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A Comparative Study of Traditional and Kullback-Leibler Divergence of Survival Functions Estimators for the Parameter of Lindley Distribution

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Abstract

A new point estimation method based on Kullback-Leibler divergence of survival functions (KLS), measuring the distance between an empirical and prescribed survival functions, has been used to estimate the parameter of Lindley distribution. The simulation studies have been carried out to compare the performance of the proposed estimator with the corresponding Least square (LS), Maximum likelihood (ML) and Maximum product spacing (MPS) methods of estimation.

Keywords: Kullback-Leibler divergence, Lindley distribution, least squares (LS) estimator, maximum likelihood (ML) estimator, maximum product spacings (MPS) estimator, survival function, simulation.

1. Introduction

The Kullback-Leibler(K-L) divergence, Kullback and Leibler (1951) or relative entropy measures the distance between two probability distributions. This divergence measure is also known as information divergence. Let X and Y be two continuous random variables with density functions f and g respectively over the same support $\mathbb{R}$. The K-L divergence of f relative to g is, then, defined as;

$$D(f||g) = \int_{\mathbb{R}} f(x) ln \frac{f(x)}{g(x)} dx \tag{1}$$

It is assumed that for all x belonging to $\mathbb{R}$, $g(x) \neq 0$. The function $D(f||g)$ is always non-negative and it is zero if and only if $f =$ g a.s..
Kullback-Leibler divergence has been used in various statistical problems including model selection and parameter estimation. For different applications of K-L divergence and related

details see, Abramov, Majda, and Kleeman (2005), Cover and Thomas (2006), Dhillon, Mallela, and Kumar (2003), Moreno, Ho, and Vasconcelos (2003), Mehrali and Asadi (2016).

Let $f(x;\theta)$ belongs to a parametric family with k-dimensional vector parameter $\theta \in \Theta \subset \mathbb{R}^k$ and f_n be kernel density estimate of f(the distribution of X) based on n random observations $\{X_1, X_2, ...X_n\}$. Basu and Lindsay, Lindsay (1994) used K-L divergence of f_n relative to f defined as;

$$D(f_n||f) = \int_{\mathbb{R}} f_n(x) ln \frac{f_n(x)}{f(x;\theta)} dx,$$

for the estimation of θ and suggested that the estimate of the parameter is that value for which $D(f_n||f)$ is least . Thus, the estimate of the parameter can be defined as;

$$\hat{\theta} = arg \inf_{\theta \in \Theta} D(f_n(x)||f(x;\theta))$$

Although, the method of estimation based on $D(f_n||f)$ possesses many interesting features, there are some limitations to apply K-L divergence measure for continuous random variables. It worthwhile to note here that the above definition is based on the density of random variables which, in general, may or may not exist, see Cover and Thomas (2006). In addition it depend on f_n which cannot be properly estimated from the sample data in the sense that even by increasing the sample size sufficiently large, one cannot guarantee that the estimated density converges to its true measure. To overcome such problems, various approaches have been suggested by different researchers to estimate the K-L divergence from sample data for continuous random variables, for details of these methods see Lee and Park (2006), Pérez-Cruz (2008), Wang, Kulkarni, and Verdú (2005), Wang, Kulkarni, and Verdú (2006). Apart from these, several alternative measures to K-L divergence have also been defined, for details reader may refer to Aczél and Daróczy (1975), Forte and Hughes (1988).

To grip over the problem, J. Liu Liu (2007) proposed a new divergence measure named as Kullback-Leibler divergence of survival functions (KLS), which measures the distance between an empirical and prescribed survival function. The key idea of using survival function instead of density function is that the survival function is more regular and can be easily estimated from sample data as compared to density function. Further, by law of large number, the estimate is convergent. Liu (2007) used this new divergence measure to estimate the parameters of uniform and exponential distributions. Although the KLS estimates are found to be biased, its convergence in mean square error to the true value of the parameter is faster than ML estimate. Yari, Mirhabibi, and Saghafi (2013) demonstrated the use of KLS method for the estimation of the parameters of Weibull distribution.

In this paper, we have used KLS method for the estimation of the parameter of Lindley distribution and compared the estimator thus obtained with few classical estimators. The reason for the choice of Lindley distribution rest in simple form of its survival function.

