.692	0.713
	AEL	0.673	0.677	0.685	0.699
	AEL.T	0.701	0.702	0.702	0.726
	EB	0.680	0.683	0.690	0.705
T=70	EL	0.706	0.706	0.699	0.699
	EL.T	0.072	0.716	0.710	0.716
	AEL	0.712	0.713	0.705	0.712
	AEL.T	0.727	0.724	0.718	0.730
	EB	0.717	0.716	0.709	0.720
T=100	EL	0.749	0.746	0.741	0.747
	EL.T	0.755	0.756	0.754	0.755
	AEL	0.753	0.754	0.750	0.755
	AEL.T	0.760	0.759	0.760	0.762
	EB	0.753	0.755	0.754	0.758
T=200	EL	0.766	0.769	0.774	0.774
	EL.T	0.769	0.773	0.778	0.776
	AEL	0.768	0.772	0.780	0.777
	AEL.T	0.774	0.780	0.783	0.779
	EB	0.771	0.777	0.784	0.788
T=1000	EL	0.906	0.908	0.913	0.917
	EL.T	0.907	0.909	0.916	0.919
	AEL	0.908	0.913	0.921	0.930
	AEL.T	0.908	0.914	0.921	0.931
	EB	0.907	0.911	0.918	0.924
		Model: $a_t \sim t_5$			
T=50	EL	0.714	0.712	0.710	0.695
	EL.T	0.732	0.722	0.722	0.707
	AEL	0.720	0.715	0.714	0.700
	AEL.T	0.738	0.733	0.730	0.720
	EB	0.725	0.716	0.716	0.709
T=70	EL	0.719	0.727	0.732	0.738
	EL.T	0.730	0.738	0.744	0.750
	AEL	0.728	0.731	0.741	0.746
	AEL.T	0.736	0.740	0.749	0.754
	EB	0.729	0.733	0.742	0.750
T=100	EL	0.753	0.760	0.765	0.774
	EL.T	0.761	0.767	0.775	0.779
	AEL	0.757	0.763	0.771	0.778
	AEL.T	0.768	0.771	0.784	0.787
	EB	0.757	0.764	0.780	0.787
T=200	EL	0.788	0.787	0.788	0.776
	EL.T	0.791	0.790	0.789	0.783
	AEL	0.789	0.788	0.789	0.791
	AEL.T	0.795	0.795	0.797	0.795
	EB	0.790	0.792	0.794	0.796
T=1000	EL	0.919	0.917	0.921	0.923
	EL.T	0.921	0.918	0.922	0.923
	AEL	0.921	0.920	0.931	0.936
	AEL.T	0.921	0.921	0.932	0.936
	EB	0.920	0.921	0.926	0.929

Table 1 – continued from previous page

n	Method	$d = 0.1$	$d = 0.2$	$d = 0.3$	$d = 0.4$
			Model: $a_t \sim \chi_5^2 - 5$		
T=50	EL	0.680	0.680	0.686	0.696
	EL.T	0.703	0.705	0.701	0.713
	AEL	0.690	0.691	0.694	0.703
	AEL.T	0.714	0.717	0.709	0.721
	EB	0.698	0.696	0.698	0.710
T=70	EL	0.722	0.726	0.723	0.722
	EL.T	0.739	0.740	0.736	0.738
	AEL	0.730	0.732	0.730	0.731
	AEL.T	0.742	0.745	0.742	0.743
	EB	0.732	0.734	0.733	0.736
T=100	EL	0.740	0.743	0.739	0.739
	EL.T	0.751	0.753	0.749	0.751
	AEL	0.746	0.751	0.745	0.750
	AEL.T	0.752	0.758	0.761	0.756
	EB	0.748	0.751	0.750	0.757
T=200	EL	0.782	0.790	0.791	0.789
	EL.T	0.786	0.796	0.795	0.794
	AEL	0.784	0.794	0.796	0.801
	AEL.T	0.787	0.799	0.802	0.802
	EB	0.785	0.795	0.799	0.808
T=1000	EL	0.928	0.931	0.926	0.932
	EL.T	0.929	0.933	0.929	0.933
	AEL	0.929	0.934	0.94	0.94
	AEL.T	0.932	0.935	0.941	0.942
	EB	0.929	0.934	0.937	0.933

EL= empirical likelihood;

