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Alpha Logarithm Transformed Fréchet Distribution: Properties and Estimation

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Abstract

In this paper, a new three-parameter distribution called the Alpha Logarithm Transformed Fréchet (ALTF) distribution is introduced which offers a more flexible distribution for modeling lifetime data. Various properties of the proposed distribution, including explicit expressions for the quantiles, moments, incomplete moments, conditional moments, moment generating function Rényi and δ-entropies, stochastic ordering, stress-strength reliability and order statistics are derived. The new distribution can have decreasing, reversed J-shaped and upside-down bathtub failure rate functions depending on its parameter values. The maximum likelihood method is used to estimate the distribution parameters. A simulation study is conducted to evaluate the performance of the maximum likelihood estimates. Finally, the proposed extended model is applied on real data sets and the results are given which illustrate the superior performance of the ALTF distribution compared to some other well-known distributions.

Keywords: logarithm transformed distribution, hazard rate function, maximum likelihood estimation, asymptotic variance-covariance matrix.

1. Introduction

The Fréchet distribution occupies an important place in describing the lifetime of components and analyzing several extreme events including earthquakes, floods, rain fall, queues in supermarkets, wind speeds and sea waves etc. The Fréchet distribution is a special case of the generalized extreme value distribution and is equivalent to taking the reciprocal of values from a standard Weibull distribution. Fréchet distribution provides a reasonable parametric fit for modeling phenomenon with non-monotone failure rates, such as the upside-down bathtub failure rates, which are common in reliability and biological studies. For example, such failure rates can be observed in the course of a disease whose mortality reaches a peak after some finite period and then declines gradually. The lifetime models that present upside-down bathtub shaped failure rates are very useful in survival analysis. For greater details, readers may refer to (Kotz and Nadarajah 2000). Moreover, (Zaharim, Najid, Razali, and Sopian 2009) applied the Fréchet model for analyzing wind speed data. (Mubarak 2011) studied the Fréchet distribution based on progressive type-II censored data with binomial removals.

Many generalizations of Fréchet distribution have been attempted by researchers. Notable among them are : (Nadarajah and Kotz 2003) pioneered the exponentiated Fréchet, (Nadarajah and Gupta 2004) and (Barreto-Souza, Cordeiro, and Simas 2011) studied the beta Fréchet, (Mahmoud and Mandouh 2013) proposed the transmuted Fréchet, (Krishna, Jose, Alice, and Ristic 2013) introduced the Marshall-Olkin Fréchet, (Silva, de Andrade, Maciel, Campos, and Cordeiro 2013) defined the gamma extended Fréchet, (Elbatal, Asha, and Raja 2014) studied the transmuted exponentiated Fréchet, (Mead and R. 2014) introduced the Kumaraswamy Fréchet and (Afify, Hamedani, Ghosh, and Mead 2015) investigated the transmuted Marshall-Olkin Fréchet distributions.

Let X follow a Fréchet random distribution with parameters $\beta,\ \lambda > 0$. Recall that the probability density function (pdf) and cumulative density function (cdf) associated to X are respectively given by

$$g(x;\beta,\lambda) = \beta\lambda^{\beta}x^{-\beta-1}\exp\left[-\left(\frac{\lambda}{x}\right)^{\beta}\right],\quad x>0,\ \beta,\lambda>0$$

and

$$G(x;\beta,\lambda) = \exp\left[-\left(\frac{\lambda}{x}\right)^{\beta}\right], \tag{1}$$

where λ and β are the scale and shape parameters, respectively.

If $G(x)$ is an absolute continuous cdf with corresponding pdf $g(x)$, (Pappas, Adamidis, and Loukas 2012) introduced a new family of distributions with the following cdf and pdf, respectively

$$F(x) = \begin{cases} 1-\frac{\log[\alpha-(\alpha-1)G(x)]}{\log(\alpha)} & \text{if } \alpha>0,\ \alpha\neq 1 \\ G(x) & \text{if } \alpha=1 \end{cases} \tag{2}$$

and

$$f(x) = \begin{cases} \frac{(\alpha-1)g(x)}{\log(\alpha)[\alpha-(\alpha-1)G(x)]} & \text{if } \alpha>0,\ \alpha\neq 1 \\ g(x) & \text{if } \alpha=1. \end{cases} \tag{3}$$

It has to be noted when $\alpha = 2$, the cdf and the pdf in (2) and (3) reduce to the cdf and pdf of the logarithmic transformed method proposed by (Maurya, Kaushik, Singh, Singh, and

Singh 2016). (Dey, Nassar, and Kumar 2017) used the method of (Pappas *et al.* 2012) to introduce a new generalization of the generalized exponential distribution and referred to as Alpha Logarithmic Transformed (ALT) generalized exponential distribution.

The main aim of this paper is to propose and study a new life time model, called the ALTF distribution as an extension of the Fréchet distribution using a similar idea as (Pappas *et al.* 2012). The new distribution is very flexible in the sense that it can be skewed depending upon the specific choices of the parameters. Furthermore, the associated cdf is available in closed form. The major motivation of introducing the ALTF distribution can be summarized as follows. (i) it contains several lifetime distributions as special cases, such as the one parameter Fréchet (inverse Weibull), two parameter Fréchet, inverse exponential, inverse Rayleigh distributions; (ii) The ALTF distribution exhibits monotone as well as non-monotone hazard rates but does not exhibit a constant hazard rate, which makes this distribution superior to other lifetime distributions, which exhibit only monotonically increasing/decreasing, or constant hazard rates. (iii) It is shown in Section 2 that the ALTF distribution can be viewed as a mixture of Fréchet distribution; (iv) it can be viewed as a suitable model for fitting skewed data which may not be properly fitted by other common distributions and can also be used in a variety of problems in different areas such as public health, biomedical studies and industrial reliability and survival analysis; and (v) The ALTF distribution outperforms several well-known lifetime distributions with respect to two real data examples.

The contents of the rest of this paper are organized as follows. In Section 2, we introduce the ALTF distribution, and discuss some properties of this distribution in Section 3. In Section 4, maximum likelihood estimators of the unknown parameters along with asymptotic confidence intervals are obtained. In Section 5, a simulation study is carried out based on small, moderate and large sample sizes to study the behavior of the proposed estimators along with confidence intervals. In Section 6, the usefulness of the ALTF distribution is illustrated by means of two real data sets. Finally, Section 7 offers some concluding remarks.

2. Model description

Combining (1) and (3), the cdf of the ALTF distribution can be written as

$$F(x;\alpha,\beta,\lambda) = 1 - \frac{\log[\alpha-(\alpha-1)e^{-\left(\frac{\lambda}{x}\right)^{\beta}}]}{\log(\alpha)}, \quad x>0, \quad \alpha>0, \alpha\neq 1, \ \beta,\lambda>0. \tag{4}$$

The corresponding pdf and hazard rate function are, respectively, given by

$$f(x;\alpha,\beta,\lambda) = \frac{(\alpha-1)\lambda^{\beta}\beta x^{-(\beta+1)}e^{-\left(\frac{\lambda}{x}\right)^{\beta}}}{\log(\alpha)\left[\alpha-(\alpha-1)e^{-\left(\frac{\lambda}{x}\right)^{\beta}}\right]}, \quad x>0, \quad \alpha>0, \alpha\neq 1, \ \beta,\lambda>0 \tag{5}$$

and

$$h(x;\alpha,\beta,\lambda)=\frac{(\alpha-1)\lambda^{\beta}\beta x^{-(\beta+1)}e^{-\left(\frac{\lambda}{x}\right)^{\beta}}}{\log\left[\alpha-(\alpha-1)e^{-\left(\frac{\lambda}{x}\right)^{\beta}}\right]\left[\alpha-(\alpha-1)e^{-\left(\frac{\lambda}{x}\right)^{\beta}}\right]} \tag{6}$$

where α and β are the shape parameters and λ is the scale parameter. Hereafter, a random variable X that has the pdf given in (5) is denoted by $X \sim ALTF(\alpha,\beta,\lambda)$. Figures 1a and 1b show the curves for pdf and hazard rate function of the ALTF distribution with various values of α and β with $\lambda = 1$. Clearly, the pdf of the ALTF distribution is decreasing or uni-modal and positively skewed while the hazard function is decreasing, upside down bathtub and reversed J-shaped failure rate shapes.