The organisation of rest of the paper is in the following manner. To introduce the model under consideration, a brief description of the Lindley distribution is given in Section 2. Different estimation procedures have been discussed in Section 3. Section 4 contains a simulation study to compare the performance of KLS estimation procedure with some alternative methods. The conclusion of the present piece of work have been provided in Section 5.

2. The Lindley distribution

Lifetime distributions describe the random behavior of the length of the life of a system or a device. These are having their applications in the field of science and technology . In statistical literature, several models such as Exponential, Gamma, Weibull have been proposed to analyze the lifetime data. The popularity of Gamma and Weibull distributions, over exponential distribution is due to the fact that these distributions have more general mathematical form than that of exponential distribution, although with an additional parameter. In recent years, the use of one parameter Lindley distribution over exponential and other distributions has increased. It was originally proposed by Lindley (1958) in the context of Bayesian statistics as a counter example of fiducial statistics and latter attracted the researchers for its use in modeling lifetime data. This distribution has wider applicability than other competitive models including exponential, log-normal and gamma etc. It is particularly most suitable model when the data show increasing failure rate. Lindely distribution can be seen as a mixture of $exp(\theta)$ and $gamma(2,\theta)$. Various statistical properties of the Lindley distribution is discussed extensively by Ghitany, Atieh, and Nadarajah (2008) and they have verified that Lindley distribution is particularly useful in modeling biological data from mortality studies, see Ghitany, Alqallaf, Al-Mutairi, and Husain (2011), Sharma, Singh, Singh, and Ul-Farhat (2017) and is also useful for estimation of the reliability of a stress-strength system, see Ghitany, Al-Mutairi, and Aboukhamseen (2015).

A random variable X is said to have Lindley distribution with parameter θ, if its probability density function is defined as;

$$f(x) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}, x > 0, \theta > 0 \tag{2}$$

The corresponding cumulative distribution function (c.d.f.) is:

$$F(x) = 1 - \frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}, x > 0, \theta > 0 \tag{3}$$

and the survival function is:

$$S(x) = \frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}, x > 0, \theta > 0 \tag{4}$$

3. Parameter estimation

In statistical literature a number of estimation procedure see Yari and Tondpour (2017), Louzada, Ramos, and Perdoná (2016), Bakouch, Dey, Ramos, and Louzada (2017), Mazucheli, Ghitany, and Louzada (2017) are available under classical paradigm. Here, we shall consider four such procedures, namely Kullback- Leibler divergence of survival functions(KLS), Least square estimation(LSE), Maximum likelihood estimation(MLE) and Maximum product spacing method(MPS).

3.1. Method of least square

Let $x_1 < x_2 < ... < x_n$ be n ordered random observations from Lindley distribution with CDF F(x), then

$$E[F(x_i)] = \frac{i}{(n+1)}$$

The least square estimate is that value of the parameter which minimizes

$$SD(\theta) = \sum_{i=1}^{n}\left(F(x_i) - \frac{i}{n+1}\right)^2 \tag{5}$$

Putting the value of $F(x)$ in equation (5) we get

$$SD(\theta) = \sum_{i=1}^{n}\left\{\left(1 - \frac{1+\theta+\theta x_i}{1+\theta}exp(-\theta x_i)\right) - \frac{i}{n+1}\right\}^2 \tag{6}$$

In order to get the value of the parameter which minimizes equation (6), we differentiate it with respect to θ and equating it to zero, which gives the following equation:

$$\sum_{i=1}^{n}\left\{1 - \frac{1+\theta+\theta x_i}{1+\theta}exp(-\theta x_i) - \frac{i}{n+1}\right\}\left\{\frac{\theta^2(1+x_i)exp(-\theta x_i)}{1+\theta}\right\} = 0$$

Since the above equation cannot be solved analytically, we propose the use of Newton-Rapson method for its numerical solution.