EB=EL with estimated Bartlett correction; AEL=Adjusted EL

'.T' represents the transformed statistics

Table 2: Coverage probabilities for the parameter of ARFIMA(0,d,1) model

n	Method	$d = 0.1$	$d = 0.2$	$d = 0.3$	$d = 0.4$
		Model: $a_t \sim N(0,1)$			
T=50	EL	0.312	0.418	0.321	0.311
	EL.T	0.483	0.494	0.465	0.496
	AEL	0.421	0.425	0.424	0.416
	AEL.T	0.512	0.516	0.517	0.523
	EB	0.410	0.411	0.413	0.417
T=70	EL	0.453	0.502	0.508	0.509
	EL.T	0.543	0.543	0.543	0.542
	AEL	0.522	0.523	0.524	0.525
	AEL.T	0.557	0.554	0.553	0.558
	EB	0.515	0.523	0.525	0.518
T=100	EL	0.546	0.622	0.624	0.623
	EL.T	0.631	0.632	0.632	0.632
	AEL	0.628	0.629	0.631	0.629
	AEL.T	0.634	0.634	0.634	0.637
	EB	0.601	0.628	0.629	0.628
T=200	EL	0.716	0.717	0.722	0.715
	EL.T	0.742	0.738	0.739	0.737
	AEL	0.736	0.734	0.732	0.73
	AEL.T	0.752	0.752	0.748	0.741
	EB	0.728	0.731	0.727	0.721
T=1000	EL	0.891	0.911	0.917	0.915
	EL.T	0.925	0.921	0.921	0.921
	AEL	0.909	0.914	0.933	0.925
	AEL.T	0.938	0.937	0.937	0.932
	EB	0.910	0.911	0.923	0.921
		Model: $a_t \sim t_5$			
T=50	EL	0.412	0.413	0.404	0.403
	EL.T	0.581	0.582	0.581	0.573
	AEL	0.599	0.599	0.597	0.596
	AEL.T	0.606	0.607	0.607	0.605
	EB	0.514	0.504	0.513	0.499
T=70	EL	0.612	0.609	0.604	0.603
	EL.T	0.644	0.645	0.641	0.642
	AEL	0.648	0.649	0.652	0.648
	AEL.T	0.652	0.651	0.653	0.653
	EB	0.629	0.633	0.626	0.622
T=100	EL	0.678	0.677	0.675	0.674
	EL.T	0.684	0.686	0.678	0.676
	AEL	0.686	0.683	0.682	0.682
	AEL.T	0.686	0.686	0.685	0.686
	EB	0.686	0.685	0.684	0.681
T=200	EL	0.778	0.777	0.775	0.772
	EL.T	0.784	0.786	0.785	0.786
	AEL	0.784	0.781	0.778	0.776
	AEL.T	0.786	0.785	0.782	0.781
	EB	0.782	0.779	0.778	0.774
T=1000	EL	0.896	0.893	0.897	0.892
	EL.T	0.902	0.904	0.904	0.904
	AEL	0.913	0.917	0.917	0.916
	AEL.T	0.935	0.936	0.936	0.927
	EB	0.909	0.904	0.906	0.906

Table 2 – continued from previous page

n	Method	$d = 0.1$	$d = 0.2$	$d = 0.3$	$d = 0.4$
			Model: $a_t \sim \chi_5^2 - 5$		
T=50	EL	0.252	0.258	0.265	0.281
	EL.T	0.544	0.555	0.555	0.555
	AEL	0.423	0.442	0.453	0.448
	AEL.T	0.628	0.639	0.635	0.644
	EB	0.410	0.421	0.432	0.445
T=70	EL	0.502	0.502	0.502	0.497
	EL.T	0.675	0.675	0.675	0.673
	AEL	0.597	0.597	0.597	0.598
	AEL.T	0.714	0.714	0.714	0.712
	EB	0.531	0.532	0.578	0.563
T=100	EL	0.708	0.709	0.706	0.704
	EL.T	0.741	0.742	0.749	0.747
	AEL	0.732	0.731	0.731	0.734
	AEL.T	0.748	0.753	0.76	0.752
	EB	0.732	0.731	0.728	0.721
T=200	EL	0.869	0.871	0.872	0.876
	EL.T	0.881	0.880	0.881	0.881
	AEL	0.875	0.879	0.878	0.882
	AEL.T	0.884	0.884	0.886	0.884
	EB	0.872	0.875	0.874	0.878
T=1000	EL	0.893	0.899	0.895	0.896
	EL.T	0.908	0.907	0.908	0.902
	AEL	0.912	0.919	0.911	0.912
	AEL.T	0.923	0.923	0.929	0.937
	EB	0.907	0.906	0.905	0.904

