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Bayesian Estimation of Shape Parameter of Gamma Family under Kullback-Leibler Type Loss

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Abstract

In statistical decision-making, the Kullback-Leibler divergence is used as distance measure that depicts the error while measuring the discrepancy between the actual and approximated distributions. In this study, this measure is employed to determine the loss function in case of two-parameter gamma distribution. The Bayes estimate and the associated risk function of the shape parameter α are obtained under the derived loss function. In this context, the truncated Poisson distribution has been chosen as the prior knowledge for the unknown parameter. An extensive simulation study is carried out to verify the performance of the Bayes estimator based on the risk values. Additionally, three real life datasets have been analyzed to investigate the applicability of the proposed procedure and the Bayes estimator has shown its efficacy.

Keywords: Bayes estimate, distance loss function, Kullback-Leibler divergence, truncated Poisson distribution, risk function.

1. Introduction

Bayesian procedure is found to be more effective compared to the classical approach for estimating the model parameters when the available sample size is small. Statistical inference using the Bayesian technique mainly depends on the chosen prior distribution of the parameter of interest and a suitable loss function. The loss which can quantify the precision of a decision is measured by a function of the parameter and its estimator.

In decision theory, most of the parameter estimation problems have been developed under the squared error loss function (SELF) and the Bayes estimate is obtained by deriving the posterior mean (Han 2017). The situations where the loss is symmetric in nature, i.e. over estimation and under estimation are equally important, then use of SELF is being well justified. Albeit, in practice the loss function is often not symmetric and this has led to the study of asymmetric loss functions. The use of asymmetric loss function is very useful when the over estimation is more serious than under estimation of same magnitude and vice-versa. Varian (1975) introduced LINEX (linear-exponential) loss function in relating to the study of real estate assessment. Several authors mentioned that LINEX loss is appropriate for estimation of the location parameter (Parsian, Sanjari Farsipour, and Nematollahi 1992; Singh, Singh, and Kumar 2013; Nematollahi and Pagheh 2017; Kumar, Kumar, Singh, and Singh 2019; Khatun, Matin *et al.* 2020). For estimating the scale parameter the weighted squared error loss function, a generalization of SELF, and the modified LINEX loss function proposed by Basu and Ebrahimi (1991), were two alternative losses for estimating the scale parameter. Zellner (1994) suggested the Balanced loss function for incorporating both goodness-of-fit and precision of estimation in the evaluation of an estimator. A parametric family of bounded asymmetric loss function known as BLINEX was proposed by Wen and Levy (2001). It seems to be more realistic to express the loss in terms of the ratio in many practical situations. In such cases a useful asymmetric loss function is the entropy loss function (James and Stein 1961). Further, a simple generalization of entropy loss function called general entropy loss function (GELF) was suggested by Calabria and Pulcini (1996). For more detailed discussion on loss functions, few suggested references are Berger (1985), Kamińska and Porosiński (2009), Ferguson (2014), Banerjee and Bhunia (2022), Banerjee, Bhunia, and Seal (2024) etc.

It is noted that, the above mentioned loss functions have their own traditional layouts. There also exist some other types of loss structures which are purely derived from the density functions of the distributions and are known as distance loss (Seal, Banerjee, Bhunia, and Ghosh 2023). In literature, Kullback-Leibler (K-L) divergence or relative entropy measure is one of the most frequently used distance measures to derive the loss functions and was introduced by Kullback and Leibler (1951). It is an asymmetric loss and like SELF, instead of using discrepancies between the true parameter value and its estimate, K-L measures the differences between two probability distributions. We derive the Bayes estimate by using a K-L divergence type loss function. The loss function is derived for the shape parameter of the gamma distribution having the following probability density function:

$$f(x \mid \alpha, p) = \frac{p^{\alpha}}{\Gamma\alpha} x^{\alpha-1} e^{-px}\,;\; x > 0. \tag{1}$$

The shape and scale parameters are $\alpha > 0$ and $p > 0$ respectively and $\Gamma\alpha$ is the complete gamma function expressed as

$$\Gamma\alpha = \int_0^{\infty} t^{\alpha-1} e^{-t} dt.$$

The gamma distribution is a lifetime distribution popularly used for modeling data with right-skewness. Although, a large amount of research are available in literature on parameter estimation of gamma distribution using classical approach but not much attention has been paid towards the Bayesian inference of the gamma shape parameter. Since the shape parameter of the gamma density mentioned in (1) includes the gamma function, deriving both maximum likelihood (ML) and Bayesian estimations become challenging problem. Several authors including Son and Oh (2006), Apolloni and Bassis (2009), Pradhan and Kundu (2011), Antonio Moala, Luiz Ramos, and Alberto Achcar (2013), Banerjee and Seal (2022), Seal, Bhunia, and Banerjee (2024) attempted the estimation of gamma parameters in different ways.

