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Estimation of Stress Strength Reliability of Inverse Weibull Distribution under Progressive First Failure Censoring

Hare Krishna
Ch. Charan Singh,
University, India

Madhulika Dube
Maharshi Dayanand,
University, India

Renu Garg
Maharshi Dayanand,
University, India

Abstract

In this article, estimation of stress-strength reliability $\delta = P(Y < X)$ based on progressively first failure censored data from two independent inverse Weibull distributions with different shape and scale parameters is studied. Maximum likelihood estimator and asymptotic confidence interval of δ are obtained. Bayes estimator of δ under generalized entropy loss function using non-informative and gamma informative priors is derived. Also, highest posterior density credible interval of δ is constructed. Markov Chain Monte Carlo (MCMC) technique is used for Bayes computation. The performance of various estimation methods are compared by a Monte Carlo simulation study. Finally, a pair of real life data is analyzed to illustrate the proposed methods of estimation.

Keywords: inverse Weibull distribution, progressive first failure censoring, stress-strength reliability, maximum likelihood estimation, Bayes estimation, MCMC technique.

1. Introduction

The cumulative distribution function (cdf) of the inverse Weibull distribution (IWD) is given by

$$F(x;\alpha,\lambda) = e^{-\lambda x^{-\alpha}}, \qquad x > 0, \quad \alpha, \lambda > 0, \tag{1}$$

where α and λ are shape and scale parameters, respectively.

The corresponding probability density function (pdf) and failure rate function, respectively, are given by

$$f(x;\alpha,\lambda) = \alpha\lambda x^{-\alpha-1} e^{-\lambda x^{-\alpha}}, \qquad x > 0, \quad \alpha, \lambda > 0, \tag{2}$$

$$h(x;\alpha,\lambda) = \frac{\alpha\lambda x^{-\alpha-1}}{\left(e^{\lambda x^{-\alpha}} - 1\right)}, \qquad x > 0, \quad \alpha, \lambda > 0. \tag{3}$$

The plot of failure rate function of IWD for different values of shape parameter α and fixed scale parameter $\lambda = 1$ is shown in Figure 1.

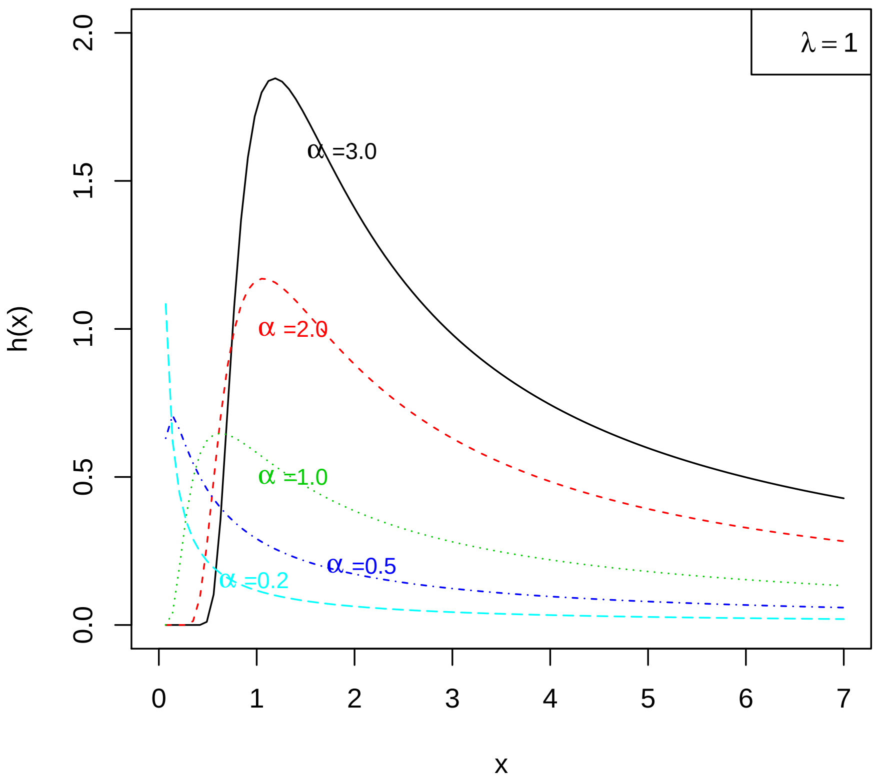

Figure 1: Graph of failure rate function of IWD.

The IWD is a useful lifetime probability distribution in reliability engineering. The IWD can be used to model a variety of failure characteristics such as infant mortality, utilisation time and wear-out periods. The IWD can also be used to determine the cost effectiveness and maintenance periods of reliability centered maintenance activities. From Figure 1 we observe that IWD has upside-down bathtub (UBT) shaped failure rate. There are various real life examples where data show the UBT failure rate, see, Sharma, Singh, and Singh (2014). If the empirical studies indicate that the failure rate function might be UBT, IWD may be an appropriate model. Several researches have been carried out for inferences on the IWD using classical and Bayesian approaches, for example, Khan, Pasha, and Pasha (2008) considered theoretical analysis of IWD. Kundu and Howlader (2010) studied Bayesian inferences and prediction of the IWD for type II censored data. Singh, Singh, and Kumar (2013) studied Bayesian estimation of parameters of IWD. Sultan, Alsadat, and Kundu (2014) discussed Bayesian and maximum likelihood estimation methods of the IWD parameters under progressive type II censoring. Singh and Tripathi (2018) studied estimation of the parameters of IWD under progressive type I interval censoring. Xiuyun and Zaizai (2016) discussed Bayesian estimation and prediction for the IWD under general progressive censoring. Kubota and Kurosawa (2017) studied Bayesian prediction of unobserved values for type II censored data in case of IWD.

The term stress-strength reliability in statistical literature typically refers to the quantity $\delta = P(Y < X)$. This term represent the reliability of a system with strength X subject to a stress Y. The system fails if the applied stress exceeds its strength. The stress-strength reliability of system is the probability that the system is strong enough to overcome the stress imposed on it. The model $P(Y < X)$ was first considered by Birnbaum and McCarty (1958). Since then several researcher have considered different choices for stress and strength distributions. An excellent monograph on the different stress-strength models is given by Kotz, Lumelskii, and Pensky (2003). Some recent studies on stress-strength models can also be found in Raqab, Madi, and Kundu (2008), Rezaei, Tahmasbi, and Mahmoudi (2010), Al-Mutairi, Ghitany, and Kundu (2013), Jia, Nadarajah, and Guo (2017), Krishna, Dube, and Garg (2017), Jovanović (2017), Sharma (2018) and the references cited therein.

Although the estimation of stress-strength model was discussed in literature frequently for complete samples, not much attention has been paid to cases when the data are censored. In life testing experiments there are many practical situations where complete sample data are not possible to collect. In such situations, we observed censored samples rather than complete samples. There are two most popular censoring schemes, type I and type II censoring schemes. In type I censoring scheme, the experiment is terminated after a pre-specified time point and in type II censoring scheme, the experiment is terminated after getting a pre-fixed number of failures. The main drawback of these censoring schemes is non-removal of live units from the experiment at point other than the final termination point. The progressive first failure censoring scheme, a generalization of progressive censoring scheme, allows the removal of live units from the experiment. Recently, it has become very popular in life testing experiments because it has an advantage in terms of shorter test time, a saving of resources and in this a specific fraction of individuals at risk may be removed from the experiment at each of several ordered failure times.

The progressive first failure censoring scheme was introduced in literature by Wu and Kus (2009). They studied the estimation methods based on progressively first failure censored Weibull distribution. Soliman, Abd-Ellah, Abou-Elheggag, and Modhesh (2012) discussed estimation of the parameters of life for Gompertz distribution using progressive first failure censored data. Soliman, Abd-Ellah, Abou-Elheggag, and Modhesh (2013) studied estimation in Burr type XII distribution using progressive first failure censored data. Dube, Garg, and Krishna (2016a) and Dube, Krishna, and Garg (2016b) described classical and Bayesian estimation in Lindley and generalized inverted exponential distributions, respectively under progressive first failure censoring. Also, Ahmed (2017) discussed estimation and prediction for the generalized inverted exponential distribution based on progressively first failure censored data with application.

Recently, some authors have also investigated the estimation of δ for some lifetime distributions based on progressively first failure censored data, see for example, Lio and Tsai (2012). They discussed the estimation of stress-strength reliability for Burr XII distribution based on progressively first failure censored sample. Kumar, Krishna, and Garg (2015) studied estimation of stress-strength reliability for Lindley distribution based on progressively first failure censored sample. The progressive first failure censoring is briefly described as follows:

Suppose that initially, n independent groups with k items within each group are put on a life testing experiment with pre-fixed progressive censoring scheme $\underset{\sim}{R} = (R_1, R_2, \ldots, R_m)$ such that after observing first failure, the group in which the first failure is observed and R_1 other live groups are randomly removed from the experiment. At the time of second failure, the group in which second failure is observed and R_2 others groups are randomly removed from the experiment and so on. This procedure continues until the group in which m^{th} failure has occurred and all remaining R_m groups are removed at the time of m^{th} failure. It is clear that $\sum_{i=1}^{m} R_i + m = n$. Note that, if $R_1 = R_2 = \ldots = R_m = 0$, progressive first failure censoring scheme is reduced to first failure censoring scheme and if $R_1 = R_2 = \ldots = R_{m-1} = 0$ and $R_m = n - m$, this scheme reduces to first failure type II censoring. With $k = 1$ in each group, progressive first failure censoring scheme becomes the progressive censoring scheme.

Let $x_1, x_2, \ldots, x_m$ be a progressively first failure censored sample with pre-fixed progressive censoring scheme $\underset{\sim}{R} = (R_1, R_2, \ldots, R_m)$ from a population with pdf $f_X(.)$ and cdf $F_X(.)$, the likelihood function is given by (see, Wu and Kus (2009))

$$L(x_1, x_2, \ldots, x_m) = Ak^m \prod_{i=1}^{m} f_X(x_i)\{1-F_X(x_i)\}^{(k(R_i+1)-1)}, \qquad 0 < x_1 < x_2 < \ldots < x_m < \infty, \tag{4}$$

where $A = n(n - R_1 - 1)(n - R_1 - R_2 - 2)\ldots(n - R_1 - R_2 - \ldots - R_{m-1} - m + 1)$.