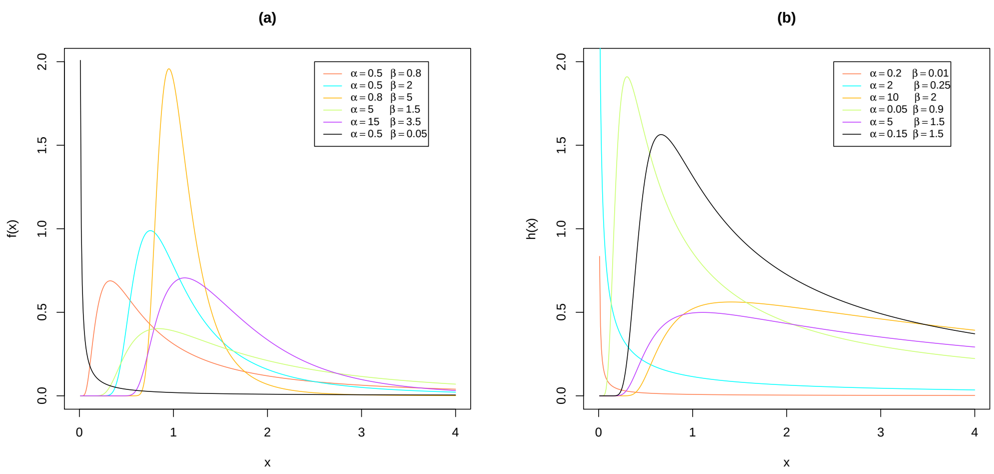

Figure 1: (a) Density and (b) hazard rate functions of ALTF distribution with $\lambda = 1$ and different values of α.

Remark 1. Using Maclaurin series and binomial expansion, the pdf of the ALTF distribution in (5) can be written in the following mixture representation

$$f(x)=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\omega_{k,j}\,g_{Fr}(x;\beta,\lambda(j+1)^{1/\beta}), \tag{7}$$

where

$$\omega_{k,j}=\frac{(\alpha-1)^{j+1}}{(\alpha+1)^{k+1}\log(\alpha)}\left(\begin{array}{c}k\\ j\end{array}\right),$$

and $g_{Fr}(x;\beta,\lambda(j+1)^{1/\beta})$ is the pdf of the Fréchet distribution with scale parameter $\lambda(j+1)^{1/\beta}$, and shape parameter β. Remark 1 is very useful to derive the various properties of the ALTF distribution directly from the Fréchet distribution. For more details about the ALT family of distributions, see (Dey *et al.* 2017).

Special cases: Let $X \sim ALTF(\alpha, \beta, \lambda)$.

1. If $\alpha \to 1$ then X reduces to the two parameter Fréchet (inverse Weibull) distribution.
2. If $\alpha \to 1$ and $\lambda = 1$, then X reduces to the one parameter Fréchet distribution.
3. If $\alpha \to 1$ and $\beta = 1$, then X reduces to the inverse exponential distribution.
4. If $\alpha \to 1$ and $\beta = 2$ then X reduces to inverse Rayleigh distribution.

3. Mathematical properties

In this section, we give some important mathematical properties of the ALTF distribution such as quantiles, moments, incomplete moments, moment generating function, Rényi and δ-entropies, stochastic ordering, stress-strength reliability and order statistics. Established algebraic expressions to determine some structural properties of the ALTF distribution can be more efficient than computing them directly by numerical integration of its density function.

3.1. Quantiles and random numbers generation

Quantiles are fundamental for estimation (for example, quantile estimators) and simulation.
The p^{th} quantile x_p of the ALTF distribution is the root of the equation

$$x_p = Q(p) = \frac{\lambda}{\left[-\log\left(\frac{\alpha-\alpha^{1-p}}{\alpha-1}\right)\right]^{1/\beta}}. \tag{8}$$

Let $U \sim uniform(0,1)$, then equation (8) can be used to simulate a random sample of size n from the ALTF distribution as follows

$$x_i = \frac{\lambda}{\left[-\log\left(\frac{\alpha-\alpha^{1-u_i}}{\alpha-1}\right)\right]^{1/\beta}}, i = 1, 2, ..., n.$$

3.2. Moments

The r^{th} ordinary moments of the ALTF distribution is given by

$$\begin{aligned} \mu_r' &= E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \omega_{k,j} \int_0^{\infty} x^r g_{Fr}(x; \beta,\ \lambda(j+1)^{1/\beta})dx \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \omega_{k,j}\ \lambda^r (j+1)^{r/\beta}\Gamma\left(1-\frac{r}{\beta}\right), \quad r < \beta. \end{aligned} \tag{9}$$

In particular,

$$\mu_1' = E(X) = \mu = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\omega_{k,j}\ \lambda(j+1)^{1/\beta}\Gamma\left(1-\frac{1}{\beta}\right)$$

and

$$\mu_2' = E(X^2) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\omega_{k,j}\ \lambda^2(j+1)^{2/\beta}\Gamma\left(1-\frac{2}{\beta}\right).$$

The quantile function in (8) and the r^{th} raw moments in (9) of the ALTF distribution are used to obtain the values of median, mean, variance, skewness and kurtosis for some selected values of α and β with scale parameter $\lambda = 1$. These values are displayed in Table 1. Table 1 indicates that, for fixed λ and α as β increases the median, mean, variance, skewness and kurtosis decreases. Also, for fixed λ and β, the median, mean and variance increases when α increases, while the skewness and kurtosis are decreasing functions of β.

Table 1: Median, mean, variance, skewness and kurtosis of the ALTF model for $\lambda = 1$ and various values of α and β.

α	β	Median	Mean	Variance	Skewness	Kurtosis
0.5	5	1.0256	1.1083	0.1120	3.7030	51.9404
	7.5	1.0170	1.0624	0.0386	2.4304	16.8982
	10	1.0127	1.0435	0.0195	2.0378	11.9526
	15	1.0085	1.0269	0.0079	1.7839	8.9137
2.5	5	1.1533	1.2513	0.1749	3.3020	42.8450
	7.5	1.0998	1.1497	0.0560	2.1087	13.8210
	10	1.0739	1.1063	0.0273	1.7361	9.7381
	15	1.0487	1.0672	0.0106	1.4653	7.2835
5	5	1.2202	1.3281	0.2192	3.1265	38.9772
	7.5	1.1419	1.1948	0.0677	1.9700	12.5374
	10	1.1046	1.1383	0.0325	1.6069	8.8292
	15	1.0686	1.0874	0.0124	1.3356	6.6177
7.5	5	1.2629	1.3779	0.2522	3.0274	36.8227
	7.5	1.1684	1.2235	0.0761	1.8921	11.8297
	10	1.1238	1.1583	0.0361	1.5345	8.3198
	15	1.0809	1.0999	0.0137	1.2638	6.2535
15	5	1.3421	1.4717	0.3242	2.8677	33.3976
	7.5	1.2167	1.2763	0.0940	1.7675	10.7150
	10	1.1585	1.1950	0.0437	1.4188	7.5408
	15	1.1030	1.1225	0.0163	1.1377	5.6843

3.3. Incomplete moments

The nth incomplete moment of ALTF distribution is given by $m_n(t) = E\left[X^n|x<t\right] = \int_0^t x^n f(x)dx$. We can write from equation (7)

$$\begin{aligned} m_n(t) &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\omega_{k,j} \int_0^t x^n \; g_{Fr}(x;\beta,\; \lambda(j+1)^{1/\beta})dx \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\omega_{k,j} \; \lambda^n (j+1)^{n/\beta} \; \gamma\left(1-\frac{n}{\beta},\; (j+1)\left(\frac{\lambda}{t}\right)^{\beta}\right), \quad n<\beta. \end{aligned}$$

The important application of the first incomplete moment is related to Bonferroni and Lorenz curves defined by $L(p) = m_1(x_p)/\mu_1'$ and $B(p) = m_1(x_p)/p\mu_1'$, respectively, where x_p can be evaluated numerically from equation (8) for a given probability. These curves are very useful in economics, demography, insurance, engineering and medicine.

3.4. Conditional moments

For the ALTF distribution, it can be easily seen that the conditional moments, $E(X^n|X > x)$, can be written as

$$E(X^n|X>x) = \frac{1}{S(x)} J_n(x),$$

where

$$\begin{aligned} J_n(x) &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\omega_{k,j} \int_x^{\infty} x^n \; g_{Fr}(y;\beta,\; \lambda(j+1)^{1/\beta})dy \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\omega_{k,j} \; \lambda^n (j+1)^{n/\beta} \; \Gamma\left(1-\frac{n}{\beta},\; (j+1)\left(\frac{\lambda}{x}\right)^{\beta}\right), \quad n<\beta, \end{aligned} \tag{10}$$

where $\Gamma(a,\; x)$ denote the upper incomplete gamma function defined by $\Gamma(a,\; x) = \int_x^{\infty} t^{a-1}e^{-t}dt$ and $S(x) = 1 - F(x)$.