The purpose of this study is to develop an estimation procedure for the shape parameter of the gamma distribution based on the K-L distance loss function and the truncated Poisson distribution as the prior knowledge. In order to derive the Bayes estimate of α, we minimize the posterior expected loss with respect to that estimator. It is observed that the risk function of the newly obtained estimator is not in an explicit form and hence an extensive simulation is performed.

The content of the article is organized in following manner. In Section 2, the Kullback-Leibler distance loss function is specified by using gamma density functions. In Section 3, under the assumption of truncated Poisson prior distribution, the posterior distribution is provided and finally the Bayes estimate of the shape parameter α is obtained in the Subsection 3.1. To evaluate the performance of the Bayes estimator, corresponding risk function is numerically worked out and results are presented in Section 4. In Section 5, three real data analyses are done for illustrative purpose. Finally, the paper is accomplished in Section 6, with the expectation of the better performance of the suggested estimator.

2. Kullback-Leibler type loss for shape parameter

The Kulback-Leibler (K-L) divergence measure becomes popular in information theory and statistical inference as it provides an asymmetric measure of how closely two probability densities are related. K-L divergence is always a non-negative measure and when two probability densities are identical, it equals to zero. For two continuous probability distributions, K-L divergence measure is expressed as

$$D_{KL}(f(x)||g(x)) = E_f\left[log\left(\frac{f(x)}{g(x)}\right)\right] = \int_{-\infty}^{\infty} f(x)\, log\left(\frac{f(x)}{g(x)}\right) dx; \tag{2}$$

where $f(x)$ and $g(x)$ are the respective probability densities and the expectation is taken with respect to the density $f(x)$ (Abufoudeha, Awwada, and Bdairb 2019). More specifically, suppose $f_1\,(x \mid \alpha_1, p)$ and $f_2\,(x \mid \alpha_2, p)$ are the two density functions from the gamma distribution as mentioned in (1), then the K-L divergence measure can be interpreted as an amount of information lost when we use the model $f_2\,(x \mid \alpha_2, p)$ rather than using $f_1\,(x \mid \alpha_1, p)$.

Now, plugging the densities of two-parameter gamma distribution with varying shape and fixed scale parameters into (2), the K-L divergence reduces to

$$\begin{aligned}
D(f(. \mid \alpha_1), f(. \mid \alpha_2)) &= E_{\alpha_1}\left[log\left(\frac{f_1(x \mid \alpha_1, p)}{f_2(x \mid \alpha_2, p)}\right)\right] \\
&= E\left[log\left\{\frac{p^{\alpha_1}}{\Gamma\alpha_1}x^{\alpha_1-1}\right\} - log\left\{\frac{p^{\alpha_2}}{\Gamma\alpha_2}x^{\alpha_2-1}\right\}\right] \\
&= E\left[\alpha_1\ log(p\ x) - \alpha_2\ log(p\ x) + log\ \Gamma\alpha_2 - log\ \Gamma\alpha_1\right] \\
&= E\left[(\alpha_1-\alpha_2)log\ p\right] + E\left[(\alpha_1-\alpha_2)log\ x\right] + E\left[log\ \Gamma\alpha_2\right] - E\left[log\ \Gamma\alpha_1\right] \\
&= (\alpha_1-\alpha_2)log\ p + log\ \Gamma\alpha_2 - log\ \Gamma\alpha_1 + (\alpha_1-\alpha_2)E(log\, x). \qquad (3)
\end{aligned}$$

Here, the term $E[log\, x]$ in the above expression (3) is derived separately and the value becomes,

$$\begin{aligned}
E\left[logx\right] &= \frac{p^{\alpha_1}}{\Gamma\alpha_1}\int_0^{\infty} x^{\alpha_1-1}\ e^{-px}\ log\, x\ dx \\
&= \frac{1}{\Gamma\alpha_1}\int_0^{\infty} e^{-z}z^{\alpha_1-1}(log\, z - log\, p)\ dz \\
&= \frac{1}{\Gamma\alpha_1}\int_0^{\infty} e^{-z}z^{\alpha_1-1}\ logz\ dz - \frac{1}{\Gamma\alpha_1}\ logp\int_0^{\infty} e^{-z}z^{\alpha_1-1}\ dz \\
&= \frac{1}{\Gamma\alpha_1}\int_0^{\infty} e^{-z}z^{\alpha_1-1}\ log\, z\ dz - \frac{1}{\Gamma\alpha_1}\ log\, p\ \Gamma\alpha_1 \\
&= \psi(\alpha_1) - log\, p.
\end{aligned}$$

After substituting this value in (3) the loss function simplifies to,

$$\begin{aligned}
D(f(.|\alpha_1), f(.|\alpha_2)) &= (\alpha_1-\alpha_2)\ logp + log\ \Gamma\alpha_2 - log\Gamma\alpha_1 + (\alpha_1-\alpha_2)\left\{\psi(\alpha_1) - logp\right\} \\
&= log\ \Gamma\alpha_2 - log\ \Gamma\alpha_1 + (\alpha_1-\alpha_2)\ \psi(\alpha_1). \qquad (4)
\end{aligned}$$