3.2. Maximum likelihood estimation

The likelihood function of θ is

$$L(\theta) = \left(\frac{\theta^2}{1+\theta}\right)^n \prod_{i=1}^{n}(1+x_i)e^{-\theta\sum x_i} \tag{7}$$

and hence, the log-likelihood function is

$$lnL(\theta) = 2n * ln\theta - n * ln(1+\theta) + \sum_{i=1}^{n} ln(1+x_i) - \theta\sum_{i=1}^{n} x_i$$

For maximazation of equation (7), we differentiate the above expression with respet to θ and equate it to zero, and get the following equation:

$$\frac{2n}{\theta} - \frac{n}{1+\theta} - \sum_{i=1}^{n} x_i = 0 \Rightarrow \bar{X}\theta^2 + (\bar{X}-1)\theta - 2 = 0, \theta > 0$$

Let $\hat{\theta}_{ML}$ be the maximum likelihood estimate of θ then solving the above equation for θ we get

$$\hat{\theta}_{ML} = \frac{(1-\bar{X}) + \sqrt{\bar{X}^2 + 6\bar{X} + 1}}{2\bar{X}} \tag{8}$$

clearly, $\hat{\theta}_{ML}$ is a function of sample mean $\bar{X}$.

3.3. Maximum product spacings method

Let $x_1 < x_2 < ... < x_n$ are be the n ordered random sample from Lindley distribution with CDF given in equation(3). The spacing are defined as follows; see Singh, Singh, and Singh (2014)

$$D_1 = F(x_1) = 1 - \left[\frac{1+\theta+\theta x_1}{1+\theta}e^{-\theta x_1}\right]$$

$$D_{n+1} = 1 - F(x_n) = \frac{1+\theta+\theta x_n}{1+\theta}e^{-\theta x_n}$$

and the general term of spacing D_i for i= 2, 3, ... ,n is given by,

$$D_i = F(x_i) - F(x_{i-1}) = \left[\frac{1+\theta+\theta x_{i-1}}{1+\theta}e^{-\theta x_{i-1}}\right] - \left[\frac{1+\theta+\theta x_i}{1+\theta}e^{-\theta x_i}\right] \tag{9}$$

Note that $\sum D_i = 1$, The geometric mean of spacings is

$$G = \left(\prod_{i=1}^{n+1} D_i\right)^{\frac{1}{n+1}}$$

The MPS method considers that value of θ as its estimate, which maximizes the logarithm of the geometric mean of the spacings i.e.

$$P = lnG = \frac{1}{1+n}\sum_{i=1}^{n+1} lnD_i$$

which reduces, after simplification to the following:

$$
\begin{aligned}
P = & \frac{1}{n+1}\left[ln\left\{1-\frac{1+\theta+\theta x_1}{1+\theta}e^{-\theta x_1}\right\}\right] \\
& + \frac{1}{n+1}\left[\sum_{i=1}^{n} ln\left\{\left(\frac{1+\theta+\theta x_{i-1}}{1+\theta}e^{-\theta x_{i-1}}\right)-\left(\frac{1+\theta+\theta x_i}{1+\theta}e^{-\theta x_i}\right)\right\}\right] \\
& + \frac{1}{n+1}\left[ln\left\{\frac{1+\theta+\theta x_n}{1+\theta}e^{-\theta x_n}\right\}\right]
\end{aligned} \tag{10}
$$

Thus, MPS estimate, denoted as $\hat{\theta}_{MPS}$, is that value of the parameter θ which maximizes P. In order to obtain the MPS estimate of θ, we have to maximised equation (10) with respect to θ which can be obtained by setting the partial derivative of P with respect to θ equal to zero and solve the resulting equation. But, it may be seen that the resulting equation will be a non-linear equation having no closed form solution, therefore it is solved numerically. This can be easily done by using Newton-Rapson method.