EL= empirical likelihood; EL.T=EL with the transformation;
AEL=Adjusted EL; AEL.T=EL with the transformation;
'.T' represents the transformed statistics

4.2. Power curves for ARFIMA (1,d,0) models

For ARFIMA(1,d,0) model, we consider the behavior of power of the transformed empircal likelihood and transformed adjusted empirical likelihood under three different distributions for the white noise process a_t: $N(0,1)$, t_5, and χ_5^2. The latter two distributions are centered around zero. In these cases, we consider $\phi = 0.2$ with four different values for d: $d = (0.1, 0.2, 0.3, 0.4)$. We also consider four different sample sizes ($n = 50, 70, 100, 200$) under each error distribution. Figure 1 below provides the power curves for ARFIMA(1,d,0) model at various sample sizes and orders of differencing (d) for given error distributions. The black solid line is for the EL, red dotted line is for the EL with transformation, blue dotdash line is for the AEL and green dotdash line is for the AEL with transformation. It also indicates the patterns we observe from the Table 1 & Table 2.
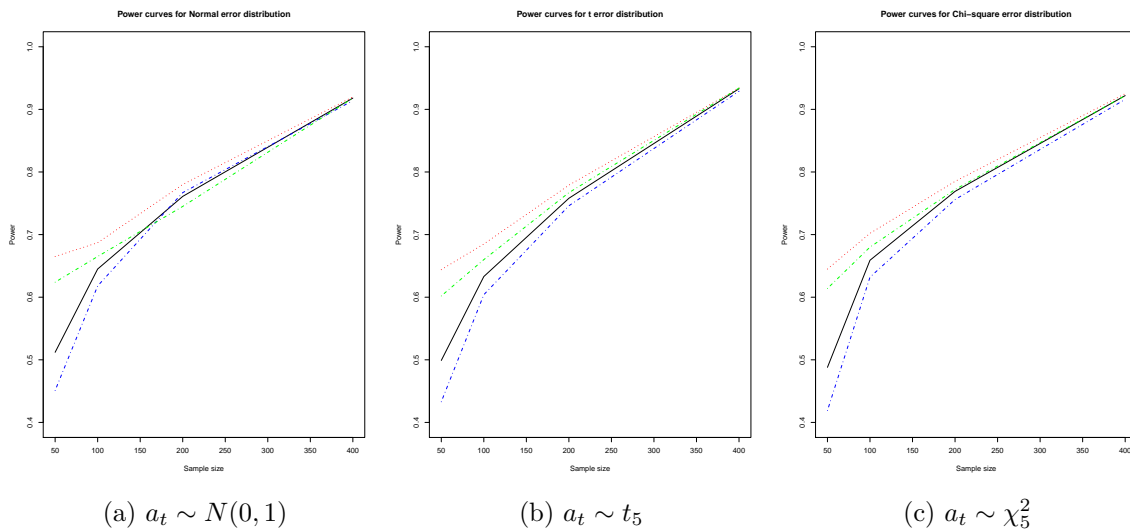

(a) $a_t \sim N(0,1)$ (b) $a_t \sim t_5$ (c) $a_t \sim \chi_5^2$

Figure 1: Power curves for ARFIMA(1,d,0) models

5. Discussion

In this paper, we propose a transformed adjusted empirical likelihood (TAEL) to extend the method by Gamage *et al.* (2017) for long-memory time series models, specifically ARFIMA models. Such a method combines the advantage of the adjusted empirical likelihood (AEL) by Chen *et al.* (2008) on ensuring the existence of the solutions in optimizing the profile empirical likelihood ratio, and the advantage of the transformed empirical likelihood (JEL) by Jing *et al.* (2017) on improving coverage probabilities while maintaining the same asymptotic properties and simple form. The asymptotic null distribution of the TAEL statistic for long-memory time series models has been established as a standard chi-square distribution. Simulations have been conducted for different ARFIMA models with various sample sizes and error distributions to compare the performance between the proposed TAEL and the other existing methods. Results indicate that the TAEL provides better coverage probabilities especially for small sample sizes.

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