The main objective of this article is the classical and Bayesian estimation of stress-strength reliability $\delta = P(Y < X)$, where, X and Y both are independent random variables having inverse Weibull distributions and samples obtained from both the distribution are progressively first failure censored samples. Let $X \sim IWD(\alpha, \lambda)$ and $Y \sim IWD(\beta, \theta)$ be independent random variables, then

$$\delta = P(Y < X) = \lambda \int_0^\infty e^{-(\lambda t + \theta t^{\frac{\beta}{\alpha}})}\, \mathrm{d}t = H(\alpha, \beta, \lambda, \theta) \tag{5}$$

The rest of the article is organized as follows: In section 2, we obtain maximum likelihood estimator (MLE) and asymptotic confidence interval of stress-strength reliability δ. Bayes estimator of δ under generalized entropy loss function (GELF) using non-informative and informative gamma priors is discussed in section 3. For Bayesian computation, Markov Chain Monte Carlo (MCMC) techniques are used. Also, highest posterior density (HPD) credible interval of δ is derived using MCMC methods. In section 4, a Monte Carlo simulation study is performed to compare different estimation methods. Section 5 deals with real data analysis for illustration purposes. Finally, discussion and concluding remarks on the article are given in section 6.

2. Maximum likelihood estimation

In this section, we obtain the MLEs of the unknown parameters and stress-strength reliability δ. Also, asymptotic confidence interval of δ is constructed. Let $x_1, x_2, \ldots, x_{m_1}$ be the progressively first failure censored sample from $IWD(\alpha, \lambda)$ with pre-fixed progressive censoring scheme $\underset{\sim}{R}_x = (R_{x,1}, R_{x,2}, \ldots, R_{x,m_1})$ and $y_1, y_2, \ldots, y_{m_2}$ be the independent progressively first failure censored sample from $IWD(\beta, \theta)$ with progressive censoring scheme $\underset{\sim}{R}_y = (R_{y,1}, R_{y,2}, \ldots, R_{y,m_2})$, the likelihood function of α, β, λ and θ is given by

$$\begin{aligned} L(\alpha, \beta, \lambda, \theta) &= \left[A_1 k_1^{m_1} \alpha^{m_1} \lambda^{m_1} e^{-\lambda \sum_{i=1}^{m_1} x_i^{-\alpha}} \prod_{i=1}^{m_1} x_i^{-\alpha-1} \prod_{i=1}^{m_1} \left(1 - e^{-\lambda x_i^{-\alpha}}\right)^{k_1(R_{x,i}+1)-1}\right] \\ &\quad \times \left[A_2 k_2^{m_2} \beta^{m_2} \theta^{m_2} e^{-\theta \sum_{j=1}^{m_2} y_j^{-\beta}} \prod_{j=1}^{m_2} y_j^{-\beta-1} \prod_{j=1}^{m_2} \left(1 - e^{-\theta y_j^{-\beta}}\right)^{k_2(R_{y,j}+1)-1}\right], \end{aligned} \tag{6}$$

where $A_1 = n_1(n_1 - R_{x,1} - 1)(n_1 - R_{x,1} - R_{x,2} - 2)\ldots(n_1 - R_{x,1} - R_{x,2} - \ldots - R_{x,m_1-1} - m_1 + 1)$ and $A_2 = n_2(n_2 - R_{y,1} - 1)(n_2 - R_{y,1} - R_{y,2} - 2)\ldots(n_2 - R_{y,1} - R_{y,2} - \ldots - R_{y,m_2-1} - m_2 + 1)$. Thus, the corresponding log-likelihood function becomes

$$\begin{aligned} l(\alpha, \beta, \lambda, \theta) &= C + m_1 \ln\alpha + m_1 \ln\lambda - \lambda \sum_{i=1}^{m_1} x_i^{-\alpha} - (\alpha+1) \sum_{i=1}^{m_1} \ln x_i \\ &\quad + \sum_{i=1}^{m_1} (k_1(R_{x,i}+1) - 1) \ln(1 - e^{-\lambda x_i^{-\alpha}}) + m_2 \ln\beta + m_2 \ln\theta \\ &\quad - \theta \sum_{j=1}^{m_2} y_j^{-\beta} - (\beta+1) \sum_{j=1}^{m_2} \ln y_j + \sum_{j=1}^{m_2} (k_2(R_{y,j}+1) - 1) \ln(1 - e^{-\theta y_j^{-\beta}}), \end{aligned} \tag{7}$$

where $C = m_1 \ln k_1 + \ln A_1 + m_2 \ln k_2 + \ln A_2$.

Differentiating log-likelihood function $l(\alpha, \beta, \lambda, \theta)$ partially with respect to the parameters α, β, λ and θ, respectively, and then equating to zero,we get

$$\frac{\partial l}{\partial \alpha} = \frac{m_1}{\alpha} + \lambda \sum_{i=1}^{m_1} x_i^{-\alpha} \ln x_i - \sum_{i=1}^{m_1} \ln x_i - \sum_{i=1}^{m_1} \frac{\left(k_1 \left(R_{x,i}+1\right)-1\right) \lambda e^{-\lambda x_i^{-\alpha}} x_i^{-\alpha} \ln x_i}{\left(1 - e^{-\lambda x_i^{-\alpha}}\right)} = 0 \tag{8}$$

$$\frac{\partial l}{\partial \beta} = \frac{m_2}{\beta} + \theta \sum_{j=1}^{m_2} y_j^{-\beta} \ln y_j - \sum_{j=1}^{m_2} \ln y_j - \sum_{j=1}^{m_2} \frac{\left(k_2 \left(R_{y,j}+1\right)-1\right) \theta e^{-\theta y_j^{-\beta}} y_j^{-\beta} \ln y_j}{\left(1 - e^{-\theta y_j^{-\beta}}\right)} = 0 \tag{9}$$

$$\frac{\partial l}{\partial \lambda} = \frac{m_1}{\lambda} - \sum_{i=1}^{m_1} x_i^{-\alpha} + \sum_{i=1}^{m_1} \frac{\left(k_1 \left(R_{x,i}+1\right)-1\right) e^{-\lambda x_i^{-\alpha}} x_i^{-\alpha}}{\left(1 - e^{-\lambda x_i^{-\alpha}}\right)} = 0 \tag{10}$$

$$\frac{\partial l}{\partial \theta} = \frac{m_2}{\theta} - \sum_{j=1}^{m_2} y_j^{-\beta} + \sum_{j=1}^{m_2} \frac{\left(k_2 \left(R_{y,j}+1\right)-1\right) e^{-\theta y_j^{-\beta}} y_j^{-\beta}}{\left(1 - e^{-\theta y_j^{-\beta}}\right)} = 0 \tag{11}$$

The MLEs $\hat{\alpha}$, $\hat{\beta}$, $\hat{\lambda}$ and $\hat{\theta}$ of the parameters α, β, λ and θ, respectively, are solutions of the equations (8), (9), (10), and (11), which cannot be solved analytically. To solve these non-linear equations we use a suitable numerical iterative procedure. In statistical software R, we may use functions such as *nlm*, *optim*, *maxLik* etc. to obtain MLEs. Once we obtain MLEs $\hat{\alpha}$, $\hat{\beta}$, $\hat{\lambda}$ and $\hat{\theta}$ using the invariance property of MLEs, the MLE of δ can be obtained as

$$\hat{\delta} = H\left(\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\theta}\right) \tag{12}$$

2.1. Asymptotic confidence interval

We obtain MLE $\hat{\delta}$ in (12) using invariance property of MLEs, but it is very difficult to obtain the exact distribution of δ. Thus, the construction of exact confidence interval of δ becomes quite difficult and it is not pursued here. Here, we suggest the use of delta method to obtain the asymptotic variance of $\hat{\delta}$, when m_1 and m_2 become large enough but $\frac{m_1}{m_2}$ remains constant, see Sharma (2018). Let $\hat{\omega} = \left(\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\theta}\right)$ be the MLEs of the unknown parameters $\omega = (\omega_1, \omega_2, \omega_3, \omega_4) = (\alpha, \beta, \lambda, \theta)$. The asymptotic variance of $\hat{\delta}$ using delta method, see Krishnamoorthy and Lin (2010), is $Var(\hat{\delta}) = [b_c^t I^{-1}(\omega) b_c]$ where, $I(\omega) = E\left[-\frac{\partial^2 l}{\partial \omega_i \partial \omega_j}\right], i = j = 1, 2, 3, 4$ is Fisher information matrix and $b_c = \left(\frac{\partial \delta}{\partial \alpha}, \frac{\partial \delta}{\partial \beta}, \frac{\partial \delta}{\partial \lambda}, \frac{\partial \delta}{\partial \theta}\right)'$. Under mild conditions the observed Fisher information is used as a consistent estimator of the Fisher information. Thus, the observed variance of $\hat{\delta}$ is given by

$$\widehat{Var}(\hat{\delta}) = [b_c^t I^{-1}(\omega) b_c]_{\omega = \hat{\omega}}$$

For second derivatives in $I(\omega)$ and elements of b_c, see, Appendix I.

Thus, $\frac{(\hat{\delta}-\delta)}{\sqrt{\widehat{Var}(\hat{\delta})}} \sim N(0,1)$. This result yields the asymptotic $100 \times (1-\gamma)\,\%$ confidence interval for δ as

$$\hat{\delta} \pm z_{\gamma/2} \sqrt{\widehat{Var}\left(\hat{\delta}\right)}$$

where $z_{\gamma/2}$ is the upper $(\gamma/2)^{th}$ quantile of standard normal distribution.