An application of the conditional moments is the mean residual life (MRL). Thus, in life testing experiments, the expected additional lifetime given that a component has survived until time x is called the MRL. The MRL function is given by

$$m_X(x) = E(X - x|X > x) = \frac{1}{S(x)}\, J_1(x) - x,$$

where $J_1(x)$ can be obtained from (10) with $n = 1$.

Let μ and M denote the mean and the median of the ALTF distribution, respectively. Thus the mean deviations about the mean and the median can be calculated as

$$\delta_\mu = \int_0^{\infty} |x-\mu|\, f(x)dx = 2\mu F(\mu) - 2\mu + 2J_1(\mu)$$

and

$$\delta_M = \int_0^\infty |x - M|\, f(x)dx = 2J_1(M) - \mu,$$

respectively, where $J_1(\mu)$ and $J_1(M)$ can be obtained from (10). Also, $F(\mu)$ can be easily obtained from (4).

3.5. Moment generating function

First, we obtain the moment generating function of Fréchet distribution by setting $y = x^{-1}$ as follows

$$\begin{aligned} M(t) &= E(e^{tx}) = \int_{-\infty}^{\infty} e^{ty} f(x)dx \\ &= \beta\lambda^\beta \int_0^\infty y^{\beta-1}\, e^{\frac{t}{y}}\, e^{-(\lambda y)^\beta} dy \\ &= \beta\lambda^\beta \sum_{p=0}^\infty \frac{\lambda^p t^p}{p!} \int_0^\infty y^{\beta-p-1}\, e^{-(\lambda y)^\beta} dy \\ &= \sum_{p=0}^\infty \frac{\lambda^p t^p}{p!} \Gamma\left(\frac{\beta - p}{\beta}\right). \end{aligned}$$

Recall the Wright generalized hyper-geometric function defined by

$${}_p\psi_q \left[\begin{matrix} (\beta_1, A_1), \ldots, (\beta_p, A_p) \\ (\lambda_1, B_1), \ldots, (\lambda_q, B_q) \end{matrix} ; z \right] = \sum_{p=0}^\infty \frac{\prod_{j=1}^p \Gamma(\beta_j + A_j p)}{\prod_{j=1}^q \Gamma(\lambda_j + B_j p)} \frac{z^p}{p!}.$$

Therefore the moment generating function of Fréchet distribution is

$$M(t) = {}_1\psi_0 \left[\begin{matrix} (1, & -\beta^{-1}) \\ & - \end{matrix} ; \lambda t \right]. \tag{11}$$

From (7) and (11), the moment generating function of ALTF distribution is given by

$$M(t) = \sum_{k=0}^\infty \sum_{j=0}^\infty \omega_{k,j}\ {}_1\psi_0 \left[\begin{matrix} (1, & -\beta^{-1}) \\ & - \end{matrix} ; \lambda (j+1)^{1/\beta} t \right].$$

3.6. Rényi entropy and $\delta-$entropies

Entropy is used to measure the randomness of systems and it is widely used in areas like physics, molecular imaging of tumors and sparse kernel density estimation. If X has the probability distribution function $f(\cdot)$, Rényi entropy can be defined as

$$H_\delta(x) = \frac{1}{1-\delta} \log\left(\int_0^\infty f^\delta(x)dx\right), \qquad \delta > 0, \qquad \delta \neq 1.$$

Using equation (5), we have

$$f^{\delta}(x) = \left(\frac{(\alpha-1)\beta\lambda^{\beta}}{\log\alpha}\right)^{\delta} \; x^{-\delta(\beta+1)} \; \frac{\exp\left\{-\delta\left(\frac{\lambda}{x}\right)^{\beta}\right\}}{\left[\alpha-(\alpha-1)\exp\left\{-\left(\frac{\lambda}{x}\right)^{\beta}\right\}\right]^{\delta}}.$$

After some algebra, we can write

$$f^{\delta}(x) = \sum_{j=0}^{\infty} \omega_j \; x^{-\delta(\beta+1)} \exp\left[-(\delta+j)\left(\frac{\lambda}{x}\right)^{\beta}\right],$$

where

$$\omega_j = \left(\frac{(\alpha-1)\beta}{\alpha\log\alpha}\right)^{\delta} \frac{\lambda^{\delta\beta} \; \delta^{(j)}}{j!},$$

and $a^{(j)} = \Gamma(a+j)/\Gamma(a)$ is the rising factorial defined for any real a.

Then, the Rényi entropy of X reduces to

$$H_{\delta}(x) \;\; = \;\; \frac{1}{1-\delta}\log\left(\sum_{j=0}^{\infty}\omega_j \; \int_0^{\infty} x^{-\delta(\beta+1)}\exp\left\{-(\delta+j)\left(\frac{\lambda}{x}\right)^{\beta}\right\}dx\right).$$

Finally, it can be expressed as

$$H_{\delta}(x) \;\; = \;\; \frac{1}{1-\delta}\log\left[\Gamma\left(\frac{\delta(\beta+1)}{\beta}\right)\sum_{j=0}^{\infty}\upsilon_j\right], \tag{12}$$

where

$$\upsilon_j = \left(\frac{\alpha-1}{\alpha\log\alpha}\right)^{\delta} \frac{\delta^{(j)}}{j!}\left(\frac{\beta}{\lambda}\right)^{\delta-1} \; (\delta+j)^{((1-\delta(\beta+1))/\beta)}.$$

The $\delta-$entropy, say $I_{\delta}(x)$, is defined by

$$I_{\delta}(x) \;\; = \;\; \frac{1}{\delta-1}\log\left[1-\int_0^{\infty} f^{\delta}(x)dx\right], \qquad \delta>0, \qquad \delta\neq 1,$$

and then it follows from equation (12).

3.7. Stochastic ordering

Ordering of distributions, particularly among lifetime distributions play an important role in the statistical literature. (Johnson, Kotz, and Balakrishnan 1995) have a major section on ordering of different lifetime distributions. Here, we consider four different stochastic orders, namely, the usual, the hazard rate, the mean residual life, and the likelihood ratio order for two independent ALTF random variables under a restricted parameter space. It may be recalled that if a family has a likelihood ratio ordering, it has the monotone likelihood ratio property. This implies there exists a uniformly most powerful test for any one sided hypothesis when the other parameters are known. If X and Y are independent random variables with CDFs F_X and F_Y respectively, then X is said to be smaller than Y in the

- stochastic order ($X \leq_{st} (Y)$) if $F_X(x) \geq F_Y(x)$ for all x
- hazard rate order ($X \leq_{hr} (Y)$) if $h_X(x) \geq h_Y(x)$ for all x
- mean residual life order ($X \leq_{mrl} (Y)$) if $m_X(x) \geq m_Y(x)$ for all x
- likelihood ratio order ($X \leq_{lr} (Y)$) if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

The following results due to (Shaked and G. 1994) are well known for establishing stochastic ordering of distributions.

$$\begin{array}{ccc} X \leq_{lr} Y \Rightarrow X & \leq_{hr} & Y \Rightarrow X \leq_{mlr} Y \\ & \Downarrow & \\ & X \leq_{st} Y & \end{array}$$

The ALTF distribution is ordered with respect to the strongest 'likelihood ratio' ordering as shown in the following theorem. It shows the flexibility of three parameter ALTF distribution.