Here, $\psi(.)$ is referred as the digamma function and that is called as logarithmic derivatives of the gamma function, i.e.

$$\psi(\alpha) = \frac{\partial}{\partial\alpha}ln\,\Gamma(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma\alpha}.$$

To find an explicit form of the expression (4), we use the approximated value of the digamma function i.e.

$$\frac{\partial}{\partial\alpha_1}log\,\Gamma\alpha_1 \approx \left[log\alpha_1 - \frac{1}{2\alpha_1}\right], \text{for large } \alpha_1$$

and finally the expression of the K-L divergence based on two gamma densities is obtained as

$$\begin{aligned} D(f(.|\alpha_1), f(.|\alpha_2)) &\approx log\Gamma\alpha_2 - log\Gamma\alpha_1 + (\alpha_1-\alpha_2)\left[log\alpha_1 - \frac{1}{2\alpha_1}\right] \\ &= log\Gamma\alpha_2 - \alpha_2\ log\alpha_1 + \frac{\alpha_2}{2\alpha_1} - log\Gamma\alpha_1 + \alpha_1\ log\alpha_1 - \frac{1}{2} \\ &= (log\Gamma\alpha_2 - \alpha_2\ log\alpha_1) - (log\Gamma\alpha_1 - \alpha_1\ log\ \alpha_1) + \frac{\alpha_2}{2\alpha_1} - \frac{1}{2} \\ &= log\left(\frac{\Gamma\alpha_2}{\alpha_1^{\alpha_2}}\right) + log\left(\frac{\alpha_1^{\alpha_1}}{\Gamma\alpha_1}\right) + \frac{\alpha_2}{2\alpha_1} - \frac{1}{2} \\ &= log\left(\frac{\alpha_1^{\alpha_1}}{\Gamma\alpha_1}\right) - log\left(\frac{\alpha_1^{\alpha_2}}{\Gamma\alpha_2}\right) + \frac{1}{2}\left(\frac{\alpha_2}{\alpha_1} - 1\right). \end{aligned} \tag{5}$$

As the digamma approximation is used for large α_1, then the ratio $\frac{\alpha_2}{\alpha_1}$ may be taken as small quantity. Therefore, in order to simplify this expression, a negligible amount of value can be ignored from (5) and hence the loss function using K-L divergence may be taken as

$$\begin{aligned} L(\partial(x),\alpha) &= \left| log\left(\frac{\partial(x)^{\partial(x)}}{\Gamma\partial(x)}\right) - log\left(\frac{\partial(x)^{\alpha}}{\Gamma\alpha}\right)\right| \\ &= \left|\partial(x)\ log\ \partial(x) - log\ \Gamma\partial(x) - \alpha\ log\ \partial(x) + log\ \Gamma\alpha\right|, \end{aligned}$$

where α is assumed to be large.

As, we take expectation with respect to the shape parameter α, the terms without α are constant and hence we neglect those terms to define the K-L loss function. Therefore, the final expression of the K-L distance loss function for the shape parameter α using two gamma densities becomes,

$$L(\partial(x),\alpha) = |log\ \Gamma\alpha - \alpha\ log\ \partial(x)| \tag{6}$$

and we utilize this loss function for further calculation to obtain the Bayes estimate for the shape parameter of the gamma distribution.

3. Posterior distribution and the Bayes estimate

In Bayesian inference, the posterior distribution is useful for making future inferences or decision and is defined as the proportional to the likelihood function for the data given the parameter values multiplied by the prior for the parameters. Therefore, for deriving the posterior probability, the basic step is to specify the appropriate prior distribution for the parameter of interest and that is what distinguishes the method from the classical approach. The likelihood function of the parameters α and p for the sample observations $x_1, x_2, \cdots, x_n$ taken from gamma distribution, mentioned in (1), has the following form

$$L(\alpha, p \mid \mathbf{x}) = \left(\frac{p^{\alpha}}{\Gamma\alpha}\right)^n \prod_{i=1}^{n} x_i^{\alpha-1} e^{-p\sum_{i=1}^{n} x_i}, \tag{7}$$

with the scale parameter p being known.

Sometimes, assumption of discrete distribution as the prior knowledge about the concerned parameter is appeared to be more convenient rather than considering the continuous prior distribution. The reason being, the information accumulated in real life are measured to only a finite number of decimal places and it is not always possible to represent them as a continuous variable. Even though the measurements are conducted on a continuous scale, the researchers' prior belief regarding the parameter of interest may come from a discretized distribution is more appropriate.