3. Bayesian estimation

In this section, we derive Bayes estimator of stress-strength reliability δ under generalized entropy loss function (GELF) using MCMC technique. In order to select the best decision in decision theory, an appropriate loss function must be specified. A symmetric loss function such as the popular squared error loss function (SELF) is well justified when over estimation and under estimation of equal magnitude has the same consequences. When the true loss is not symmetric with respect to over estimation and under estimation, asymmetric loss functions are used to represent the consequences of different errors. A general purpose loss function is GELF. The GELF is proposed by Calabria and Pulcini (1996). This loss function is a generalization of the entropy loss function and is given by

$$L\left(\theta, \hat{\theta}\right) \propto \left[\left(\frac{\hat{\theta}}{\theta}\right)^{q} - q\ln\left(\frac{\hat{\theta}}{\theta}\right) - 1\right]; \qquad q \neq 0,$$

where $\hat{\theta}$ is the decision rule which estimates θ. For $q > 0$, a positive error has a more serious effect than a negative error and for $q < 0$, a negative error has a more serious effect than a positive error. Under GELF the Bayes estimator is given by

$$\hat{\theta} = \left[E\left(\theta^{-q}|data\right)\right]^{-1/q} \tag{13}$$

It should be mentioned that for $q = 1, -1, -2$, the Bayes estimator given in (13) simplifies to the Bayes estimator under entropy loss function, SELF and precautionary loss function, respectively. Here, we assume that the unknown parameters α, β, λ and θ are a-priori independent and distributed as gamma distributions with their respective, pdfs

$$\begin{aligned} g_1(\alpha) &\propto \alpha^{a_1-1}e^{-b_1\alpha}; && \alpha > 0, a_1, b_1 > 0, \\ g_2(\beta) &\propto \beta^{a_2-1}e^{-b_2\beta}; && \beta > 0, a_2, b_2 > 0, \\ g_3(\lambda) &\propto \lambda^{a_3-1}e^{-b_3\lambda}; && \lambda > 0, a_3, b_3 > 0, \\ g_4(\theta) &\propto \theta^{a_4-1}e^{-b_4\theta}; && \theta > 0, a_4, b_4 > 0, \end{aligned}$$

where hyper-parameters $a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4$ are chosen to reflect prior knowledge about the parameters α, β, λ and θ, respectively. Many authors have used independent gamma priors in case of IWD, for example, Kundu and Howlader (2010), Singh *et al.* (2013), Sultan *et al.* (2014), Singh and Tripathi (2018). Thus, the joint prior distribution of α, β, λ and θ can be written as

$$g(\alpha, \beta, \lambda, \theta) \propto \alpha^{a_1-1}\beta^{a_2-1}\lambda^{a_3-1}\theta^{a_4-1}e^{-(b_1\alpha+b_2\beta+b_3\lambda+b_4\theta)} \tag{14}$$

The class of gamma prior distributions is quite flexible, as it can model a variety of prior information. It also includes non-informative prior distribution taking hyper-parameters $a_i = b_i = 0$; $i = 1, 2, 3, 4$. Thus, using the likelihood function in (6) and the joint prior distribution in (14), joint posterior distribution of α, β, λ and θ is given by

$$\pi(\alpha, \beta, \lambda, \theta|data) = \frac{L(data|\alpha, \beta, \lambda, \theta)g(\alpha, \beta, \lambda, \theta)}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty L(data|\alpha, \beta, \lambda, \theta)g(\alpha, \beta, \lambda, \theta)\,\mathrm{d}\alpha\mathrm{d}\beta\mathrm{d}\lambda\mathrm{d}\theta} \tag{15}$$

The joint posterior distribution of the unknown parameters given in (15) is complicated and no closed form estimates appear to be possible. We, therefore, consider MCMC techniques namely, Gibbs sampler, (see Smith and Roberts (1993)) and Metropolis-Hastings (M-H) algorithm (Hastings (1970)) to obtain the sample based Bayes estimator of the stress-strength reliability δ and to construct the corresponding HPD credible interval. The full posterior conditional distributions for α, β, λ and θ respectively, are given by

$$\pi_1(\alpha|\lambda, data) \propto \alpha^{m_1+a_1-1}e^{-\alpha(b_1+\sum_{i=1}^{m_1}\ln x_i)-\lambda\sum_{i=1}^{m_1}x_i^{-\alpha}}\prod_{i=1}^{m_1}\left(1-e^{-\lambda x_i^{-\alpha}}\right)^{k_1(R_{x,i}+1)-1} \tag{16}$$

$$\pi_2(\beta|\theta, data) \propto \beta^{m_2+a_2-1} e^{-\beta(b_2+\sum_{j=1}^{m_2}\ln y_j)-\theta\sum_{j=1}^{m_2} y_j^{-\beta}} \prod_{j=1}^{m_2}\left(1-e^{-\theta y_j^{-\beta}}\right)^{k_2(R_{y,j}+1)-1} \tag{17}$$

$$\pi_3(\lambda|\alpha, data) \propto \lambda^{m_1+a_3-1} e^{-\lambda(b_3+\sum_{i=1}^{m_1} x_i^{-\alpha})} \prod_{i=1}^{m_1}\left(1-e^{-\lambda x_i^{-\alpha}}\right)^{k_1(R_{x,i}+1)-1} \tag{18}$$

$$\pi_4(\theta|\beta, data) \propto \theta^{m_2+a_4-1} e^{-\theta(b_4+\sum_{j=1}^{m_2} y_j^{-\beta})} \prod_{j=1}^{m_2}\left(1-e^{-\theta y_j^{-\beta}}\right)^{k_2(R_{y,j}+1)-1} \tag{19}$$

We use following hybrid algorithm to generate samples from the full conditional posterior densities:

Step 1 Start with an initial guess of λ and θ say $\lambda^{(0)}$ and $\theta^{(0)}$.

Step 2 Set $j = 1$.

Step 3 Generate $\alpha^{(j)}$ from $\pi_1(\alpha|\lambda^{(j-1)}, data)$ in (16) using M-H algorithm with normal proposal distribution.

Step 4 Generate $\beta^{(j)}$ from $\pi_2(\beta|\theta^{(j-1)}, data)$ in (17) using M-H algorithm with normal proposal distribution.

Step 5 Generate $\lambda^{(j)}$ from $\pi_3(\lambda|\alpha^{(j)}, data)$ in (18) using M-H algorithm with normal proposal distribution.

Step 6 Generate $\theta^{(j)}$ from $\pi_4(\theta|\beta^{(j)}, data)$ in (19) using M-H algorithm with normal proposal distribution.

Step 7 Compute $\delta^{(j)} = H(\alpha^{(j)}, \beta^{(j)}, \lambda^{(j)}, \theta^{(j)})$ using (5).

Step 8 Set $j = j + 1$.

Step 9 Repeat steps 3-8, $(M - 1)$ times.

The selection of appropriate initial values and the normal distribution as proposal density is an important issue in the M-H algorithm. The rapid convergence is facilitated by selecting appropriate initial values and proposal normal distributions. The MLEs of the parameters may be considered as the starting values. Also, the MLEs and observed variances of MLEs obtained from posterior distribution in (15) may be a good choice as the mean and variance of the proposal normal distributions (Ntzoufras (2009), pp 44-45).

Now, the Bayes estimator of δ under GELF using MCMC technique is obtained as

$$\hat{\delta}_{Bayes} = \left[E(\delta^{-q}|data)\right]^{-1/q} = \left[\frac{1}{M_1}\sum_{j=M_0+1}^{M}(\delta^{(j)})^{-q}\right]^{-1/q}, \tag{20}$$

where M_0 is the burn-in i.e. we discard the first $\delta^{(1)}, \delta^{(2)}, \ldots, \delta^{(M_0)}$ observations and work with the remaining $M_1 = (M - M_0)$ observations, which are viewed as being an independent sample from the stationary distribution of the Markov chain which is typically the posterior distribution.

3.1. HPD credible interval

Once we have desired posterior MCMC sample, the HPD credible interval for δ can be constructed by using the algorithm proposed by Chen and Shao (1999). Let $\delta_{(1)} < \delta_{(2)} < \ldots <$

$\delta_{(M_1)}$ denote the ordered values of $\delta^{(M_0+1)}, \delta^{(M_0+2)}, \ldots, \delta^{(M)}$, the $100 \times (1-\gamma)\%$, $0 < \gamma < 1$, HPD credible interval for δ is given by

$$\left(\delta_{(j)}, \delta_{(j+[(1-\gamma)M_1])}\right),$$

where j is chosen such that

$$\delta_{(j+[(1-\gamma)M_1])} - \delta_{(j)} = min_{1 \le i \le \gamma M_1}\left(\delta_{(i+[(1-\gamma)M_1])} - \delta_{(i)}\right), j = 1, 2, \ldots, M_1,$$

here, $[x]$ is the integral part of x .

4. Monte Carlo simulation study

In this section, we present a Monte Carlo simulation study to compare the performance of the different methods of estimation for different censoring schemes, and for different parameter values. We compare the performances of the ML and Bayes estimates under GELF using non-informative and informative gamma priors in terms of average absolute biases (AB), and mean squared errors (MSE). We also compare asymptotic confidence and HPD credible intervals in terms of average length (AL) and coverage probability (CP). For the ease of simulation, we consider same group sizes $k = k_1 = k_2$, same number of groups $n = n_1 = n_2$, and same number of failures $m = m_1 = m_2$ with same pre-fixed censoring schemes $\underset{\sim}{R} = \underset{\sim}{R}_x = \underset{\sim}{R}_y$. A large number $N = 1,000$ of progressively first failure censored samples of varying group sizes $k = 3, 5$, number of groups in a sample $n = 30, 50$ and $m = 40\%, 50\%$ and 80% of n and different combinations of progressive censoring schemes $\underset{\sim}{R}$ are generated using algorithm proposed by Balakrishnan and Sandhu (1995) with distribution function $1-(1-F(x))^k$ from IWD. Note that different censoring schemes used in this article have been represented by short notations such as $(0*2)$ denotes $(0, 0)$ and $((2, 0, 0)*2)$ denotes $(2, 0, 0, 2, 0, 0)$.

The non-informative and informative gamma priors are denoted by $Prior0$ and $Prior1$, respectively. Hyper-parameters are taken so that the means of prior distributions are exactly equal to the true values of the parameters, in case of proper gamma priors. We take $a_i = b_i = 0; i = 1, 2, 3, 4$ for non-informative priors. We consider two sets of true values of the parameters $\alpha = 3, \beta = 2, \lambda = 1, \theta = 2$ so that $\delta = 0.2762$ with corresponding informative hyper-parameters $a_1 = 6, b_1 = 2, a_2 = 4, b_2 = 2, a_3 = 2, b_3 = 2, a_4 = 4, b_4 = 2$, and $\alpha = 2, \beta = 3, \lambda = 2, \theta = 1$ so that $\delta = 0.7238$ with corresponding informative hyper-parameters $a_1 = 4, b_1 = 2, a_2 = 6, b_2 = 2, a_3 = 4, b_3 = 2, a_4 = 2, b_4 = 2$. For GELF we take three different choices of $q = 1, -1, -2$. The MCMC technique is used for Bayesian computational procedure. We obtain average length of 95% HPD credible interval for stress-strength reliability δ. Here, we generate $M = 10,000$ MCMC samples and take $M_0 = 2,000$ burn-in-period. All calculations are performed on the statistical software R. The main results of the Monte Carlo simulation study are listed in Tables 3-8, see, Appendix II.

From Tables 3, 4, 6 and 7 we observe that for both ML and Bayes estimation methods, as the number of groups n and the effective sample size m increase, AB and MSEs decrease. It verifies the consistency properties of both of estimation methods. The AB and MSE increase with an increase in group size k. Also, as the true value of stress-strength reliability δ increases AB increases while MSE decreases for all combinations of k, n and m. Similar pattern is observed for Bayes estimator of δ. Bayes estimate is better than MLE even for non-informative priors and the best for informative priors in terms of AB and MSE in all cases.