Theorem 1: Let $X \sim ALTF(\alpha_1, \beta_1, \lambda_1)$ and $Y \sim ALTF(\alpha_2, \beta_2, \lambda_2)$. If $\beta_1 = \beta_2 = \beta$ and $\lambda_1 \geq \lambda_2$, then $X \leq_{lr} Y$, $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

Proof. The likelihood ratio is

$$\frac{f_X(x)}{f_Y(x)} = \frac{(\alpha_1 - 1)\beta_1\lambda_1^{\beta_1}\ e^{-(\lambda_1/x)^{\beta_1}} \log\alpha_2[\alpha_2 - (\alpha_2 - 1)e^{-(\lambda_2/x)^{\beta_2}}]}{(\alpha_2 - 1)\beta_2\lambda_2^{\beta_2}\ e^{-(\lambda_2/x)^{\beta_2}} \log\alpha_1[\alpha_1 - (\alpha_1 - 1)e^{-(\lambda_1/x)^{\beta_1}}]}$$

thus,

$$\begin{aligned} \frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} &= -\frac{(\beta_1+1)}{x} + \frac{\beta_1\lambda_1^{\beta_1}}{x^{\beta_1+1}} + \frac{(\alpha_2-1)\beta_2\lambda_2^{\beta_2}e^{-(\lambda_2/x)^{\beta_2}}}{x^{\beta_2+1}[\alpha_2-(\alpha_2-1)e^{-(\lambda_2/x)^{\beta_2}}]} \\ &+ \frac{(\beta_2+1)}{x} - \frac{\beta_2\lambda_2^{\beta_2}}{x^{\beta_2+1}} - \frac{(\alpha_1-1)\beta_1\lambda_1^{\beta_1}e^{-(\lambda_1/x)^{\beta_1}}}{x^{\beta_1+1}[\alpha_1-(\alpha_1-1)e^{-(\lambda_1/x)^{\beta_1}}]}. \end{aligned}$$

Now if $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$ and $\lambda_1 \geq \lambda_2$ then $\frac{d}{dx} \log \frac{f_X(x)}{f_Y(x)} \leq 0$, which implies that $X \leq_{lr} Y$ and hence X is similar than Y also hazard rate order, in mean residual lifetime order and in stochastic order. $\square$

3.8. Stress strength reliability

Let X be the strength of a system which is subjected to a stress Y, and if X follows $ALTF(\alpha_1, \beta_1, \lambda_1)$ and Y follows $ALTF(\alpha_2, \beta_2, \lambda_2)$, provided X and Y are statistically independent random variables, then $R = P(Y < X)$, the measure of system performance (stress-strength reliability measure) is given by

$$
\begin{aligned}
R &= P(Y < X) \\
&= \int_0^\infty f_1(x) F_2(x) dx \\
&= \frac{(\alpha_1 - 1)\beta_1 \lambda_1^{\beta_1}}{\log \alpha_1} \int_0^\infty \frac{x^{-(\beta_1+1)} e^{-(\lambda_1/x)^{\beta_1}}}{[\alpha_1 - (\alpha_1 - 1)e^{-(\lambda_1/x)^{\beta_1}}]} \left[1 - \frac{\log\left\{\alpha_2 - (\alpha_2 - 1)e^{-(\lambda_2/x)^{\beta_2}}\right\}}{\log \alpha_2} \right] dx \\
&= 1 - \frac{(\alpha_1 - 1)}{\log \alpha_1 \log \alpha_2} \sum_{j=1}^\infty \sum_{k=0}^\infty \frac{\alpha_2(\alpha_2 - 1)}{j\alpha_2(j + k + 1)} \left(\frac{\alpha_1 - 1}{\alpha_1}\right)^k,
\end{aligned}
$$

where $f_1(x) \sim X$ and $F_2(x) \sim Y$.

3.9. Order statistics

Let $X_{1:n} \leq \cdots \leq X_{n:n}$ denote the order statistics of a random sample $X_1, \ldots, X_n$ from a continuous population with cdf $F(x)$ and pdf $f(x)$, then the pdf of $X_{j:n}$ is given by

$$
f_{j:n}(x) = \frac{n!}{(j-1)!(n-j)!} f(x) \left(F(x)\right)^{j-1} \left(1 - F(x)\right)^{n-j},
$$

for $j = 1, \ldots, n$. The pdf of the j^{th} order statistic of the ALTF distribution is given by

$$
\begin{aligned}
f_{j:n}(x) &= \frac{n!}{(j-1)!(n-j)!} \sum_{u=0}^{j-1} (-1)^u \binom{j-1}{u} \left[\frac{\log[\alpha - (\alpha - 1)e^{-(\lambda/x)^\beta}]}{\log \alpha} \right]^{n-j+u} \\
&\times \frac{(\alpha - 1)\beta\lambda^\beta x^{-(\beta+1)} e^{-(\lambda/x)^\beta}}{\log(\alpha)[\alpha - (\alpha - 1)e^{-(\lambda/x)^\beta}]}, \\
&= \sum_{k=0}^\infty \omega_k \, g_{Fr}(x; \beta, \ \lambda(n - j + u + v + k + 1)^{1/\beta}), \qquad (13)
\end{aligned}
$$

where

$$\omega_k = \frac{n!}{(j-1)!(n-j)!}\sum_{u=0}^{j-1}\sum_{v=0}^{\infty}\frac{\phi_v(n-j+u)(\alpha-1)^{n-j+u+v+1}}{\alpha(\log\alpha)^{n-j+u+1}(n-j+u+v+k+1)}$$
$$\times \quad (-1)^{n-j+u}\binom{j-1}{u}\left(\frac{\alpha-1}{\alpha}\right)^k$$

and $\phi_p(n-j+u)$ is the coefficient of $(\alpha-1)e^{-(\lambda/x)^\beta}$ in the expansion of $\left(\sum_{p=1}^{\infty}\frac{(\alpha-1)e^{-(\lambda/x)^\beta}}{p}\right)^{n-j+u}$ (see, Balakrishnan and Cohen (1991)), $g_{Fr}(x;\beta,\ \lambda(n-j+u+v+k+1)^{1/\beta})$ denotes the Fréchet density function with parameters $\lambda(n-j+u+v+k+1)^{1/\beta}$ and β. Thus, the density function of the ALTF order statistics is a linear mixture of the Fréchet densities. Based on equation (13), we can obtain some structural properties of $X_{j:n}$ from those Fréchet properties. The corresponding cdf is

$$\begin{aligned}
F_{j:n}(x) &= \sum_{l=j}^{n}\binom{n}{l}[F(x)]^l[1-F(x)]^{n-l}\\
&= \sum_{l=j}^{n}\sum_{u=0}^{l}(-1)^u\binom{n}{l}\binom{l}{u}[1-F(x)]^{n-l+u}\\
&= \sum_{l=j}^{n}\sum_{u=0}^{l}(-1)^u\binom{n}{l}\binom{l}{u}\left[\frac{\log[\alpha-(\alpha-1)e^{-(\lambda/x)^\beta}]}{\log\alpha}\right]^{n-l+u}.
\end{aligned}$$

The qth moments of $X_{j:n}$ for $(q<\alpha)$ is given by

$$E[X_{j:n}^q] = \sum_{k=0}^{\infty}\omega_k E[Y^q_{\lambda(n-j+u+v+k+1)}], \tag{14}$$

where $Y^q_{\lambda(n-j+u+v+k+1)} \sim$ Fréchet distribution with scale parameter $\lambda(n-j+u+v+k+1)^{1/\beta}$ and shape parameter β. Other useful measures for lifetime models are the L-moments L_k due to (Hoskings 1990). They can be obtained as a linear function of expected order statistics defined by

$$\lambda_k = \frac{1}{k}\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}E(X_{k-i:k}),\quad k\geq 1.$$

Based on (14), the explicit expressions for L-moments of X can be obtained as infinite weighted linear combinations of suitable ALTF means.

4. Maximum likelihood estimation

Parameter estimation is of utmost importance for any probability distribution. Among all the estimation methods, the most frequently used method is the maximum likelihood (see Lehmann and Casella (1998); Pawitan (2001); Rohde (2014)) method. Its underlying motivation is simple and intuitive. For example, they are asymptotically unbiased, consistent, and asymptotically normally distributed. In this section, we describe the maximum likelihood estimation method for estimating the parameters, α , β and λ of the ALTF distribution.