Also, it has been found that, for the shape parameter of gamma distribution there exists no proper conjugate prior. Hence, by taking these points into consideration, here for the shape parameter α, we assume the Poisson distribution as the prior knowledge. It is well known that both the prior and baseline distribution belong to the exponential family of distributions. However, as the range of α starts from zero, there may be a drawback in using the Poisson prior. Since the shape parameter α of gamma distribution is always positive and the value zero is neither considered to be positive nor negative, we use the truncated Poisson distribution, truncated at $\alpha = 0$ as the prior probability distribution to make the prior positive. Therefore, the probability mass function of the prior distribution can be written as

$$\pi(\alpha) = \frac{e^{-\lambda}\lambda^{\alpha}}{\alpha!(1-e^{-\lambda})}; \quad \alpha = 1, 2, ..., \infty; \lambda > 0. \tag{8}$$

Now, according to the Bayes' theorem, the posterior distribution of α given the data $\mathbf{x}$, $\pi(\theta \mid \mathbf{x})$, is obtained by combining the likelihood function (7) and the prior probability (8) as follows

$$\begin{aligned}\pi(\alpha \mid \mathbf{x}) &= \frac{f(\mathbf{x} \mid \alpha)\pi(\alpha)}{\sum_{\alpha} f(\mathbf{x} \mid \alpha)\pi(\alpha)} \\ &= \frac{\left(\frac{p^{\alpha}}{\Gamma\alpha}\right)^{n} \prod_{i=1}^{n} x_i^{\alpha-1} \frac{\lambda^{\alpha}}{\alpha!}}{\sum_{\alpha}\left(\frac{p^{\alpha}}{\Gamma\alpha}\right)^{n} \prod_{i=1}^{n} x_i^{\alpha-1} \frac{\lambda^{\alpha}}{\alpha!}} \\ &= \frac{\left(p^{n}\prod_{i=1}^{n} x_i\lambda\right)^{\alpha}/(\Gamma\alpha)^{n}\alpha!}{\sum_{\alpha}\left(p^{n}\prod_{i=1}^{n} x_i\lambda\right)^{\alpha}/(\Gamma\alpha)^{n}\alpha!} \\ &\propto \frac{\left(p^{n}\prod_{i=1}^{n} x_i\lambda\right)^{\alpha}}{(\Gamma\alpha)^{n}\alpha!}. \end{aligned} \tag{9}$$

Actually, the posterior distribution is nothing but the conditional distribution of α after the data have been observed. Now, the K-L distance loss function is combined with the posterior distribution to obtain the posterior expectation of the loss function and the $\hat{\alpha}$ value that minimizes the expected loss is selected as the Bayes estimator.

3.1. Posterior expected loss

In this subsection, our focus is on the derivation of the posterior expected loss at first and then the Bayes estimate of the shape parameter α is obtained for which this expected loss attains the minimum value. Therefore, the posterior expected loss function can be obtained as,

$$\begin{aligned}&E_{\alpha} \mid log\ \Gamma\alpha - \alpha\ log\ (\partial(x)) \mid \\ &= \sum_{\alpha} \mid log\ \Gamma\alpha - \alpha\ log\ (\partial(x)) \mid \pi(\alpha|x) \\ &= \sum_{\alpha} \mid log\ \Gamma\alpha - \alpha\ log\ (\partial(x)) \mid \frac{(p^{n}\prod_{i=1}^{n} x_i\ \lambda)^{\alpha}}{(\Gamma\alpha)^{n}\ \alpha!} \\ &= \sum_{\alpha} \mid log\ \Gamma\alpha - \alpha\ log\ (\partial(x)) \mid \frac{(p^{n}\prod_{i=1}^{n} x_i\ \lambda)^{\alpha}}{(\Gamma\alpha)^{n+1}\ \alpha} \\ &= \sum_{\alpha} \left| \frac{log\ \Gamma\alpha}{\alpha} - log\ (\partial(x)) \right| \frac{(p^{n}\prod_{i=1}^{n} x_i\ \lambda)^{\alpha}}{(\Gamma\alpha)^{n+1}}. \end{aligned} \tag{10}$$

The above expression (10) will be minimized when it is measured about the median. It is noticed that if $\frac{log\,\Gamma\alpha}{\alpha}$ is a monotone function, then $log\,\partial(x)$ will be the median of $\frac{log\,\Gamma\alpha}{\alpha}$ or, equivalently, median of α. So, the Bayes estimator will be determined from the median of the posterior distribution.

Lemma 3.1. $\frac{log\,\Gamma\alpha}{\alpha}$ *is a monotonically increasing function.*

Proof. Let us suppose, $g(\alpha) = \frac{log\,\Gamma\alpha}{\alpha}$; where $\alpha > 0$ and show that $g(\alpha)$ is a monotonic increasing function. Now to show that we proceed as follows.

$$\begin{aligned}
g(\alpha+1) \geq g(\alpha) \quad &\Rightarrow \quad \frac{g(\alpha+1)}{g(\alpha)} \geq 1 \\
&\Rightarrow \quad \left(\frac{log\,\alpha}{log\,\Gamma\alpha}+1\right)\left(\frac{\alpha}{\alpha+1}\right) \geq 1 \\
&\Rightarrow \quad \frac{log\,\alpha}{log\,\Gamma\alpha}+1 \geq 1+\frac{1}{\alpha} \\
&\Rightarrow \quad \frac{log\,\alpha^{\alpha}}{log\,\Gamma\alpha} \geq 1 \\
&\Rightarrow \quad \alpha^{\alpha} \geq \Gamma\alpha, \text{which is obvious.}
\end{aligned}$$

It supports our assumption that $g(\alpha+1) \geq g(\alpha)\ \forall\alpha$. So, $g(\alpha)$ is a monotonically increasing function.