According to Tables 5 and 8 average lengths of asymptotic confidence and HPD credible intervals narrow down as failure proportion m/n increases for fixed k. HPD credible intervals with informative gamma prior has smaller average length and higher coverage probability than asymptotic confidence interval or HPD credible interval with non-informative prior. Also, as the group size k increases average lengths of confidence interval as well as HPD credible intervals also increase. The coverage probability for ML and Bayes estimates of δ obtained here, rarely attain their prescribed confidence coefficient.

5. Data analysis

This section deals with real life applications of the methodology developed in this article. Xia, Yu, Cheng, Liu, and Wang (2009) studied breaking strength of jute fiber at four different gauge lengths i.e. 5 mm, 10 mm, 15 mm and 20 mm. Saraçoglu, Kinaci, and Kundu (2012) considered two data sets of gauge lengths 10 mm and 20 mm out of these four data sets for estimation of stress-strength reliability in exponential distribution under progressive type II censoring. Here, we are considering data of breaking strength of jute fiber of gauge lengths 10 mm and 20 mm. For the convenience of readers the data are presented in Tables 1 and 2, respectively.

The following notations have been used: X is the breaking strength of jute fiber with 10 mm, and Y is the breaking strength of jute fiber with 20 mm. Now, we check that whether the IWD can be used or not to analyse these data sets. We obtain the ML estimates of the shape and scale parameters, Kolmogorov-Smirnov (K-S) distances (D) between the empirical and the fitted distribution functions with associated p-values. The ML estimates of α, β, λ and θ are 1.1833, 491.6463, 1.0845 and 228.5011, respectively. The Kolmogorov-Smirnov distance D values are 0.1701 and 0.1567 and the associated p values are 0.3140 and 0.4108, respectively. Based on the p values, we accept the hypothesis that the data are coming from IW distributions.

For illustrative purposes, we consider number of rows and number of columns of Tables 1 and 2 as group sizes and number of groups, respectively. Note that the bold observation in each column represents the smallest breaking strength (first failure) in a group. Thus, two first failure censored samples from Tables 1 and 2, respectively with $k_1 = k_2 = 3$ and $n_1 = n_2 = 10$ are given below:

X: 43.93, 50.16, 101.15, 108.94, 123.06, 151.48, 183.16, 257.44, 262.90, 303.90.

Y: 36.75, 45.58, 48.01, 71.46, 99.72, 116.99, 145.96, 166.49, 187.85, 284.64.

Next, we generate two different progressively first failure censored samples of sizes $m_1 = m_2 = 7$, respectively, from the above first failure censored data. The progressive censoring schemes and the corresponding generated data are obtained as follows:

$$\underset{\sim}{R}_x = (1, 0, 0, 1, 0, 0, 1), \quad \underset{\sim}{x} = (43.93, 101.15, 108.94, 123.06, 183.16, 257.44, 262.90)$$

$$\underset{\sim}{R}_y = (1, 0, 0, 1, 0, 0, 1), \quad \underset{\sim}{y} = (36.75, 48.01, 71.46, 99.72, 145.96, 166.49, 187.85).$$

The MLE and the 95% asymptotic confidence interval of δ result in 0.5797 and $(0.3706, 0.7888)$, respectively.

To compute the Bayes estimates, since we do not have any prior information, we use non-informative priors taking hyper-parameters $a_i = b_i = 0; i = 1, 2, 3, 4$. We run the Gibbs sampler within M-H algorithm to generate a Markov chains with $1,00,000$ observations from the conditional posterior distributions in (16), (17), (18) and (19). We check the convergence of the generated MCMC samples using diagnostic plots, such as trace plots, posterior density plots with histogram and autocorrelation function plots, see, Appendix III. We plot the Markov chains, posterior densities with Gaussian kernel and autocorrelation functions for all parameters without thinning and burn-in-period in Figures 2, 3 and 4, respectively. For reducing autocorrelation among the generated chains of the parameters, we take every 10^{th} observation as iid observation and discarded first $2,000$ observations as burn-in and again plots trace plots, histograms with Gaussian kernel and ACF plots in Figures 5, 6 and 7, respectively. Initially, a strong autocorrelation is observed among the generated chains. However, after thining and taking burn-in-period, the chains show fine mixing, histograms become more symmetric and autocorrelation is minimized.

The Bayes estimates of δ with $q = -1, 1, 2$ become 0.4985, 0.5356 and 0.5513, respectively. Also, the 95% HPD credible interval of δ results in $(0.2887, 0.7848)$.

Table 1: Breaking strength of jute fiber of gauge length 10 mm.

693.73	704.66	323.83	778.17	**123.06**	637.66	383.43	**151.48**	**108.94**	**50.16**
671.49	**183.16**	**257.44**	727.23	291.27	**101.15**	376.42	163.4	141.38	700.74
262.9	353.24	422.11	**43.93**	590.48	212.13	**303.9**	506.6	530.55	177.25

Table 2: Breaking strength of jute fiber of gauge length 20 mm.

71.46	419.02	**284.64**	585.57	456.6	113.85	**187.85**	688.16	662.66	**45.58**
578.62	756.7	594.29	**166.49**	**99.72**	707.36	765.14	187.13	**145.96**	350.7
547.44	**116.99**	375.81	581.6	119.86	**48.01**	200.16	**36.75**	244.53	83.55

6. Discussion and concluding remarks

In this article, we address the problem of estimation of stress-strength reliability $\delta = P(Y < X)$ for the inverse Weibull distributions using progressively first failure censored samples. We consider the case when scale as well as shape parameters of IWD are different and unknown. We first derive the MLE and asymptotic confidence intervals of δ and then, Bayes estimators of δ under GELF using non-informative and informative gamma priors. Because the Bayes estimators are not in explicit form, the MCMC technique is used to compute the Bayes estimate and associated HPD credible interval. The performance of the point and interval estimates of δ is examined by a Monte Carlo simulation study. Simulation results suggest that the Bayes estimation is more precise than the ML estimation and these can be used for all practical purposes. A real data example is also discussed for illustration purposes.

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Appendix I

The partial derivatives in $I(\omega)$ are given by

$$\frac{\partial^2 l}{\partial\alpha^2} = -\frac{m_1}{\alpha^2} - \lambda\sum_{i=1}^{m_1} x_i^{-\alpha}(\ln x_i)^2 - \sum_{i=1}^{m_1}\frac{(k_1(R_{x,i}+1)-1)\,x_i^{-\alpha}(\ln x_i)^2\,\lambda e^{-\lambda x_i^{-\alpha}}\,[e^{-\lambda x_i^{-\alpha}}+\lambda x_i^{-\alpha}-1]}{\left(1-e^{-\lambda x_i^{-\alpha}}\right)^2},$$

$$\frac{\partial^2 l}{\partial\beta^2} = -\frac{m_2}{\beta^2} - \theta\sum_{j=1}^{m_2} y_j^{-\beta}(\ln y_j)^2 - \sum_{j=1}^{m_2}\frac{(k_2(R_{y,j}+1)-1)\,y_j^{-\beta}(\ln y_j)^2\,\theta e^{-\theta y_j^{-\beta}}\,[e^{-\theta y_j^{-\beta}}+\theta y_j^{-\beta}-1]}{\left(1-e^{-\theta y_j^{-\beta}}\right)^2},$$

$$\frac{\partial^2 l}{\partial\lambda^2} = -\frac{m_1}{\lambda^2} - \sum_{i=1}^{m_1}\frac{(k_1(R_{x,i}+1)-1)\,x_i^{-2\alpha}e^{-\lambda x_i^{-\alpha}}}{\left(1-e^{-\lambda x_i^{-\alpha}}\right)^2},$$

$$\frac{\partial^2 l}{\partial\theta^2} = -\frac{m_2}{\theta^2} - \sum_{j=1}^{m_2}\frac{(k_2(R_{y,j}+1)-1)\,y_j^{-2\beta}e^{-\theta y_j^{-\beta}}}{\left(1-e^{-\theta y_j^{-\beta}}\right)^2},$$

$$\frac{\partial^2 l}{\partial\alpha\partial\beta}=\frac{\partial^2 l}{\partial\beta\partial\alpha}=0,$$

$$\frac{\partial^2 l}{\partial\alpha\partial\lambda}=\frac{\partial^2 l}{\partial\lambda\partial\alpha}=\sum_{i=1}^{m_1}x_i^{-\alpha}\ln x_i+\sum_{i=1}^{m_1}\frac{\left(k_1\left(R_{x,i}+1\right)-1\right)x_i^{-\alpha}\left(\ln x_i\right)^2 e^{-\lambda x_i^{-\alpha}}\left[e^{-\lambda x_i^{-\alpha}}+\lambda x_i^{-\alpha}-1\right]}{\left(1-e^{-\lambda x_i^{-\alpha}}\right)^2},$$

$$\frac{\partial^2 l}{\partial\alpha\partial\theta}=\frac{\partial^2 l}{\partial\theta\partial\alpha}=0,$$

$$\frac{\partial^2 l}{\partial\beta\partial\lambda}=\frac{\partial^2 l}{\partial\lambda\partial\beta}=0,$$

$$\frac{\partial^2 l}{\partial\beta\partial\theta}=\frac{\partial^2 l}{\partial\theta\partial\beta}=\sum_{j=1}^{m_2}y_j^{-\beta}\ln y_j-\sum_{j=1}^{m_2}\frac{\left(k_2\left(R_{y,j}+1\right)-1\right)y_j^{-\beta}\left(\ln y_j\right)^2 e^{-\theta y_j^{-\beta}}\left[e^{-\theta y_j^{-\beta}}+\theta y_j^{-\beta}-1\right]}{\left(1-e^{-\theta y_j^{-\beta}}\right)^2},$$

$$\frac{\partial^2 l}{\partial\lambda\partial\theta}=\frac{\partial^2 l}{\partial\theta\partial\lambda}=0,$$

and the elements of b_c are given by

$$\frac{\partial\delta}{\partial\alpha}=\frac{\beta\lambda\theta}{\alpha^2}\int_0^\infty e^{-\lambda t}e^{-\theta t^{\frac{\beta}{\alpha}}}t^{\frac{\beta}{\alpha}}\ln t\,\mathrm{d}t,$$

$$\frac{\partial\delta}{\partial\beta}=-\frac{\lambda\theta}{\alpha}\int_0^\infty e^{-\lambda t}e^{-\theta t^{\frac{\beta}{\alpha}}}t^{\frac{\beta}{\alpha}}\ln t\,\mathrm{d}t,$$

$$\frac{\partial\delta}{\partial\lambda}=\int_0^\infty e^{-\lambda t}e^{-\theta t^{\frac{\beta}{\alpha}}}\left[1-\lambda t\right]\mathrm{d}t,$$

$$\frac{\partial\delta}{\partial\theta}=-\lambda\int_0^\infty e^{-\lambda t}e^{-\theta t^{\frac{\beta}{\alpha}}}t^{\frac{\beta}{\alpha}}\,\mathrm{d}t.$$

Appendix II

Results of the the simulation study are given in following Tables:

Table 3: AB of the ML and Bayes estimates of δ under GELF when $\delta = 0.2762$.