3.4. Kullback-Leibler divergence of survival functions method

Let $x_1, x_2, ...x_n$ be i.i.d. sample of size n from the Lindley distribution having distribution function $F(x;\theta)$ with unknown parameter θ. Let $S(x,\theta)$ be the corresponding true survival function and $G_n(x)$ be the emprical survival function based on a random samples of size n. The Kullback-Leibler divergence of Survival functions $G_n(x)$ and $S(x)$ is defined by

$$
KLS(G_n||S) = \int_0^\infty G_n(x) ln\frac{G_n(x)}{S(x)} - [G_n(x) - S(x)]dx \tag{11}
$$

where $G_n(x)$ is empirical survival function of a random sample of size n, defined by

$$
G_n(x) = \sum_{i=0}^{n-1}\left(1-\frac{i}{n}\right) I_{[X(i),X(i+1)]}(x) \tag{12}
$$

in the above expression, I is an indicator function and $0 = X_{(0)} \leq X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ is ordered sample.
Since G_n and S are integrable , it is easy to verify that equation (11) can be simplified to

$$
KLS(G_n||S) = \int_0^\infty G_n(x) ln\frac{G_n(x)}{S(x)}dx - [\bar{X_n} - E(X_1)] \tag{13}
$$

$$
= \int_0^\infty G_n(x) lnG_n(x)dx - \int_0^\infty G_n(x) lnS(x)dx - [\bar{X_n} - E(X_1)] \tag{14}
$$

using equation(12), we get

$$
\int_0^\infty G_n(x) lnG_n(x)dx = \sum_{i=1}^{n-1}(1-\frac{i}{n}) ln(1-\frac{i}{n})\triangle x_{i+1} \tag{15}
$$

Now using the definition of $G_n(x)$ given in (12), as n $\longrightarrow \infty$ for a fixed n, we can write

$$
\int_0^\infty G_n(x) lnS(x)dx = \sum_{i=0}^{n-1}(1-\frac{i}{n})\int_{x_{(i)}}^{x_{(i+1)}} lnS(x)dx \tag{16}
$$

Let us define $h(x) = \int_0^x lnS(t)dt$ for $x \in S_x$. h is well defined because S is monotone on its support S_x and $h(0) = 0$. Using above substution in equation (16) and solving it, we get

$$
\int_0^\infty G_n(x) lnS(x)dx = \frac{1}{n}\sum_{i=1}^{n} h(x_i) \tag{17}
$$

For details of the above,one can refer Rao, Chen, Vemuri, and Wang (2004); Yari *et al.* (2013). Now, in order to estimate the parameter of Lindley distribution using KLS, we simplify equation (14) using equations (4),(15) and (17), we get

$$\begin{aligned} KLS(G_n(x)||S) = & \sum_{i=1}^{n-1}(1-\frac{i}{n})ln(1-\frac{i}{n})\triangle x_{i+1} \\ & -\frac{1}{n}\sum_{i=1}^{n}\left[\int_0^{x_i} ln(1+\theta+\theta t)dt - ln(1+\theta)\int_0^{x_i} dt - \theta\int_0^{x_i} tdt\right] \\ & -\left[\bar{X}_n - \frac{2+\theta}{\theta(1+\theta)}\right] \end{aligned} \tag{18}$$

Where $\triangle x_{i+1} = x_{i+1} - x_i, x_0 = 0$. After simplifying equation (18), we get

$$\begin{aligned} KLS(G_n(x)||S) = & \frac{1}{n}\sum_{i=1}^{n}\left[(1+x_i+\frac{1}{\theta})ln(1+\theta) - (1+x_i+\frac{1}{\theta})ln(1+x_i+\theta x_i)\right] \\ & \sum_{i=1}^{n-1}(1-\frac{i}{n})ln(1-\frac{i}{n})\triangle x_{i+1} + \frac{1}{n}\sum_{i=1}^{n}\left[x_i+\frac{\theta}{2}x_i^2\right] - \left[\bar{X}_n - \frac{2+\theta}{\theta(1+\theta)}\right] \end{aligned} \tag{19}$$

Let $\hat{\theta}_{KLS}$ denotes the KLS estimator of parameter θ. In order to obtain the KLS estimator of θ, we have to minimize equation (19) with respect to θ. Therefore by setting its partial derivative with respect to θ equal to zero, we get an equation. The solution of that equation provides the desired estimate. But it may be noted here that the resulting equation is non-linear equation and an analytical solution of it does not exist. However one can use numerical method for its solution, we have used Newton-Rapson method.