Hence, the posterior expected loss will be minimum when,

$$\begin{aligned}
log\,\partial(x) &= \frac{log\,\Gamma median(\alpha \mid x)}{median(\alpha \mid x)} \\
\partial(x) &= e^{log\ (\Gamma median(\alpha|x))^{\frac{1}{median(\alpha|x)}}} \\
&= (\Gamma median(\alpha \mid x))^{\frac{1}{median(\alpha|x)}}.
\end{aligned}$$

Therefore, the Bayes estimate of α under Kullback-Leibler distance loss function becomes,

$$\partial(x) = (\Gamma median(\alpha \mid x))^{\frac{1}{median(\alpha|x)}}. \tag{11}$$

4. Evaluation of risk of the Bayes estimator

Risk function is used as a statistical tool to evaluate the performance of an estimator. Here, in order to see the behaviour of the proposed estimator, the risk function is obtained using Kullback-Leibler distance loss function. Let us denote $\hat{\alpha}_{BE}$ as the proposed Bayes estimator and then the associated risk of that estimator is written as $E[L(\hat{\alpha}_{BE}, \alpha)]$, where the expectation is taken over the sample observations. Thus, the risk function has been obtained by taking the loss (6) and the density function (1) as follows,

$$\begin{aligned}
\text{Risk function} &= E_{x|\alpha}\left[L\left(\alpha, \hat{\alpha}_{BE}\right)\right] \\
&= E_{x|\alpha}\left|log\,\Gamma\alpha - \alpha\,log\,\hat{\alpha}_{BE}\right| \\
&= \int_{x=0}^{\infty}\left|log\,\Gamma\alpha - \alpha\,log\,\hat{\alpha}_{BE}\right| f(x \mid \alpha)dx \\
&= \int_{x=0}^{\infty}\left|log\,\Gamma\alpha - \alpha\,log\,\hat{\alpha}_{BE}\right|\left(\frac{p^{\alpha}}{\Gamma\alpha}\right)^{n}\prod_{i=1}^{n} x_i^{\alpha-1}e^{-p\sum_{i=1}^{n}x_i}dx \\
&= \left(\frac{p^{\alpha}}{\Gamma\alpha}\right)^{n}\int_{x=0}^{\infty}\left|log\,\Gamma\alpha - \alpha\,log\,\hat{\alpha}_{BE}\right|\prod_{i=1}^{n} x_i^{\alpha-1}e^{-p\sum_{i=1}^{n}x_i}dx.
\end{aligned} \tag{12}$$

From the above expression (12), it is clear that the derivation of the risk function in closed form is difficult. Therefore, we perform a simulation study using R Core Team (2024) to evaluate the performance of the estimator in terms of risk values. For this purpose, some known values of hyperparameter λ are initially selected and for each chosen λ, $\alpha_i's$ are randomly generated from the prior distribution. It is observed that the randomness in α is high for large choices of λ. So, we take $\lambda = 4, 4.5, 5, 5.5, 6$ and for each selected λ, 30 $\alpha_i's$ are generated.

Table 1: Risk values of the Bayes estimator of the shape parameter α for $\lambda = 4$

Generated	Risk values			
α	when n=10	when n=25	when n=40	when n=50
2	0.092934	0.026617	0.007948	0.004529
6	0.607539	0.354553	0.204078	0.150497
3	0.145218	0.051995	0.019356	0.011543
3	0.153326	0.049280	0.021023	0.014137
4	0.488779	0.225247	0.120172	0.079146
2	0.085275	0.027818	0.008410	0.003512
5	0.168387	0.092697	0.051025	0.043894
2	0.088417	0.027911	0.007856	0.003882
4	0.472429	0.239167	0.109187	0.076196
3	0.135431	0.051410	0.017336	0.013763
5	0.174061	0.083161	0.058339	0.040947
3	0.147187	0.047206	0.023198	0.011622
3	0.151603	0.057275	0.021645	0.013605
3	0.150509	0.052460	0.023971	0.013087
7	0.771855	0.308955	0.180278	0.127178
5	0.180999	0.090667	0.052309	0.040962
9	0.966785	0.237555	0.073378	0.027892
2	0.085830	0.022920	0.007948	0.004159
5	0.180027	0.087357	0.054622	0.046608
5	0.171907	0.093733	0.057447	0.037057
1	0.000000	0.000000	0.000000	0.000000
6	0.603623	0.363756	0.223141	0.150959
5	0.178594	0.084977	0.064279	0.040237
1	0.000000	0.000000	0.000000	0.000000
6	0.615006	0.360301	0.218467	0.157289
5	0.167236	0.089151	0.050584	0.038685
2	0.090815	0.026894	0.008410	0.004251
7	0.775510	0.314579	0.184984	0.147580
4	0.490001	0.229551	0.115451	0.067278
5	0.174260	0.087118	0.056377	0.041987