(k,n,m)	Censoring Scheme	$\hat{\delta}_{MLE}$	$\hat{\delta}_{Bayes}$					
			$q=1$		$q=-1$		$q=-2$	
			Prior0	Prior1	Prior0	Prior1	Prior0	Prior1
(3,30,12)	(18,0*11)	0.0724	0.0535	0.0502	0.0567	0.0522	0.0567	0.0531
(3,30,12)	(2*4,0*3,3*6)	0.0686	0.0446	0.0422	0.0487	0.0438	0.0479	0.0443
(3,30,12)	(0*11,18)	0.0696	0.0437	0.0397	0.0432	0.0406	0.0453	0.0401
(3,30,15)	(15,0*14)	0.0659	0.0527	0.0485	0.0562	0.0517	0.0557	0.0520
(3,30,15)	(1*15)	0.0623	0.0463	0.0425	0.0494	0.0443	0.0499	0.0446
(3,30,15)	(0*14,15)	0.0613	0.0456	0.0413	0.0435	0.0410	0.0465	0.0404
(3,30,24)	(6,0*23)	0.0529	0.0482	0.0459	0.0504	0.0474	0.0497	0.0476
(3,30,24)	(1*3,0*18,1*3)	0.0529	0.0470	0.0464	0.0476	0.0444	0.0475	0.0467
(3,30,24)	(0*23,6)	0.0517	0.0475	0.0458	0.0485	0.0466	0.0477	0.0469
(3,50,20)	(30,0*19)	0.0543	0.0478	0.0461	0.0509	0.0490	0.0513	0.0510
(3,50,20)	(2*8,0*5,2*7)	0.0533	0.0344	0.0370	0.0402	0.0370	0.0392	0.0379
(3,50,20)	(0*19,30)	0.0538	0.0346	0.0336	0.0346	0.0340	0.0346	0.0338
(3,50,25)	(25,0*24)	0.0500	0.0472	0.0467	0.0471	0.0482	0.0492	0.0457
(3,50,25)	(1*25)	0.0496	0.0420	0.0399	0.0404	0.0397	0.0406	0.0420
(3,50,25)	(0*24,25)	0.0461	0.0375	0.0374	0.0381	0.0366	0.0395	0.0365
(3,50,40)	(10,0*39)	0.0412	0.0396	0.0381	0.0405	0.0376	0.0395	0.0361
(3,50,40)	(1*5,0*30,1*5)	0.0412	0.0395	0.0371	0.0397	0.0378	0.0366	0.0387
(3,50,40)	(0*39,10)	0.0388	0.0397	0.0393	0.0409	0.0400	0.0410	0.0397
(5,30,12)	(18,0*11)	0.0671	0.0473	0.0445	0.0482	0.0457	0.0493	0.0437
(5,30,12)	(2*4,0*3,3*6)	0.0728	0.0461	0.0420	0.0453	0.0448	0.0473	0.0431
(5,30,12)	(0*11,18)	0.0781	0.0452	0.0393	0.0448	0.0440	0.0462	0.0455
(5,30,15)	(15,0*14)	0.0738	0.0401	0.0397	0.0403	0.0399	0.0415	0.0376
(5,30, 15)	(1*15)	0.0636	0.0371	0.0358	0.0376	0.0364	0.0384	0.0368
(5,30,15)	(0*14,15)	0.0674	0.0346	0.0356	0.0381	0.0376	0.0391	0.0363
(5,30,24)	(6,0*23)	0.0460	0.0451	0.0415	0.0420	0.0410	0.0424	0.0413
(5,30,24)	(1*3,0*18,1*3)	0.0466	0.0380	0.0408	0.0404	0.0392	0.0406	0.0395
(5,30,24)	(0*23,6)	0.0473	0.0399	0.0377	0.0400	0.0383	0.0401	0.0392
(5,50,20)	(30,0*19)	0.0505	0.0404	0.0413	0.0440	0.0428	0.0447	0.0401
(5,50,20)	(2*8,0*5,2*7)	0.0558	0.0326	0.0329	0.0340	0.0343	0.0347	0.0313
(5,50,20)	(0*19,30)	0.0601	0.0353	0.0365	0.0381	0.0383	0.0390	0.0347
(5,50,25)	(25,0*24)	0.0446	0.0417	0.0412	0.0424	0.0417	0.0429	0.0406
(5,50,25)	(1*25)	0.0476	0.0293	0.0298	0.0323	0.0323	0.0329	0.0300
(5,50,25)	(0*24,25)	0.0506	0.0324	0.0308	0.0310	0.0314	0.0317	0.0306
(5,50,40)	(10,0*39)	0.0372	0.0422	0.0436	0.0441	0.0433	0.0441	0.0427
(5,50,40)	(1*5,0*30,1*5)	0.0349	0.0436	0.0414	0.0420	0.0414	0.0420	0.0416
(5,50,40)	(0*39,10)	0.0368	0.0411	0.0385	0.0399	0.0394	0.0400	0.0400

Table 4: MSE of the ML and Bayes estimates of δ under GELF when $\delta = 0.2762$.

(k,n,m)	Censoring Scheme	$\hat{\delta}_{MLE}$	$\hat{\delta}_{Bayes}$					
			$q=1$		$q=-1$		$q=-2$	
			Prior0	Prior1	Prior0	Prior1	Prior0	Prior1
(3,30,12)	(18,0*11)	0.0081	0.0046	0.0040	0.0051	0.0043	0.0051	0.0044
(3,30,12)	(2*4,0*3,3*6)	0.0072	0.0033	0.0029	0.0038	0.0032	0.0038	0.0031
(3,30,12)	(0*11,18)	0.0078	0.0031	0.0025	0.0031	0.0026	0.0033	0.0027
(3,30,15)	(15,0*14)	0.0067	0.0044	0.0036	0.0047	0.0041	0.0047	0.0041
(3,30,15)	(1*15)	0.0059	0.0035	0.0029	0.0039	0.0031	0.0039	0.0031
(3,30,15)	(0*14,15)	0.0059	0.0034	0.0028	0.0031	0.0028	0.0035	0.0027
(3,30,24)	(6,0*23)	0.0042	0.0034	0.0031	0.0037	0.0034	0.0037	0.0034
(3,30,24)	(1*3,0*18,1*3)	0.0043	0.0034	0.0032	0.0033	0.0030	0.0033	0.0032
(3,30,24)	(0*23,6)	0.0042	0.0033	0.0031	0.0035	0.0032	0.0034	0.0032
(3,50,20)	(30,0*19)	0.0046	0.0034	0.0032	0.0039	0.0036	0.0039	0.0038
(3,50,20)	(2*8,0*5,2*7)	0.0043	0.0023	0.0022	0.0026	0.0022	0.0025	0.0368
(3,50,20)	(0*19,30)	0.0046	0.0020	0.0018	0.0020	0.0019	0.0020	0.0020
(3,50,25)	(25,0*24)	0.0040	0.0033	0.0032	0.0032	0.0034	0.0035	0.0031
(3,50,25)	(1*25)	0.0038	0.0027	0.0025	0.0026	0.0024	0.0026	0.0027
(3,50,25)	(0*24,25)	0.0034	0.0023	0.0023	0.0024	0.0022	0.0026	0.0022
(3,50,40)	(10,0*39)	0.0026	0.0023	0.0021	0.0024	0.0021	0.0023	0.0019
(3,50,40)	(1*5,0*30,1*5)	0.0026	0.0023	0.0020	0.0023	0.0021	0.0020	0.0022
(3,50,40)	(0*39,10)	0.0024	0.0023	0.0022	0.0024	0.0023	0.0024	0.0023
(5,30,12)	(18,0*11)	0.0071	0.0035	0.0032	0.0037	0.0033	0.0038	0.0030
(5,30,12)	(2*4,0*3,3*6)	0.0084	0.0032	0.0026	0.0033	0.0031	0.0035	0.0028
(5,30,12)	(0*11,18)	0.0092	0.0030	0.0023	0.0031	0.0028	0.0032	0.0030
(5,30,15)	(15,0*14)	0.0082	0.0026	0.0024	0.0026	0.0024	0.0028	0.0022
(5,30,15)	(1*15)	0.0065	0.0022	0.0021	0.0024	0.0022	0.0025	0.0021
(5,30,15)	(0*14,15)	0.0072	0.0019	0.0021	0.0023	0.0022	0.0024	0.0021
(5,30,24)	(6,0*23)	0.0033	0.0030	0.0025	0.0027	0.0026	0.0028	0.0026
(5,30,24)	(1*3,0*18,1*3)	0.0034	0.0023	0.0026	0.0025	0.0023	0.0025	0.0024
(5,30,24)	(0*23,6)	0.0035	0.0026	0.0023	0.0025	0.0023	0.0025	0.0025
(5,50,20)	(30,0*19)	0.0041	0.0026	0.0026	0.0029	0.0027	0.0030	0.0025
(5,50,20)	(2*8,0*5,2*7)	0.0049	0.0017	0.0017	0.0019	0.0019	0.0020	0.0016
(5,50,20)	(0*19,30)	0.0057	0.0019	0.0020	0.0022	0.0022	0.0023	0.0018
(5,50,25)	(25,0*24)	0.0032	0.0027	0.0025	0.0027	0.0026	0.0027	0.0025
(5,50,25)	(1*25)	0.0035	0.0014	0.0014	0.0017	0.0016	0.0017	0.0015
(5,50,25)	(0*24,25)	0.0040	0.0016	0.0015	0.0015	0.0015	0.0016	0.0014
(5,50,40)	(10,0*39)	0.0021	0.0024	0.0025	0.0026	0.0025	0.0026	0.0025
(5,50,40)	(1*5,0*30,1*5)	0.0019	0.0026	0.0024	0.0025	0.0024	0.0025	0.0024
(5,50,40)	(0*39,10)	0.0021	0.0025	0.0022	0.0024	0.0023	0.0023	0.0023

Table 5: AL and CP of 95% asymptotic confidence and HPD credible intervals of δ when $\delta = 0.2762$.