4.1. Maximum likelihood estimation

We assume throughout the paper that $\underline{x} = (x_1, x_2, \ldots, x_n)$ is a random sample of size n from the ALTF distribution with unknown parameters α, β and λ, the log-likelihood function $[\ell = \ell(\alpha, \beta, \lambda; \underline{x})]$ of the density (5) without constant term is given by

$$\ell = n \log \frac{(\alpha-1)\lambda^{\beta}\beta}{\log(\alpha)} - \sum_{i=1}^{n}\left(\frac{\lambda}{x_i}\right)^{\beta} - (\beta+1)\sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}\log\left(\alpha - (\alpha-1)e^{-\left(\frac{\lambda}{x_i}\right)^{\beta}}\right). \tag{15}$$

To obtain the normal equations for the unknown parameters, we differentiate (15) partially with respect to α, β and λ and equate to zero. The resulting equations are

$$0 = \frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha-1} - \frac{n}{\alpha \log \alpha} - \sum_{i=1}^{n} \frac{1-e^{-y_i^{\beta}}}{\psi_i},$$

$$0 = \frac{\partial \ell}{\partial \beta} = n\log\lambda + \frac{n}{\beta} - \sum_{i=1}^{n} y_i^{\beta}\log(y_i) - \sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}\frac{(\alpha-1)e^{-y_i^{\beta}}y_i^{\beta}\log y_i}{\psi_i}$$

and

$$0 = \frac{\partial \ell}{\partial \lambda} = \frac{n\beta}{\lambda} - \frac{\beta}{\lambda}\sum_{i=1}^{n} y_i^{\beta} - \frac{\beta}{\lambda}\sum_{i=1}^{n}\frac{(\alpha-1)e^{-y_i^{\beta}}y_i^{\beta}}{\psi_i},$$

where

$$y_i = \left(\frac{\lambda}{x_i}\right) \quad \text{and} \quad \psi_i = \alpha - (\alpha-1)e^{-\left(\frac{\lambda}{x_i}\right)^{\beta}}.$$

The MLEs of the parameters α, β and λ are denoted by $\hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$ and are obtained by solving the above nonlinear system of equations. It is usually more convenient to use nonlinear optimization algorithms such as the quasi-Newton algorithm to numerically maximize the sample likelihood function. To check whether a global maximum has been attained, a number of starting values have been used.

4.2. Approximate confidence intervals

Since the MLE of the vector of unknown parameters $\Theta = (\alpha, \beta, \lambda)$ cannot be derived in closed forms, it is not easy to derive the exact distributions of the MLEs. Hence, we cannot obtain exact confidence intervals for the parameters. The idea is to use large sample approximation. It is known that the asymptotic distribution of the MLE

Θ is $[\sqrt{n}(\hat{\alpha}_{MLE}-\alpha), \sqrt{n}(\hat{\beta}_{MLE}-\beta), \sqrt{n}(\hat{\lambda}_{MLE}-\lambda)] \to N_3(0, I^{-1}(\alpha,\beta,\lambda))$, see (Lawless 1982), where $I^{-1}(\alpha,\beta,\lambda)$ is the inverse of the observed information matrix of the unknown parameters $\Theta = (\alpha,\beta,\lambda)$ and obtained as

$$\begin{aligned} I^{-1}(\hat{\Theta}) &= \begin{pmatrix} -I_{11} & -I_{12} & -I_{13} \\ -I_{21} & -I_{22} & -I_{23} \\ -I_{31} & -I_{32} & -I_{33} \end{pmatrix}^{-1} \Bigg|_{(\alpha,\beta,\lambda)=(\hat{\alpha},\hat{\beta},\hat{\lambda})} \\ &= \begin{pmatrix} var(\hat{\alpha}) & cov(\hat{\alpha},\hat{\beta}) & cov(\hat{\alpha},\hat{\lambda}) \\ cov(\hat{\beta},\hat{\alpha}) & var(\hat{\beta}) & cov(\hat{\beta},\hat{\lambda}) \\ cov(\hat{\lambda},\hat{\alpha}) & cov(\hat{\lambda},\hat{\beta}) & var(\hat{\lambda}) \end{pmatrix}, \end{aligned}$$

where

$$I_{11} = \frac{\partial^2 \ell}{\partial \alpha^2} = -\frac{n}{(\alpha-1)^2} + \frac{n[1+\log(\alpha)]}{(\alpha\log(\alpha))^2} + \sum_{i=1}^{n} \psi_i^{-2}\left(1-e^{-y_i^{\beta}}\right)^2,$$

$$I_{22} = \frac{\partial^2 \ell}{\partial \beta^2} = -\frac{n}{\beta^2} - \sum_{i=1}^{n} y_i^{\beta}\log^2(y_i) - \sum_{i=1}^{n}(\alpha-1)\psi_i^{-2}e^{-y_i^{\beta}}y_i^{\beta}\log^2(y_i)\left[\alpha-\alpha y_i^{\beta}-(\alpha-1)e^{-y_i^{\beta}}\right],$$

$$I_{33} = \frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{n\beta}{\lambda^2} - \frac{\beta(\beta-1)}{\lambda^2}\sum_{i=1}^{n} y_i^{\beta} - \frac{\beta}{\lambda^2}\sum_{i=1}^{n}(\alpha-1)\psi_i^{-2}e^{-y_i^{\beta}}y_i^{\beta}[(\beta-1)\psi_i-\beta\alpha y_i^{\beta}],$$

$$I_{12} = I_{21} = \frac{\partial^2 \ell}{\partial \alpha \partial \beta} = -\sum_{i=1}^{n}\psi_i^{-2}e^{-y_i^{\beta}}y_i^{\beta}\log(y_i),$$

$$I_{13} = I_{31} = \frac{\partial^2 \ell}{\partial \beta \partial \lambda} = -\frac{\beta}{\lambda}\sum_{i=1}^{n}\psi_i^{-2}e^{-y_i^{\beta}}y_i^{\beta}$$

and

$$I_{23} = I_{32} = \frac{n}{\lambda} - \frac{1}{\lambda}\sum_{i=1}^{n} y_i^{\beta}\eta_i - \frac{\alpha-1}{\lambda}\sum_{i=1}^{n}\psi_i^{-2}e^{-y_i^{\beta}}y_i^{\beta}[\psi_i\eta_i-\beta\alpha y_i^{\beta}\log(y_i)],$$

where

$$\eta_i = [1+\beta\log(y_i)].$$

The above approach is used to derive approximate $100(1-\tau)\%$ confidence intervals of the parameters $\Theta = (\alpha,\beta,\lambda)$ as in the following forms

$$\hat{\alpha} \pm z_{\tau/2}\sqrt{var(\hat{\alpha})},$$

$$\hat{\beta} \pm z_{\tau/2}\sqrt{var(\hat{\beta})}$$

and

$$\hat{\lambda} \pm z_{\tau/2}\sqrt{var(\hat{\lambda})},$$

where $z_{\tau/2}$ is the upper $(\tau/2)$th percentile of the standard normal distribution.

5. Simulation study

In this section, a simulation study has been conducted using Mathcad program version 2007 to illustrate the behavior of the MLEs in terms of the sample size n. We consider the values 1 and 2 for the parameter λ, 0.5, 2.5 for parameter β and 0.5, 2.5 and 5 for parameter α. For a total of 12 parameter combinations and 1000 replications for sample sizes 25, 50, 100, 150, 200 and 250, we obtain the average values of the estimates, mean squared errors (MSEs) and the approximate confidence intervals (CIs) lengths and the corresponding coverage probabilities. The simulation results are reported in Tables 2 and 3. Based on the simulation results, it is observed that as the sample size increases the MSEs decreases which show the property of consistency of the estimates. Also, the lengths of the approximate confidence intervals decreases as the sample size increases in all cases. The simulation results also show that the MLEs of the unknown parameters are quite stable and quite close to the true parameter values, therefore, the maximum likelihood method works well to estimate the unknown parameters of the ALTF distribution.

6. Data analysis

In this section, we provide two applications to show the flexibility of the ALTF distribution. The first real data set represents the survival times in weeks, of patients suffering from acute Myelogeneous Leukaemia. These data sets have also been analyzed by (Feigl and Zelen 1965) and (Mead, Afify, Hamedani, and Ghosh 2017). The data consists of 33 observations and are given by:

65	156	100	134	16	108	121	4	39	143	56	26	22	1	1	5	65	56	65	17
7	16	22	3	4	2	3	8	4	3	30	4	43							

The second data set represents 46 observations reported on active repair times (hours) for an airborne communication transceiver analysed by (Alven 1964), (Chhikara and Folks 1977) and (Dimitrakopoulou, Adamidis, and Loukas 2007). The data are:

0.2	0.3	0.5	0.5	0.5	0.5	0.6	0.6	0.7	0.7	0.7	0.8	0.8	1.0	1.0	1.0	1.0	1.1	1.3	1.5
1.5	1.5	1.5	2.0	2.0	2.2	2.5	2.7	3.0	3.0	3.3	3.3	4.0	4.0	4.5	4.7	5.0	5.4	5.4	7.0
7.5	8.8	9.0	10.3	22.0	24.5														

We compare the ALTF distribution with the Fréchet (Fr), weighted Fréchet (WFr),

Table 2: Average values of estimates, MSEs (in parentheses), lengths of the confidence intervals and the coverage probabilities (in parentheses) for $n = 25, 50$ and 100.