Table 2: Risk values of the Bayes estimator of the shape parameter α for $\lambda = 4.5$

Generated	Risk values			
α	when n=10	when n=25	when n=40	when n=50
3	0.162697	0.063637	0.026972	0.016315
7	0.390948	0.178022	0.109750	0.086018
3	0.159273	0.057481	0.026191	0.013079
3	0.161377	0.056405	0.024763	0.015404
4	0.351162	0.149429	0.076267	0.049798
2	0.093876	0.028373	0.009242	0.005083
6	0.451108	0.264759	0.154137	0.125753
2	0.095060	0.028280	0.008687	0.003604
4	0.332506	0.153382	0.077648	0.048680
4	0.340120	0.147774	0.070418	0.050966
6	0.445003	0.258839	0.160492	0.125828
3	0.158457	0.057733	0.023139	0.015186
3	0.158772	0.065838	0.024822	0.015316
3	0.162754	0.061877	0.025784	0.015829
7	0.400761	0.193149	0.115933	0.080452
6	0.455714	0.268336	0.161635	0.115505
9	1.188451	0.329988	0.103845	0.042911
2	0.092091	0.023012	0.009427	0.003974
5	0.388298	0.197050	0.106306	0.080910
5	0.395510	0.196103	0.112332	0.076853
1	0.000046	0.000000	0.000000	0.000000
7	0.401044	0.188904	0.119149	0.084818
5	0.382474	0.193318	0.113762	0.071187
2	0.094136	0.025785	0.008965	0.004436
7	0.408761	0.178647	0.107954	0.086273
6	0.444301	0.262739	0.166042	0.120229
2	0.092489	0.025877	0.009427	0.003789
7	0.405850	0.181309	0.112753	0.094359
4	0.349032	0.149251	0.074999	0.046054
5	0.389887	0.190792	0.111043	0.075214

Table 3: Risk values of the Bayes estimator of the shape parameter α for $\lambda = 5$

Generated	Risk values			
α	when n=10	when n=25	when n=40	when n=50
3	0.178168	0.069634	0.029255	0.018624
7	0.421265	0.255861	0.165098	0.125196
4	0.283724	0.110273	0.053862	0.035194
3	0.173086	0.064103	0.026836	0.016898
5	0.614686	0.342838	0.193434	0.145599
2	0.104215	0.029852	0.010721	0.005360
6	0.277848	0.151854	0.092717	0.082397
3	0.169614	0.067354	0.028231	0.015178
5	0.592588	0.356943	0.197355	0.134187
4	0.288885	0.117136	0.054677	0.035414
7	0.416452	0.254339	0.163576	0.136407
4	0.299878	0.110921	0.057432	0.036963
3	0.178145	0.068681	0.029012	0.016505
3	0.174977	0.066460	0.027546	0.018570
8	0.858989	0.451122	0.285851	0.215738
6	0.277668	0.151855	0.100579	0.078039
10	1.088661	0.287732	0.093538	0.050631
2	0.102616	0.028650	0.010628	0.003974
6	0.289832	0.146152	0.100724	0.074314
6	0.283154	0.148094	0.095955	0.071711
1	0.000000	0.000000	0.000000	0.000000
7	0.421838	0.267066	0.171969	0.126122
6	0.281124	0.154113	0.104474	0.072586
2	0.105475	0.027449	0.009242	0.003697
7	0.433849	0.266407	0.169516	0.133283
7	0.421689	0.256866	0.172492	0.125054
2	0.097186	0.027726	0.010906	0.003882
8	0.869316	0.444712	0.276901	0.203547
4	0.293401	0.105835	0.056001	0.036972
6	0.268678	0.151004	0.096441	0.071516

Table 4: Risk values of the Bayes estimator of the shape parameter α for $\lambda = 5.5$