(k,n,m)	Censoring Scheme	$\hat{\delta}_{MLE}$		$\hat{\delta}_{Bayes}$			
				Prior0		Prior1	
		AL	CP	AL	CP	AL	CP
(3,30,12)	(18,0*11)	0.3765	0.923	0.2529	0.924	0.2385	0.927
(3,30,12)	(2*4,0*3,3*6)	0.3719	0.927	0.2278	0.930	0.2180	0.945
(3,30,12)	(0*11,18)	0.3693	0.929	0.2155	0.953	0.2087	0.935
(3,30,15)	(15,0*14)	0.3409	0.926	0.2322	0.929	0.2237	0.927
(3,30,15)	(1*15)	0.3355	0.925	0.2152	0.926	0.2071	0.931
(3,30,15)	(0*14,15)	0.3320	0.927	0.2058	0.910	0.1995	0.937
(3,30,24)	(6,0*23)	0.2734	0.929	0.1928	0.929	0.1875	0.931
(3,30,24)	(1*3,0*18,1*3)	0.2824	0.956	0.1837	0.943	0.1765	0.945
(3,30,24)	(0*23,6)	0.2843	0.932	0.1832	0.920	0.1762	0.923
(3,50,20)	(30,0*19)	0.2925	0.938	0.1981	0.919	0.1908	0.931
(3,50,20)	(2*8,0*5,2*7)	0.2911	0.942	0.1691	0.919	0.1717	0.928
(3,50,20)	(0*19,30)	0.2914	0.946	0.1649	0.945	0.1606	0.940
(3,50,25)	(25,0*24)	0.2644	0.931	0.1808	0.920	0.1759	0.945
(3,50,25)	(1*25)	0.2624	0.935	0.1636	0.922	0.1597	0.935
(3,50,25)	(0*24,25)	0.2607	0.960	0.1548	0.928	0.1522	0.940
(3,50,40)	(10,0*39)	0.2121	0.941	0.1408	0.921	0.1371	0.942
(3,50,40)	(1*5,0*30,1*5)	0.2181	0.948	0.1343	0.935	0.1312	0.931
(3,50,40)	(0*39,10)	0.2214	0.956	0.1311	0.940	0.1279	0.951
(5,30,12)	(18,0*11)	0.3328	0.911	0.2255	0.927	0.2152	0.928
(5,30,12)	(2*4,0*3,3*6)	0.3547	0.902	0.2028	0.940	0.2010	0.925
(5,30,12)	(0*11,18)	0.3771	0.906	0.1996	0.956	0.1904	0.966
(5,30,15)	(15,0*14)	0.3529	0.904	0.2026	0.956	0.1937	0.949
(5,30,15)	(1*15)	0.3170	0.913	0.1895	0.953	0.1824	0.938
(5,30,15)	(0*14,15)	0.3318	0.909	0.1829	0.955	0.1759	0.946
(5,30,24)	(6,0*23)	0.2542	0.963	0.1718	0.930	0.1678	0.924
(5,30,24)	(1*3,0*18,1*3)	0.2536	0.951	0.1656	0.932	0.1615	0.932
(5,30,24)	(0*23,6)	0.2561	0.955	0.1616	0.934	0.1573	0.930
(5,50,20)	(30,0*19)	0.2649	0.945	0.1788	0.929	0.1746	0.935
(5,50,20)	(2*8,0*5,2*7)	0.2788	0.927	0.1541	0.939	0.1500	0.942
(5,50,20)	(0*19,30)	0.2976	0.932	0.1510	0.923	0.1464	0.922
(5,50,25)	(25,0*24)	0.2413	0.955	0.1635	0.924	0.1594	0.930
(5,50,25)	(1*25)	0.2490	0.941	0.1439	0.925	0.1408	0.938
(5,50,25)	(0*24,25)	0.2620	0.940	0.1373	0.939	0.1340	0.925
(5,50,40)	(10,0*39)	0.1971	0.959	0.1269	0.939	0.1239	0.935
(5,50,40)	(1*5,0*30,1*5)	0.1982	0.970	0.1209	0.935	0.1207	0.939
(5,50,40)	(0*39,10)	0.2022	0.964	0.1189	0.942	0.1181	0.946

Table 6: AB of the ML and Bayes estimates of δ under GELF when $\delta = 0.7238$.

(k,n,m)	Censoring Scheme	$\hat{\delta}_{MLE}$	$\hat{\delta}_{Bayes}$					
			$q=1$		$q=-1$		$q=-2$	
			Prior0	Prior1	Prior0	Prior1	Prior0	Prior1
(3,30,12)	(18,0*11)	0.0676	0.0536	0.0487	0.0523	0.0475	0.0528	0.0482
(3,30,12)	(2*4,0*3,3*6)	0.0683	0.0538	0.0466	0.0541	0.0455	0.0515	0.0468
(3,30,12)	(0*11,18)	0.0752	0.0034	0.0482	0.0568	0.0460	0.0553	0.0488
(3,30,15)	(15,0*14)	0.0587	0.0503	0.0463	0.0508	0.0447	0.0509	0.0453
(3,30,15)	(1*15)	0.0604	0.0477	0.0447	0.0491	0.0443	0.0483	0.0441
(3,30,15)	(0*14,15)	0.0597	0.0508	0.0473	0.0530	0.0456	0.0527	0.0465
(3,30,24)	(6,0*23)	0.0487	0.0418	0.0421	0.0423	0.0393	0.0412	0.0407
(3,30,24)	(1*3,0*18,1*3)	0.0478	0.0419	0.0412	0.0429	0.0378	0.0438	0.0396
(3,30,24)	(0*23,6)	0.0456	0.0401	0.0405	0.0426	0.0401	0.0438	0.0392
(3,50,20)	(30,0*19)	0.0509	0.0422	0.0392	0.0411	0.0391	0.0411	0.0384
(3,50,20)	(2*8,0*5,2*7)	0.0527	0.0406	0.0384	0.0421	0.0369	0.0435	0.0390
(3,50,20)	(0*19,30)	0.0566	0.0442	0.0414	0.0452	0.0379	0.0432	0.0401
(3,50,25)	(25,0*24)	0.0439	0.0413	0.0368	0.0390	0.0362	0.0364	0.0358
(3,50,25)	(1*25)	0.0455	0.0379	0.0355	0.0389	0.0365	0.0381	0.0368
(3,50,25)	(0*24,25)	0.0461	0.0427	0.0376	0.0431	0.0397	0.0425	0.0384
(3,50,40)	(10,0*39)	0.0358	0.0338	0.0324	0.0337	0.0324	0.0327	0.0325
(3,50,40)	(1*5,0*30,1*5)	0.0366	0.0315	0.0318	0.0305	0.0320	0.0323	0.0315
(3,50,40)	(0*39,10)	0.0365	0.0320	0.0325	0.0314	0.0324	0.0323	0.0300
(5,30,12)	(18,0*11)	0.0639	0.0479	0.0432	0.0464	0.0410	0.0466	0.0423
(5,30,12)	(2*4,0*3,3*6)	0.0724	0.0465	0.0426	0.0463	0.0418	0.0452	0.0405
(5,30,12)	(0*11,18)	0.0738	0.0480	0.0417	0.0469	0.0401	0.0484	0.0418
(5,30,15)	(15,0*14)	0.0588	0.0435	0.0408	0.0443	0.0405	0.0449	0.0427
(5,30,15)	(1*15)	0.0668	0.0459	0.0383	0.0461	0.0391	0.0458	0.0427
(5,30,15)	(0*14,15)	0.0674	0.0455	0.0391	0.0470	0.0404	0.0446	0.0482
(5,30,24)	(6,0*23)	0.0468	0.0387	0.0350	0.0382	0.0357	0.0371	0.0354
(5,30,24)	(1*3,0*18,1*3)	0.0489	0.0394	0.0365	0.0410	0.0365	0.0405	0.0378
(5,30,24)	(0*23,6)	0.0481	0.0404	0.0370	0.0414	0.0378	0.0418	0.0380
(5,50,20)	(30,0*19)	0.0507	0.0382	0.0371	0.0384	0.0353	0.0379	0.0337
(5,50,20)	(2*8,0*5,2*7)	0.0552	0.0354	0.0337	0.0357	0.0330	0.0364	0.0319
(5,50,20)	(0*19,30)	0.0585	0.0372	0.0327	0.0360	0.0345	0.0359	0.0346
(5,50,25)	(25,0*24)	0.0451	0.0361	0.0330	0.0351	0.0343	0.0343	0.0327
(5,50,25)	(1*25)	0.0477	0.0336	0.0307	0.0334	0.0318	0.0325	0.0315
(5,50,25)	(0*24,25)	0.0500	0.0341	0.0318	0.0337	0.0317	0.0340	0.0332
(5,50,40)	(10,0*39)	0.0363	0.0293	0.0297	0.0301	0.0288	0.0301	0.0305
(5,50,40)	(1*5,0*30,1*5)	0.0369	0.0310	0.0291	0.0311	0.0293	0.0305	0.0291
(5,50,40)	(0*39,10)	0.0357	0.0322	0.0295	0.0299	0.0291	0.0313	0.0296

Table 7: MSE of the ML and Bayes estimates of δ under GELF when $\delta = 0.7238$.