Parameters			MLEs			CIs		
λ	β	α	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\alpha}$	λ	β	α
				$n = 25$				
1	0.5	0.5	1.884 (2.395)	0.506 (0.010)	0.382 (0.179)	8.056 (95.2)	0.378 (92.8)	2.426 (90.4)
		2.5	1.883 (2.254)	0.505 (0.010)	1.872 (3.744)	7.651 (98.2)	0.481 (90.4)	7.096 (90.7)
		5	1.645 (2.114)	0.526 (0.013)	6.482 (11.31)	7.296 (96.0)	0.490 (90.0)	13.56 (91.6)
	2.5	0.5	1.244 (0.230)	2.519 (0.352)	0.642 (1.001)	0.737 (92.5)	1.843 (90.6)	1.904 (93.1)
		2.5	1.296 (0.327)	2.437 (0.330)	2.887 (10.22)	0.822 (92.6)	2.044 (92.4)	11.56 (90.0)
		5	1.296 (0.355)	2.464 (0.350)	6.242 (9.451)	0.844 (91.6)	2.236 (91.5)	13.06 (91.1)
2	0.5	0.5	2.632 (3.688)	0.538 (0.014)	0.757 (1.019)	10.15 (91.1)	0.425 (90.8)	2.571 (90.5)
		2.5	3.013 (4.010)	0.520 (0.010)	2.328 (3.750)	9.411 (96.9)	0.470 (91.9)	8.735 (92.1)
		5	2.617 (2.230)	0.528 (0.011)	6.535 (10.39)	8.244 (98.0)	0.604 (93.9)	11.92 (94.2)
	2.5	0.5	2.382 (0.643)	2.529 (0.293)	0.661 (1.429)	1.636 (90.9)	1.986 (91.3)	2.266 (92.7)
		2.5	2.559 (0.990)	2.394 (0.326)	2.577 (7.201)	1.723 (91.6)	1.899 (96.6)	7.560 (92.0)
		5	2.615 (1.221)	2.398 (0.343)	5.044 (10.78)	1.666 (93.9)	1.946 (93.4)	10.56 (92.2)
				$n = 50$				
1	0.5	0.5	1.671 (1.025)	0.486 (0.004)	0.327 (0.115)	6.506 (97.5)	0.286 (90.3)	1.703 (92.6)
		2.5	1.367 (0.807)	0.503 (0.005)	2.653 (2.825)	5.620 (94.2)	0.369 (93.5)	6.494 (92.1)
		5	1.353 (0.696)	0.501 (0.005)	5.307 (8.580)	5.734 (94.9)	0.393 (92.4)	10.08 (93.3)
	2.5	0.5	1.147 (0.107)	2.449 (0.177)	0.547 (0.453)	0.649 (92.1)	1.406 (91.9)	0.808 (92.3)
		2.5	1.183 (0.142)	2.399 (0.197)	2.592 (9.008)	0.659 (95.4)	1.484 (93.3)	9.556 (93.4)
		5	1.171 (0.146)	2.411 (0.216)	6.106 (7.201)	0.681 (93.3)	1.640 (92.0)	10.57 (93.3)
2	0.5	0.5	2.558 (2.283)	0.513 (0.005)	0.535 (0.220)	9.963 (93.4)	0.350 (92.0)	2.185 (93.7)
		2.5	2.578 (1.544)	0.500 (0.004)	2.376 (2.624)	8.046 (96.5)	0.390 (92.2)	10.93 (96.5)
		5	2.383 (0.988)	0.508 (0.004)	4.898 (6.189)	7.240 (99.0)	0.457 (96.5)	10.83 (96.7)
	2.5	0.5	2.312 (0.363)	2.423 (0.152)	0.492 (0.506)	1.393 (95.2)	1.381 (92.8)	1.129 (94.9)
		2.5	2.429 (0.569)	2.337 (0.231)	2.426 (5.709)	1.326 (92.2)	1.301 (92.8)	5.206 (92.5)
		5	2.424 (0.609)	2.340 (0.244)	5.306 (8.353)	1.302 (93.5)	1.388 (94.3)	8.388 (95.8)
				$n = 100$				
1	0.5	0.5	1.339 (0.361)	0.494 (0.002)	0.400 (0.097)	4.295 (95.1)	0.244 (91.7)	1.678 (94.2)
		2.5	1.253 (0.315)	0.494 (0.002)	2.211 (1.944)	4.266 (96.1)	0.289 (93.7)	3.029 (94.7)
		5	1.264 (0.308)	0.492 (0.002)	4.325 (7.333)	4.545 (97.2)	0.329 (93.1)	6.522 (94.0)
	2.5	0.5	1.110 (0.056)	2.432 (0.112)	0.479 (0.298)	0.501 (93.7)	1.013 (93.2)	0.729 (94.0)
		2.5	1.127 (0.072)	2.387 (0.137)	2.491 (6.330)	0.516 (95.3)	1.128 (96.5)	8.328 (96.2)
		5	1.128 (0.073)	2.378 (0.151)	5.009 (6.659)	0.542 (94.4)	1.194 (94.3)	9.318 (94.8)
2	0.5	0.5	2.199 (0.627)	0.503 (0.002)	0.514 (0.095)	6.720 (95.5)	0.251 (93.4)	1.878 (95.3)
		2.5	2.386 (0.664)	0.495 (0.002)	2.199 (1.252)	6.088 (97.6)	0.349 (94.1)	7.918 (96.3)
		5	2.197 (0.284)	0.504 (0.002)	4.874 (2.835)	6.421 (99.6)	0.406 (97.1)	9.612 (99.4)
	2.5	0.5	2.231 (0.213)	2.415 (0.115)	0.495 (0.495)	1.103 (92.8)	1.038 (93.8)	0.953 (93.5)
		2.5	2.355 (0.351)	2.319 (0.178)	2.201 (4.198)	0.995 (95.5)	0.933 (93.6)	3.480 (94.7)
		5	2.342 (0.367)	2.323 (0.198)	4.788 (6.487)	1.032 (95.6)	1.043 (94.0)	5.558 (96.1)

Table 3: Average values of estimates, MSEs (in parentheses), lengths of the confidence intervals and the coverage probabilities (in parentheses)for $n = 150, 200$ and 250.