4. Simulation study

In this section, we have studied the performances of the estimators, discussed in the previous section based on a simulation study. For this purpose, we generated pseudo-random samples from Lindley distribution for different sample size n (=10[10]70). Since the relative behavior of the estimators are observed to be invariant with respect to θ, therefore the results are reported here are for $\theta = 1$. The estimate of the parameter is obtained for each case and the whole process was repeated $N = 50{,}000$ times. The algorithm is coded in R software Team (2015) and Leisch (2002). The simulated values were then used to compute average estimates and MSEs of estimates. The formula for computing them are

$$\bar{\theta} = \frac{1}{N}\sum_{j=1}^{N}\theta_j, \qquad MSE(\hat{\theta}) = \frac{1}{N}\sum_{j=1}^{N}(\hat{\theta}_j - \theta)^2,$$

respectively. It is clear that MSEs will depend on sample size n and parameter θ.

From Figure 1 it can be seen that the three methods i.e. KLS, MLE and LSE methods overestimate, while MPS method underestimates the Lindley parameter θ. The KLS method provides more biased estimator as compared with the other three methods. Though, its bias decreseas more rapidely with incresing sample size n. From this point of view obviously the LSE method provides a better estimator but precision of the estimator is not only related to the bais, also to the mean square error of estimator.

5. Conclusion

In this paper, we have considered the problem of comparison of different method of estimations. A new method (KLS) is used for the estimation of Lindley parameter, which minimizes an entropy based distance between empirical survival function and the Lindley survival

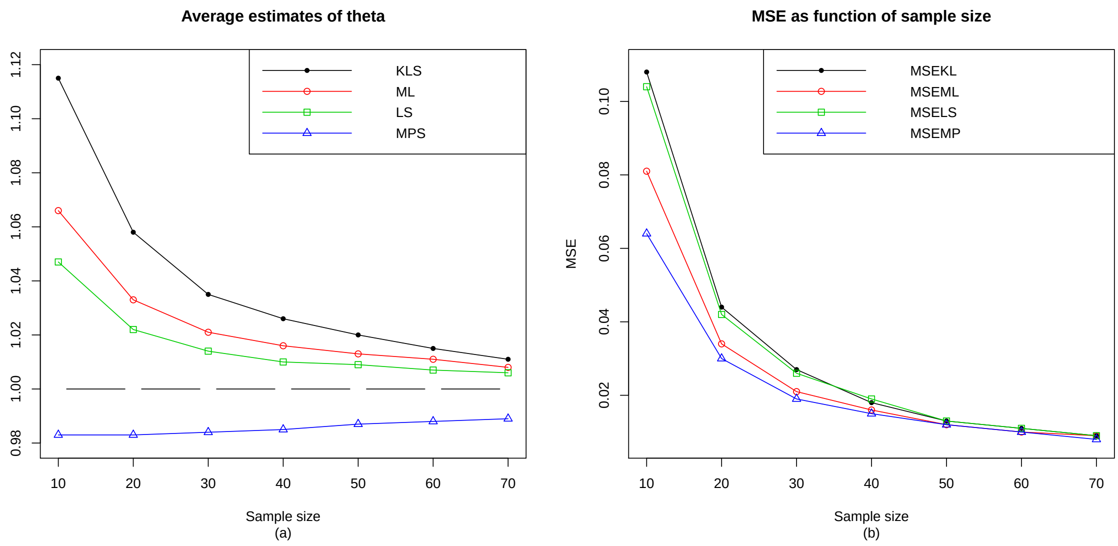

Figure 1: Figure (a) shows the average estimates of θ for the four different estimation methods with the variation of sample size n. Figure (b) shows the MSE of the four different estimation methods.

function. The general procedure to obtain the estimators are provided and then Lindley distribution is used to validate the results. The simulation study shows that among all the four considred estimation methods i.e. KLS, MLE, LSE and MPS, MPS method performs efficiently in term of MSE criterion. But, for sample size greater than 40 one can estimate the Lindley parameter θ using KLS method as efficient as the MLE and MPS. However, the KLS method is quite flexible since it uses survival function instead of density function and survival function can be easily estimated from the observed sample data . The KLS methodology can be used with any distribution and will be very useful in the field of science, engineering and medical science. Application of KLS method in goodness-of-fit tests and modeling censored experimental data could be a fruitful future research.

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