(k,n,m)	Censoring Scheme	$\hat{\delta}_{MLE}$	$\hat{\delta}_{Bayes}$					
			$q=1$		$q=-1$		$q=-2$	
			Prior0	Prior1	Prior0	Prior1	Prior0	Prior1
(3,30,12)	(18,0*11)	0.0072	0.0477	0.0038	0.0043	0.0036	0.0044	0.0036
(3,30,12)	(2*4,0*3,3*6)	0.0072	0.0044	0.0033	0.0044	0.0032	0.0041	0.0034
(3,30,12)	(0*11,18)	0.0089	0.0045	0.0036	0.0048	0.0033	0.0045	0.0036
(3,30,15)	(15,0*14)	0.0055	0.0040	0.0034	0.0041	0.0031	0.0040	0.0032
(3,30,15)	(1*15)	0.0057	0.0035	0.0031	0.0036	0.0030	0.0036	0.0030
(3,30,15)	(0*14,15)	0.0057	0.0039	0.0034	0.0043	0.0032	0.0042	0.0033
(3,30,24)	(6,0*23)	0.0036	0.0028	0.0028	0.0029	0.0024	0.0027	0.0026
(3,30,24)	(1*3,0*18,1*3)	0.0035	0.0028	0.0026	0.0028	0.0023	0.0029	0.0025
(3,30,24)	(0*23,6)	0.0032	0.0025	0.0025	0.0028	0.0026	0.0029	0.0024
(3,50,20)	(30,0*19)	0.0041	0.0028	0.0025	0.0027	0.0024	0.0027	0.0023
(3,50,20)	(2*8,0*5,2*7)	0.0044	0.0026	0.0023	0.0027	0.0021	0.0027	0.0023
(3,50,20)	(0*19,30)	0.0052	0.0030	0.0026	0.0031	0.0023	0.0028	0.0024
(3,50,25)	(25,0*24)	0.0030	0.0026	0.0021	0.0024	0.0021	0.0021	0.0020
(3,50,25)	(1*25)	0.0032	0.0022	0.0019	0.0023	0.0021	0.0022	0.0020
(3,50,25)	(0*24,25)	0.0033	0.0028	0.0021	0.0027	0.0023	0.0027	0.0022
(3,50,40)	(10,0*39)	0.0020	0.0018	0.0017	0.0018	0.0016	0.0017	0.0017
(3,50,40)	(1*5,0*30,1*5)	0.0022	0.0017	0.0016	0.0016	0.0016	0.0017	0.0016
(3,50,40)	(0*39,10)	0.0022	0.0016	0.0015	0.0017	0.0015	0.0016	0.0014
(5,30,12)	(18,0*11)	0.0063	0.0036	0.0030	0.0033	0.0026	0.0035	0.0028
(5,30,12)	(2*4,0*3,3*6)	0.0080	0.0034	0.0028	0.0033	0.0027	0.0032	0.0025
(5,30,12)	(0*11,18)	0.0087	0.0034	0.0027	0.0034	0.0026	0.0036	0.0027
(5,30,15)	(15,0*14)	0.0054	0.0029	0.0026	0.0031	0.0026	0.0032	0.0030
(5,30,15)	(1*15)	0.0070	0.0032	0.0024	0.0033	0.0024	0.0032	0.0029
(5,30,15)	(0*14,15)	0.0070	0.0032	0.0023	0.0033	0.0025	0.0031	0.0035
(5,30,24)	(6,0*23)	0.0033	0.0022	0.0019	0.0022	0.0020	0.0022	0.0020
(5,30,24)	(1*3,0*18,1*3)	0.0037	0.0023	0.0021	0.0025	0.0021	0.0025	0.0022
(5,30,24)	(0*23,6)	0.0036	0.0025	0.0021	0.0026	0.0021	0.0026	0.0022
(5,50,20)	(30,0*19)	0.0042	0.0022	0.0023	0.0023	0.0019	0.0022	0.0018
(5,50,20)	(2*8,0*5,2*7)	0.0047	0.0019	0.0018	0.0020	0.0017	0.0020	0.0016
(5,50,20)	(0*19,30)	0.0056	0.0021	0.0017	0.0020	0.0018	0.0020	0.0018
(5,50,25)	(25,0*24)	0.0032	0.0021	0.0017	0.0019	0.0018	0.0018	0.0017
(5,50,25)	(1*25)	0.0035	0.0017	0.0015	0.0017	0.0016	0.0016	0.0016
(5,50,25)	(0*24,25)	0.0041	0.0018	0.0016	0.0018	0.0016	0.0018	0.0017
(5,50,40)	(10,0*39)	0.0020	0.0013	0.0014	0.0014	0.0013	0.0014	0.0014
(5,50,40)	(1*5,0*30,1*5)	0.0021	0.0015	0.0013	0.0015	0.0013	0.0014	0.0013
(5,50,40)	(0*39,10)	0.002	0.0016	0.0014	0.0014	0.0013	0.0015	0.0013

Table 8: AL and CP of 95% asymptotic confidence and HPD credible intervals of δ when $\delta = 0.7238$.

(k,n,m)	Censoring Scheme	$\hat{\delta}_{MLE}$		$\hat{\delta}_{Bayes}$			
				Prior0		Prior1	
		AL	CP	AL	CP	AL	CP
(3,30,12)	(18,0*11)	0.2883	0.921	0.2613	0.945	0.2506	0.951
(3,30,12)	(2*4,0*3,3*6)	0.2665	0.927	0.2303	0.927	0.2232	0.933
(3,30,12)	(0*11,18)	0.2697	0.929	0.2161	0.922	0.2145	0.919
(3,30,15)	(15,0*14)	0.2605	0.932	0.2483	0.949	0.2403	0.951
(3,30,15)	(1*15)	0.2423	0.922	0.2239	0.937	0.2190	0.936
(3,30,15)	(0*14,15)	0.2406	0.926	0.2166	0.930	0.2115	0.938
(3,30,24)	(6,0*23)	0.2111	0.936	0.2101	0.947	0.2098	0.958
(3,30,24)	(1*3,0*18,1*3)	0.1988	0.924	0.1981	0.945	0.1945	0.951
(3,30,24)	(0*23,6)	0.1985	0.926	0.1954	0.960	0.1935	0.958
(3,50,20)	(30,0*19)	0.2256	0.932	0.2097	0.946	0.2033	0.962
(3,50,20)	(2*8,0*5,2*7)	0.2117	0.922	0.1782	0.941	0.1744	0.921
(3,50,20)	(0*19,30)	0.2135	0.938	0.1668	0.923	0.1652	0.936
(3,50,25)	(25,0*24)	0.2039	0.939	0.1964	0.937	0.1916	0.933
(3,50,25)	(1*25)	0.1917	0.926	0.1720	0.926	0.1696	0.931
(3,50,25)	(0*24,25)	0.1875	0.937	0.1616	0.925	0.1596	0.928
(3,50,40)	(10,0*39)	0.1636	0.947	0.1625	0.955	0.1614	0.952
(3,50,40)	(1*5,0*30,1*5)	0.1531	0.940	0.1511	0.939	0.1502	0.952
(3,50,40)	(0*39,10)	0.1554	0.942	0.1532	0.945	0.1521	0.944
(5,30,12)	(18,0*11)	0.2752	0.924	0.2257	0.933	0.2191	0.930
(5,30,12)	(2*4,0*3,3*6)	0.2775	0.922	0.2026	0.928	0.1949	0.932
(5,30,12)	(0*11,18)	0.2872	0.930	0.1996	0.926	0.1922	0.940
(5,30,15)	(15,0*14)	0.2444	0.928	0.2105	0.946	0.2038	0.942
(5,30,15)	(1*15)	0.2463	0.923	0.1874	0.922	0.1847	0.938
(5,30,15)	(0*14,15)	0.2501	0.924	0.1828	0.926	0.1792	0.920
(5,30,24)	(6,0*23)	0.1958	0.922	0.1823	0.937	0.1791	0.949
(5,30,24)	(1*3,0*18,1*3)	0.1930	0.931	0.1682	0.929	0.1654	0.925
(5,30,24)	(0*23,6)	0.1915	0.939	0.1668	0.925	0.1651	0.937
(5,50,20)	(30,0*19)	0.2130	0.919	0.1776	0.931	0.1745	0.935
(5,50,20)	(2*8,0*5,2*7)	0.2177	0.922	0.1526	0.937	0.1499	0.929
(5,50,20)	(0*19,30)	0.2292	0.932	0.1513	0.933	0.1475	0.927
(5,50,25)	(25,0*24)	0.1912	0.935	0.1650	0.927	0.1607	0.932
(5,50,25)	(1*25)	0.1924	0.932	0.1436	0.922	0.1408	0.938
(5,50,25)	(0*24,25)	0.1978	0.931	0.1403	0.934	0.1377	0.939
(5,50,40)	(10,0*39)	0.1536	0.948	0.1390	0.937	0.1372	0.945
(5,50,40)	(1*5,0*30,1*5)	0.1509	0.939	0.1317	0.942	0.1290	0.942
(5,50,40)	(0*39,10)	0.1505	0.944	0.1265	0.951	0.1255	0.949

Appendix III

Convergence plots of generated MCMC samples

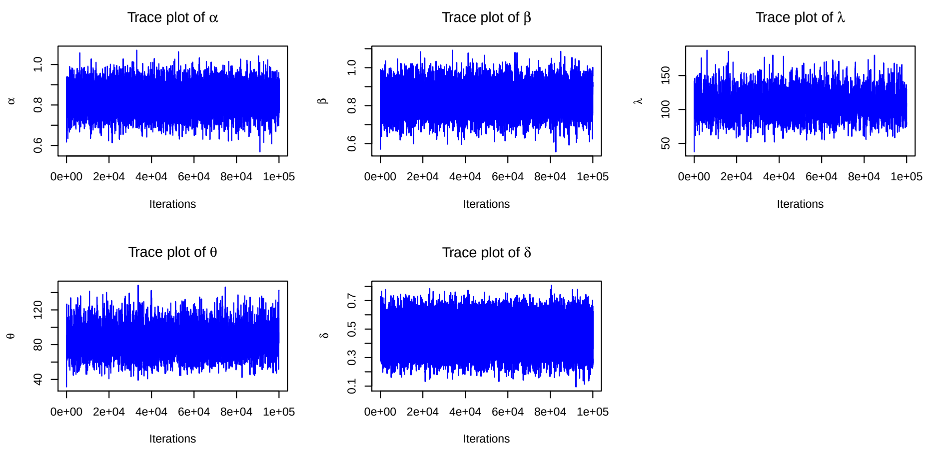

Figure 2: Trace plots of MCMC samples of the parameters without thinning and burn-in-period.

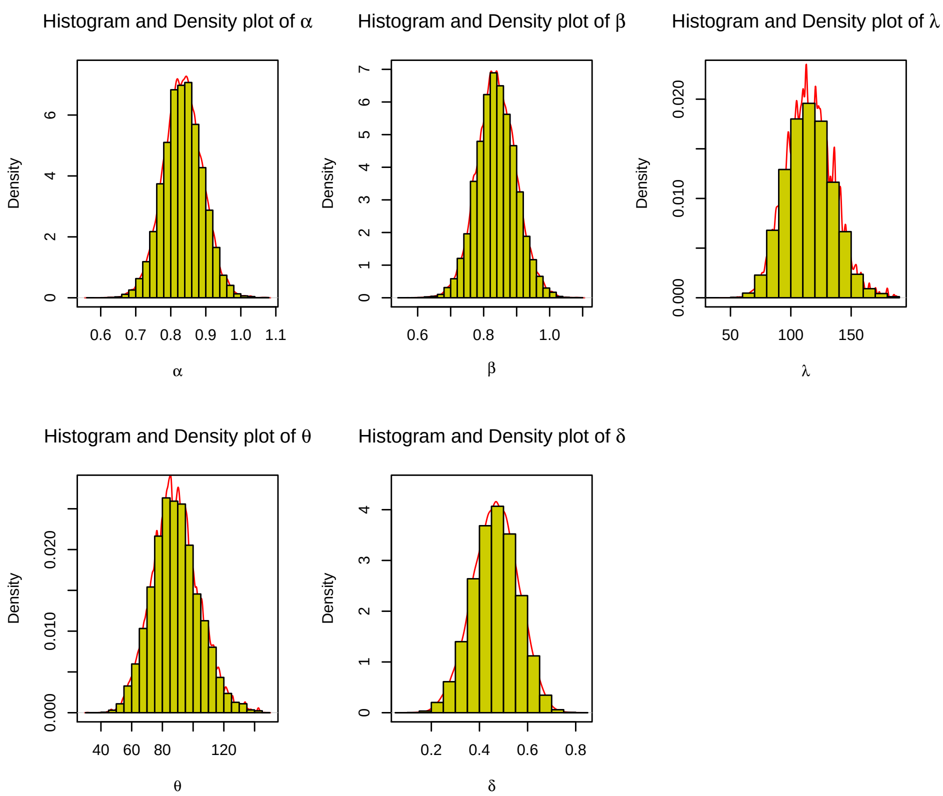

Figure 3: Histograms with Gaussian kernel density plots of MCMC samples of the parameters without thinning and burn-in-period.