Parameters			MLEs			CIs		
λ	β	α	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\alpha}$	λ	β	α
					$n = 150$			
1	0.5	0.5	1.329 (0.267)	0.489 (0.002)	0.373 (0.055)	3.969 (95.6)	0.220 (94.6)	1.607 (93.7)
		2.5	1.277 (0.289)	0.492 (0.002)	2.188 (1.984)	3.990 (95.9)	0.274 (94.2)	2.517 (94.2)
		5	1.216 (0.189)	0.491 (0.002)	4.187 (5.355)	3.729 (97.9)	0.258 (94.3)	5.487 (96.0)
	2.5	0.5	1.081 (0.036)	2.442 (0.086)	0.509 (0.276)	0.478 (94.9)	0.941 (94.9)	0.671 (96.9)
		2.5	1.099 (0.039)	2.387 (0.098)	2.217 (4.166)	0.481 (95.9)	1.006 (95.6)	6.400 (96.5)
		5	1.091 (0.037)	2.397 (0.104)	4.776 (6.043)	0.524 (95.6)	1.156 (95.9)	8.449 (97.4)
2	0.5	0.5	2.026 (0.336)	0.509 (0.002)	0.543 (0.060)	6.261 (95.1)	0.208 (94.3)	1.586 (97.3)
		2.5	2.290 (0.335)	0.495 (0.001)	2.233 (0.890)	5.351 (98.7)	0.293 (97.7)	6.025 (98.8)
		5	2.128 (0.230)	0.503 (0.001)	4.929 (1.881)	5.443 (99.3)	0.368 (97.6)	7.929 (98.7)
	2.5	0.5	2.150 (0.111)	2.445 (0.079)	0.517 (0.352)	0.961 (95.4)	0.906 (94.1)	0.824 (97.5)
		2.5	2.283 (0.225)	2.339 (0.141)	2.107 (2.842)	0.881 (96.6)	0.832 (95.5)	2.996 (98.7)
		5	2.286 (0.259)	2.337 (0.162)	4.585 (4.837)	0.868 (95.1)	0.889 (95.8)	4.568 (97.8)
					$n = 200$			
1	0.5	0.5	1.319 (0.228)	0.487 (0.001)	0.364 (0.053)	3.866 (96.3)	0.207 (95.7)	1.450 (95.5)
		2.5	1.163 (0.153)	0.494 (0.001)	2.227 (1.118)	2.857 (96.2)	0.217 (96.8)	2.168 (96.0)
		5	1.204 (0.142)	0.489 (0.001)	4.054 (4.080)	3.464 (97.6)	0.235 (96.0)	3.624 (96.6)
	2.5	0.5	1.067 (0.025)	2.447 (0.067)	0.500 (0.220)	0.422 (96.5)	0.823 (95.3)	0.488 (95.8)
		2.5	1.087 (0.031)	2.392 (0.090)	2.226 (3.939)	0.459 (95.4)	0.954 (95.5)	6.077 (96.7)
		5	1.077 (0.027)	2.410 (0.087)	4.833 (4.847)	0.458 (94.5)	1.032 (96.7)	6.978 (97.9)
2	0.5	0.5	2.030 (0.244)	0.505 (0.001)	0.533 (0.035)	5.803 (96.8)	0.184 (97.4)	1.368 (98.4)
		2.5	2.269 (0.256)	0.493 (0.001)	2.143 (0.524)	4.790 (99.4)	0.276 (96.3)	5.260 (98.6)
		5	2.079 (0.138)	0.502 (0.001)	5.003 (1.192)	4.681 (99.6)	0.342 (98.5)	5.589 (99.4)
	2.5	0.5	2.154 (0.091)	2.419 (0.068)	0.461 (0.235)	0.904 (95.5)	0.782 (95.0)	0.646 (95.7)
		2.5	2.257 (0.187)	2.334 (0.124)	2.070 (2.758)	0.819 (96.2)	0.755 (95.9)	2.659 (97.1)
		5	2.240 (0.203)	2.364 (0.140)	4.779 (4.127)	0.832 (95.7)	0.879 (97.2)	4.485 (96.9)
					$n = 250$			
1	0.5	0.5	1.207 (0.119)	0.491 (0.001)	0.397 (0.039)	3.363 (95.8)	0.202 (95.7)	1.336 (95.7)
		2.5	1.130 (0.102)	0.494 (0.001)	2.251 (0.766)	1.907 (97.7)	0.131 (96.3)	1.520 (97.9)
		5	1.171 (0.106)	0.488 (0.001)	4.133 (3.724)	3.156 (97.8)	0.206 (97.6)	2.638 (98.3)
	2.5	0.5	1.060 (0.021)	2.441 (0.059)	0.487 (0.185)	0.386 (95.2)	0.742 (93.4)	0.330 (95.0)
		2.5	1.075 (0.026)	2.411 (0.076)	2.341 (3.823)	0.462 (96.1)	0.726 (95.8)	4.683 (96.0)
		5	1.076 (0.023)	2.398 (0.078)	4.465 (3.058)	0.414 (94.1)	0.904 (95.6)	6.420 (96.5)
2	0.5	0.5	2.001 (0.157)	0.505 (0.001)	0.527 (0.025)	4.832 (97.2)	0.142 (96.6)	0.944 (97.8)
		2.5	2.235 (0.204)	0.493 (0.001)	2.202 (0.394)	4.183 (99.6)	0.257 (98.4)	4.745 (99.1)
		5	2.009 (0.102)	0.504 (0.001)	5.208 (1.579)	4.153 (99.6)	0.312 (98.8)	4.776 (99.5)
	2.5	0.5	2.134 (0.074)	2.423 (0.060)	0.472 (0.206)	0.809 (95.1)	0.695 (95.9)	0.589 (97.0)
		2.5	2.213 (0.147)	2.366 (0.109)	2.237 (2.697)	0.758 (95.5)	0.736 (96.6)	2.409 (97.6)
		5	2.184 (0.146)	2.394 (0.110)	4.961 (3.903)	0.745 (96.2)	0.823 (96.6)	3.817 (97.1)

Marshall Olkin Fréchet (MOFr), exponentiated Fréchet (EFr) and beta exponentiated Fréchet (BEFr) distributions. Their density functions (for $x > 0$) are given by:

$$\text{WFr} : f(x; \alpha, \beta) = \frac{\alpha(2\beta)^{1-1/\alpha} x^{-\alpha} e^{-2\beta x^{-\alpha}}}{\Gamma(1 - 1/\alpha)},$$

$$\text{MOFr} : f(x; \alpha, \beta, \lambda) = \alpha\beta\lambda^{\beta} x^{-(\beta+1)} e^{-(\lambda/x)^{\beta}} \left[\alpha + (1-\alpha)e^{-(\lambda/x)^{\beta}}\right]^{-2},$$

$$\text{EFr} : f(x; \alpha, \beta, \lambda) = \alpha\beta\lambda^{\beta} x^{-(\beta+1)} e^{-(\lambda/x)^{\beta}} \left[1 - e^{-(\lambda/x)^{\beta}}\right]^{\alpha-1},$$

$$\text{BEFr} : f(x; \alpha, \beta, \lambda) = \frac{\alpha\beta\lambda^{\beta}}{B(a,b)} x^{-(\beta+1)} e^{-(\lambda/x)^{\beta}} \left[1 - e^{-(\lambda/x)^{\beta}}\right]^{\alpha b-1} \left\{1 - \left[1 - e^{-(\lambda/x)^{\beta}}\right]^{\alpha}\right\}^{a-1}.$$

Tables 4 and 6 provide the MLEs and the corresponding standard errors (in parentheses) of the parameters of all the models for data sets 1 and 2, respectively. In order to compare our proposed distribution, Tables 5 and 7 list the values of $-2\hat{\ell}$, Akike information criterion (AIC), Kolmogorov-Smirnov (K-S)and the corresponding p-value, Anderson-Darling (A^*) and Cramér von Mises (W^*) goodness of fit statistics for the fitted models for data sets 1 and 2, respectively. From Tables 5 and 7, it is to be noted that the ALTF distribution gives the lowest values of these statistics among all fitted models for both the data sets, and can be used as a competitive model to the other considered models. The histogram and the fitted ALTF distribution are displayed in Figure 2(a) and 3(a) for data sets 1 and 2, respectively. Also, the plot of the fitted ALTF survival function and empirical survival function are displayed in Figures 2(b) and 3(b) for the data sets 1 and 2, respectively. Figures 4 and 5 show that the log-likelihood functions for α, β and λ, for data sets 1 and 2, respectively are log-concave, and therefore a global maximum exists.

Table 4: MLEs and standard errors (in parentheses).

Model	Estimates				
Fr(β,λ)	0.6944 (0.0915)	7.8651 (2.0913)			
WFr(α,β)	0.0070 (0.0838)	0.2461 (0.4961)			
ALTF(α,β,λ)	188.55 (214.77)	1.3504 (0.2438)	2.0364 (0.1237)		
MOFr(α,β,λ)	8.0682 (0.5111)	0.9875 (0.1215)	1.9073 (0.6347)		
EFr(α,β,λ)	5.4684 (2.3151)	0.3336 (0.0623)	152.35 (37.235)		
BEFr(α,β,λ,a,b)	1.7253 (4.5780)	0.1107 (0.1460)	29.588 (201.77)	21.0415 (64.5050)	19.7308 (62.632)

Table 5: Goodness-of fit statistics.

Model	$-2\hat{\ell}$	AIC	$K-S$	$p-value$	A^*	W^*
Fr(β,λ)	311.9971	315.997	0.1187	0.7412	1.0475	0.1731
WFr(α,β)	316.5028	320.503	0.1649	0.3308	1.3565	0.2136
ALTF(α,β,λ)	306.2983	312.298	0.0977	0.9111	0.6399	0.0976
MOFr(α,β,λ)	309.3784	315.378	0.1360	0.5749	0.7926	0.1279
EFr(α,β,λ)	308.5659	314.566	0.1285	0.5749	0.7291	0.1166
BEFr(α,β,λ,a,b)	308.1170	318.117	0.1302	0.6306	0.7049	0.1121

Table 6: MLEs and standard errors (in parentheses).