Generated	Risk values			
α	when n=10	when n=25	when n=40	when n=50
3	0.217995	0.083038	0.037734	0.023396
8	0.599684	0.383132	0.256199	0.202278
4	0.254639	0.112054	0.059393	0.037566
4	0.253652	0.117248	0.057441	0.039471
5	0.397106	0.194236	0.108776	0.080322
2	0.142138	0.037892	0.014325	0.005360
7	0.364281	0.192330	0.132589	0.115803
3	0.214706	0.079482	0.032788	0.018728
5	0.374345	0.197878	0.112033	0.072019
5	0.391185	0.205826	0.102350	0.076178
7	0.354017	0.189532	0.131034	0.108392
4	0.258443	0.116921	0.059967	0.041947
4	0.262550	0.108760	0.064018	0.037806
4	0.256583	0.121889	0.059567	0.037519
9	0.860662	0.400989	0.247316	0.189218
7	0.369094	0.196005	0.137219	0.104876
11	1.168307	0.347090	0.120021	0.061725
3	0.212483	0.078251	0.033119	0.024089
6	0.557176	0.317981	0.202551	0.147440
6	0.562959	0.325063	0.204013	0.142254
1	0.000000	0.000000	0.000000	0.000000
8	0.598023	0.392444	0.268036	0.206129
6	0.554204	0.320851	0.204791	0.146175
2	0.143143	0.036875	0.014140	0.004898
8	0.606829	0.386360	0.263853	0.209616
7	0.357613	0.186463	0.129603	0.108091
3	0.210445	0.077869	0.032667	0.021503
9	0.874018	0.402561	0.254880	0.196539
5	0.383679	0.186278	0.101528	0.075777
7	0.354894	0.203104	0.130107	0.103570

Table 5: Risk values of the Bayes estimator of the shape parameter α for $\lambda = 6$

Generated	Risk values			
α	when n=10	when n=25	when n=40	when n=50
4	0.279080	0.130680	0.074857	0.051195
9	0.585914	0.224464	0.134599	0.110909
5	0.308142	0.123429	0.068976	0.047977
4	0.284302	0.127270	0.067951	0.040345
6	0.804945	0.626027	0.416322	0.309485
3	0.213569	0.080528	0.040381	0.020482
7	0.300079	0.120506	0.077101	0.066809
4	0.272533	0.135555	0.062979	0.045693
5	0.312353	0.125645	0.068575	0.046309
5	0.311402	0.134471	0.069503	0.045639
8	0.637113	0.478610	0.330220	0.256117
5	0.321639	0.117607	0.066556	0.048109
4	0.275631	0.127044	0.066089	0.045206
4	0.282533	0.132766	0.068031	0.044262
9	0.609866	0.223370	0.155200	0.120991
7	0.312700	0.122922	0.078202	0.061096
12	1.164089	0.152000	0.024212	0.004708
3	0.209371	0.078491	0.033962	0.021054
7	0.297123	0.112894	0.076519	0.062442
7	0.313540	0.121329	0.079133	0.062469
2	0.197533	0.055452	0.016636	0.008965
8	0.641951	0.463920	0.331901	0.246807
7	0.311379	0.127491	0.074713	0.061641
3	0.211229	0.078783	0.031255	0.023031
9	0.598166	0.235484	0.144152	0.111136
8	0.640876	0.467090	0.313525	0.259960
3	0.202840	0.077221	0.032401	0.022295
9	0.609756	0.232320	0.151956	0.110527
5	0.303904	0.128045	0.067895	0.046496
7	0.306941	0.115806	0.076612	0.067292

The samples have been generated of sizes $n = 10, 25, 40 \text{ and } 50$ by using each $\alpha_i' s$ from the baseline distribution $Gamma(\alpha_i, p)$. Note that, p is the scale parameter and throughout this study it is assumed to be known. Also, it has been found that the scale parameter has no effect on the posterior probabilities and hence we fixed it at $p = 5$ in all cases. In order to obtain the risk values, we have considered the number of replications as $K = 5000$ times in this simulation process. The steps to calculate the Bayes estimators and the associated risk values are described in the following.

Step 1 : Initialize some known values of λ.

Step 2 : Generate discrete random numbers $\alpha_i' s$; $i = 1, 2, ..., m$ from the truncated Poisson distribution.

Step 3 : Calculate the prior probabilities $\pi(\alpha_1), \pi(\alpha_2), ..., \pi(\alpha_m)$ for the selected λ.

Step 4 : For fixed p, generate random samples $\boldsymbol{X_j' s}$ of size n from $Gamma(\alpha_i, p)$; $\forall i = j$.

Step 5 : Construct a matrix by calculating the likelihood for each combination of $(\alpha_i, \boldsymbol{X_j})$, where $i, j = 1, 2, ..., m$ and $\underset{\sim}{\boldsymbol{X_j}} = (X_{j1}, X_{j2}, \cdots, X_{jn})'$.

$$\begin{bmatrix} f(\underset{\sim}{\boldsymbol{x_1}} \mid \alpha_1) & f(\underset{\sim}{\boldsymbol{x_2}} \mid \alpha_1) \cdots & f(\underset{\sim}{\boldsymbol{x_m}} \mid \alpha_1) \\ f(\underset{\sim}{\boldsymbol{x_1}} \mid \alpha_2) & f(\underset{\sim}{\boldsymbol{x_2}} \mid \alpha_2) \cdots & f(\underset{\sim}{\boldsymbol{x_m}} \mid \alpha_2) \\ \vdots & \ddots & \vdots \\ f(\underset{\sim}{\boldsymbol{x_1}} \mid \alpha_m) & f(\underset{\sim}{\boldsymbol{x_2}} \mid \alpha_m) \cdots & f(\underset{\sim}{\boldsymbol{x_m}} \mid \alpha_m) \end{bmatrix}$$