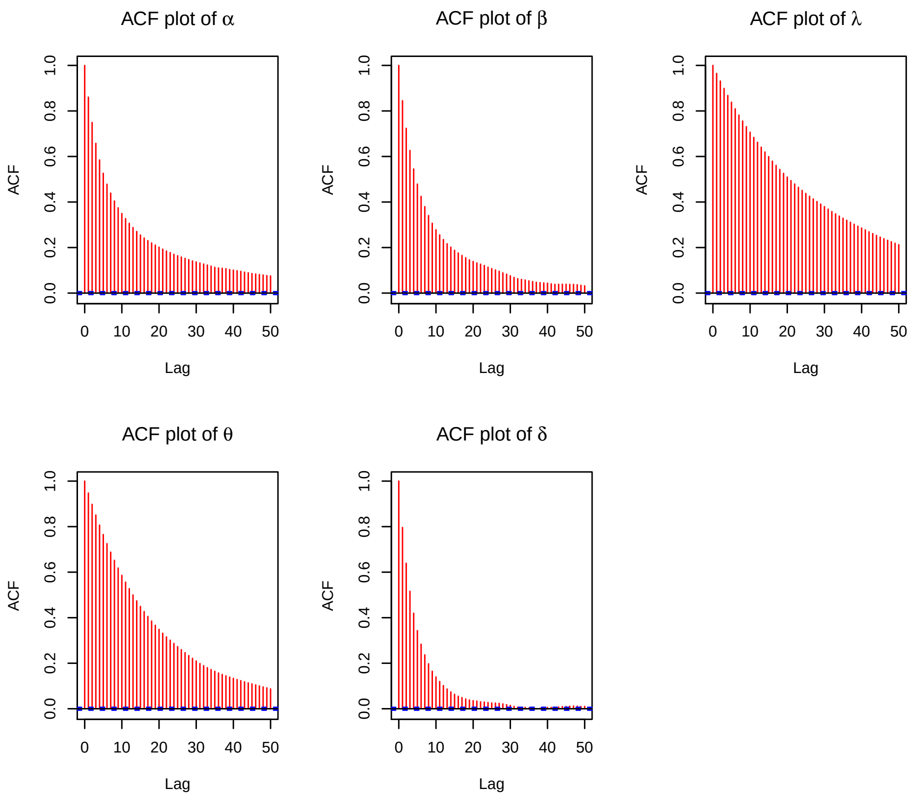

Figure 4: ACF plots of MCMC samples of the parameters without thinning and burn-in-period.

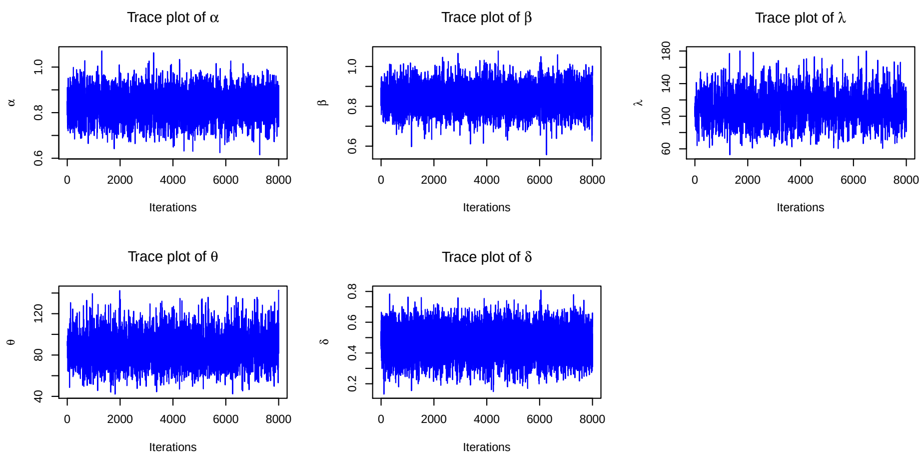

Figure 5: Trace plots of MCMC samples of the parameters with thinning= 10 and burn-in-period= 2, 000.

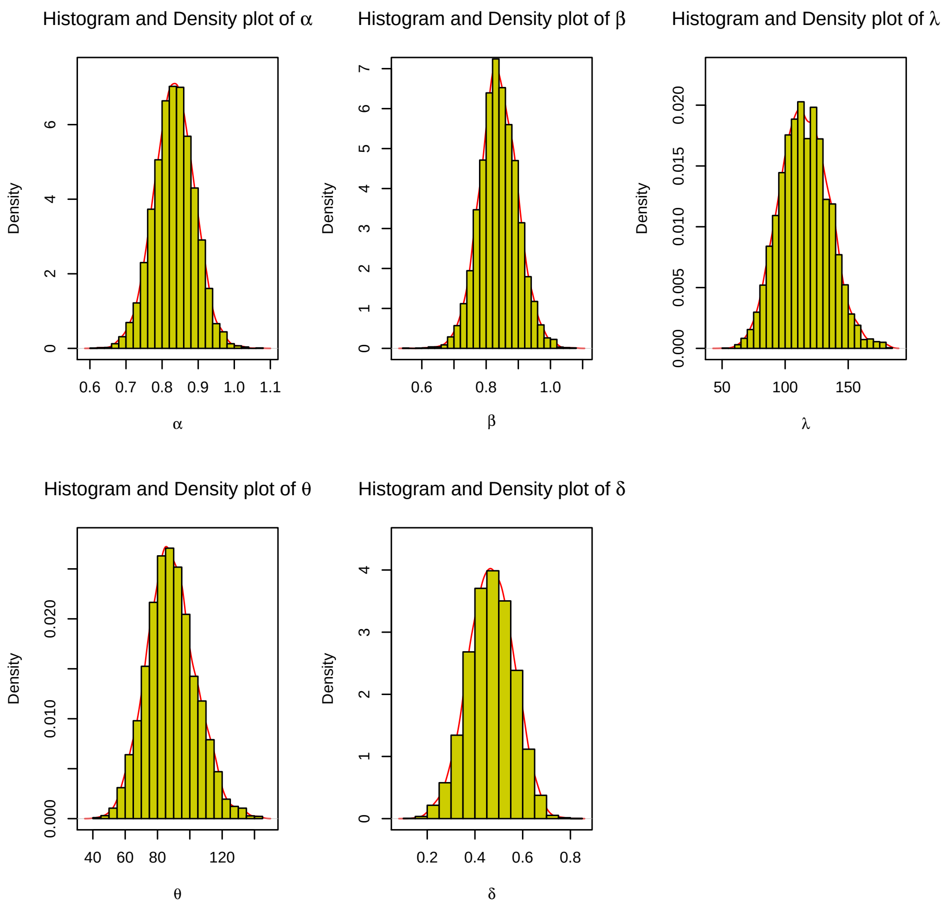

Figure 6: Histograms with Gaussian kernel density plots of MCMC samples of the parameters with thinning= 10 and burn-in-period= 2, 000.

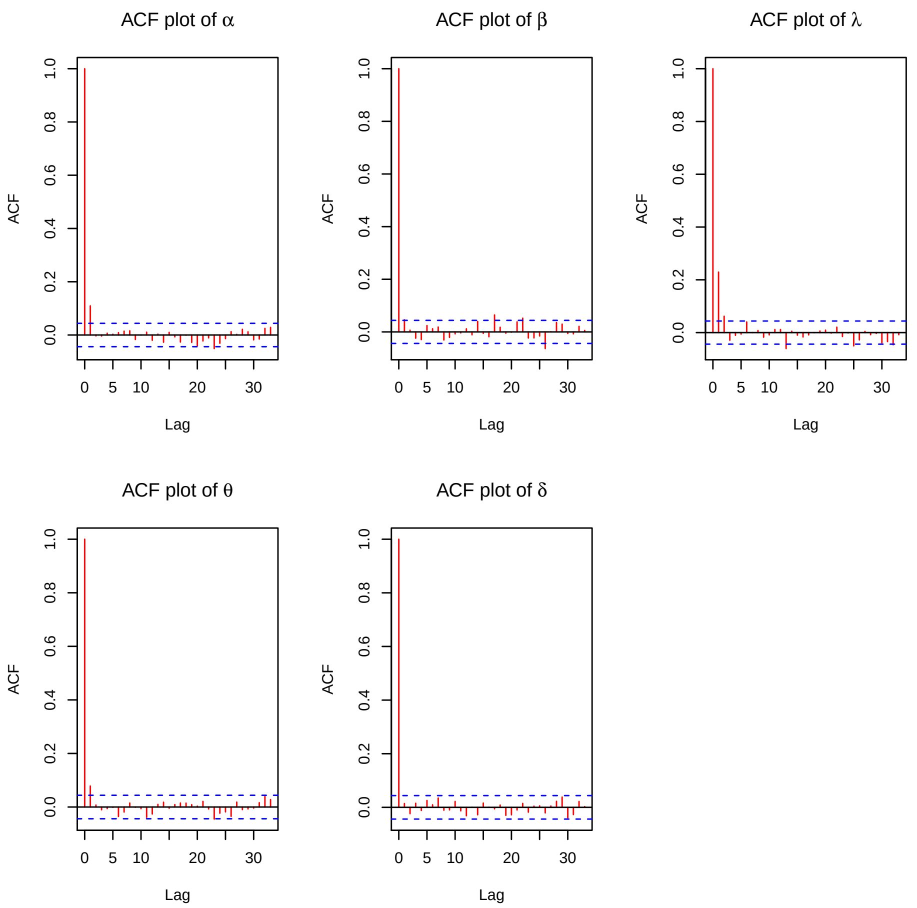

Figure 7: ACF plots of MCMC samples of the parameters with thinning= 10 and burn-in-period= $2,000$.

Affiliation:

Hare Krishna
Department of Statistics
Ch. Charan Singh University
Meerut-250004, India
E-mail: hkrishnastats@yahoo.com

Madhulika Dube
Department of Statistics
Maharshi Dayanand University
Rohtak-124001, India
E-mail: madhulikadube@gmail.com

Renu Garg
Department of Statistics
Maharshi Dayanand University
Rohtak-124001, India
E-mail: renustats@gmail.com

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