Model	Estimates				
Fr(β,λ)	1.0127 (0.1130)	1.1298 (0.1740)			
WFr(α,β)	0.1599 (0.0361)	1.7408 (0.1049)			
ALTF(α,β,λ)	298.97 (1093.2)	1.9532 (0.7469)	0.4093 (0.1485)		
MOFr(α,β,λ)	1.3382 (0.1396)	0.5068 (0.0866)	4.9123 (0.0784)		
EFr(α,β,λ)	0.5403 (0.0673)	5.3057 (0.4844)	3.8005 (1.0695)		
BEFr(α,β,λ,a,b)	1.7282 (19.063)	0.5405 (0.0022)	5.2086 (0.0883)	1.0116 (0.0346)	2.1887 (24.2151)

Table 7: Goodness-of fit statistics.

Model	$-2\hat{\ell}$	AIC	$K-S$	$p-value$	A^*	W^*
Fr(β,λ)	201.3814	205.381	0.0590	0.9972	0.4053	0.0634
WFr(α,β)	205.7220	209.722	0.0797	0.9320	0.6331	0.0930
ALTF(α,β,λ)	198.7777	204.778	0.0588	0.9973	0.2095	0.0260
MOFr(α,β,λ)	199.8606	205.861	0.0785	0.9394	0.2978	0.0479
EFr(α,β,λ)	199.4399	205.440	0.0827	0.9115	0.2735	0.0440
BEFr(α,β,λ,a,b)	199.4403	209.440	0.0825	0.9129	0.2735	0.0441

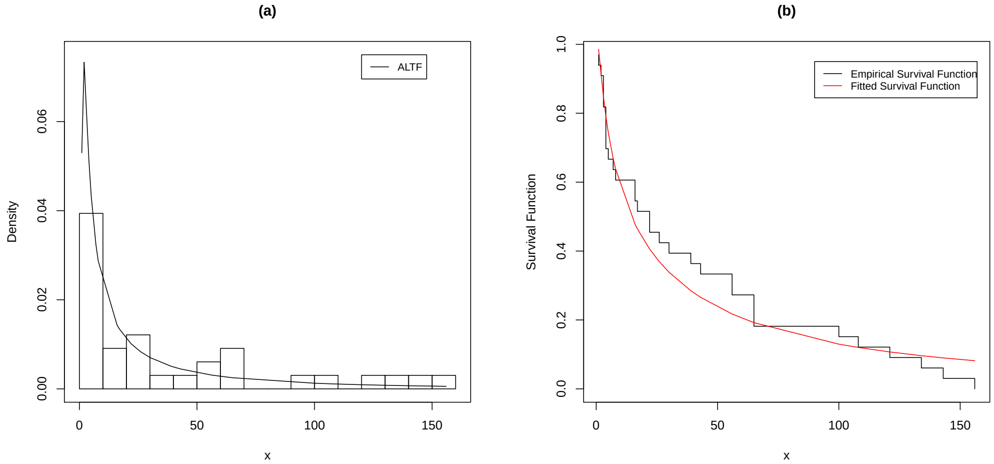

Figure 2: (a) the histogram and the fitted ALTF distribution and (b) the fitted ALTF survival function and empirical survival function for data 1.

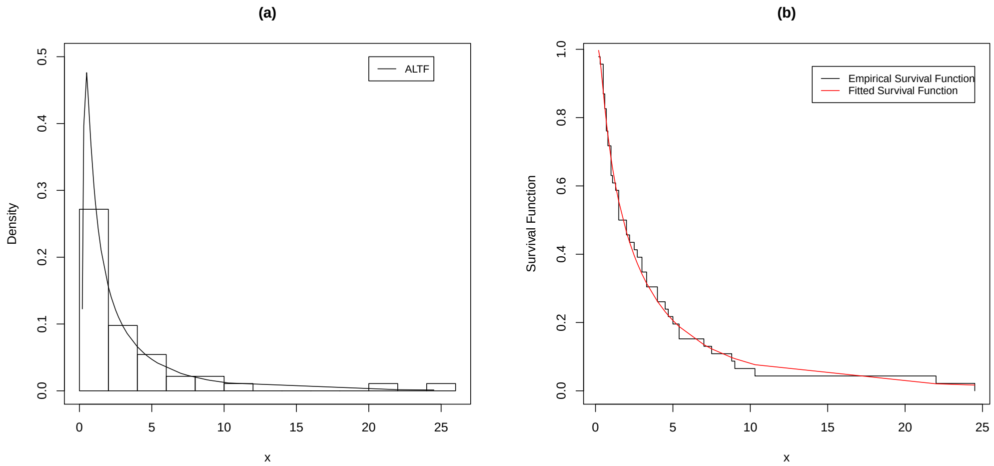

Figure 3: (a) the histogram and the fitted ALTF distribution and (b) the fitted ALTF survival function and empirical survival function for data 2.

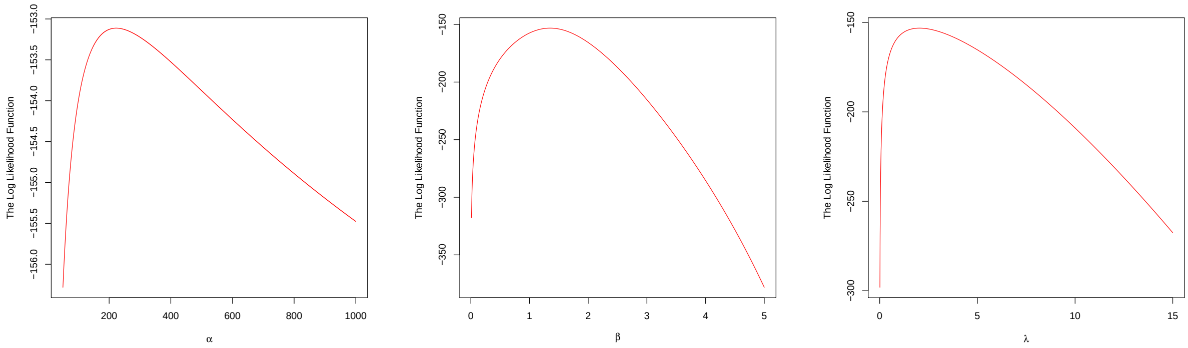

Figure 4: The log-likelihood functions of α, β and λ for data 1.

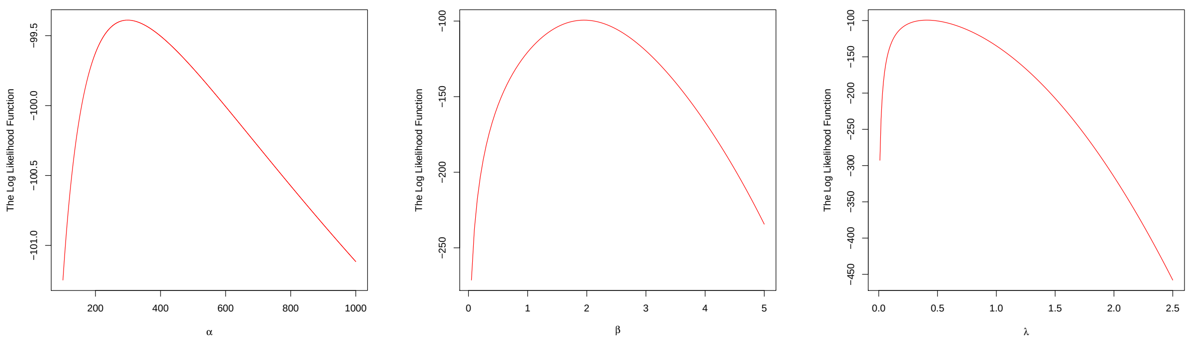

Figure 5: The log-likelihood functions of α, β and λ for data 2.

7. Conclusion

In this paper we have proposed a new three-parameter family of distributions, so-called the ALTF distribution. The proposed ALTF model has two shape parameters and one scale parameter. It includes as special sub-models: the one parameter Fréchet (inverse Weibull), two parameter Fréchet (inverse Weibull), inverse exponential and inverse Rayleigh distributions. The ALTF density function can take various forms depending on its shape parameters. Moreover, the LTF failure rate function can have the following forms depending on its shape parameters: (i) decreasing (ii) upside down bathtub and (iii) reversed J-shaped shaped. Therefore, it can be used quite effectively in analyzing lifetime data. Additionally, the new ALTF model can be used as an alternative to the

Fréchet, weighted Fréchet, Marshall Olkin Fréchet, exponentiated Fréchet and beta exponentiated Fréchet distributions and is expected that in some situations it might work better (in terms of model fitting) than the models stated, although it cannot be always guaranteed. In this paper, we have analyzed two real data sets and the proposed ALTF distribution provides a very good fit to the data sets. We hope our new distribution might attract wider sets of applications in lifetime data and reliability analysis.

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