Step 6 : Calculate prior $\times$ likelihood. i.e. multiply $\pi(\alpha_i)$ in i^{th} row, $i = 1, 2, ..., m$.

$$\begin{bmatrix} f(\underset{\sim}{\boldsymbol{x_1}} \mid \alpha_1)\pi(\alpha_1) & f(\underset{\sim}{\boldsymbol{x_2}} \mid \alpha_1)\pi(\alpha_1) \cdots & f(\underset{\sim}{\boldsymbol{x_m}} \mid \alpha_1)\pi(\alpha_1) \\ f(\underset{\sim}{\boldsymbol{x_1}} \mid \alpha_2)\pi(\alpha_2) & f(\underset{\sim}{\boldsymbol{x_2}} \mid \alpha_2)\pi(\alpha_2) \cdots & f(\underset{\sim}{\boldsymbol{x_m}} \mid \alpha_2)\pi(\alpha_2) \\ \vdots & \ddots & \vdots \\ f(\underset{\sim}{\boldsymbol{x_1}} \mid \alpha_m)\pi(\alpha_m) & f(\underset{\sim}{\boldsymbol{x_2}} \mid \alpha_m)\pi(\alpha_m) \cdots & f(\underset{\sim}{\boldsymbol{x_m}} \mid \alpha_m)\pi(\alpha_m) \end{bmatrix}$$

Step 7: Calculate the marginal probability of $\boldsymbol{x_j}$ as

$$m(\boldsymbol{x_j}) = \sum_{i=1}^{m} f(\boldsymbol{x_j} | \alpha_i)\pi(\alpha_i) \text{ for } j = 1, 2, \cdots, m.$$

Step 8: Divide j^{th} column of the matrix in **Step 6** by the corresponding $m(\boldsymbol{x_j})$; $j = 1, 2, \cdots, m$ to obtain the posterior probabilities.

Step 9: By using the each column of the posterior probability, we will get the corresponding median of α_i; $i = 1, 2, \cdots, m$.

Step 10: Calculate the Bayes estimate $\hat{\alpha}_i$; $i = 1, 2, \cdots, m$ by using (11).

Step 11: Repeat **Step 4 - Step 10** to obtain K times $\hat{\alpha}_i$; $i = 1, 2, \cdots, m$.

Step 12: Calculate loss values $L(\hat{\alpha}_i, \alpha_i)$, for K times by using (6); $i = 1, 2, \cdots, m$.

Step 13: Calculate the risk by taking expectation over the loss values with respect to the density function.

The simulation results are provided in Tables 1-5. The risk values reported for different parameter combinations and sample sizes are considerably small. Moreover, as the size of sample increases the risk values decrease, which verifies the consistency property of the estimator. From these results, it is observed that the Bayes estimator performs well under the truncated Poisson prior distribution while considering the K-L distance loss function.

5. Real data applications

In this section, the previously mentioned estimation procedure is illustrated by analyzing three real life datasets from different applied field. Before proceeding further, the first task is to check whether the gamma distribution fits these datasets or not. For any analysis, assessment of the goodness-of-fit is a vital step as it identifies the discrepancy between a statistical model and the available data. Among several accessible approaches in the literature for determining a model's goodness-of-fit to a specific data, the Kolmogorov-Smirnov (K-S) test is widely used since it performs better for limited sample observations. Therefore, to assess the goodness-of-fit for each data set, we take the K-S test statistic along with their P-values. Also, some diagnostic plots like histogram with estimated pdf, estimated cdf, P-P plot and Q-Q plot are displayed to verify graphically the fitting of the datasets. For each of the considered data, we calculate the posterior medians and their corresponding Bayes estimates respectively. Further, we also split each of the datasets into several groups and find the Bayes estimates as well in order to see the performance of the estimate.

Data I: The Relief times data

In the first example, we consider a lifetime data related to relief times (in minutes) of 20 patients receiving an analgesic as reported by Gross and Clark (1975). The data consists of a random sample of 20 patients and is given as: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0. The sample mean and variance for this relief time data are 1.90 and 0.496 respectively. The maximum likelihood estimates of the unknown parameters α and p are obtained as $\hat{\alpha}_{mle} = 9.670$ and $\hat{p}_{mle} = 5.089$. We also obtain the K-S test value along with the associated P-value (reported within the bracket) as 0.1734 (0.5845), which confirms that the considered data set fits the gamma distribution at 5% level of significance. The histogram with estimated pdf, P-P plot, estimated cdf and Q-Q plot for the relief times data are shown in Figure 1.

Table 6: Posterior median and the Bayes estimate (BE) for Relief times data

Data set	$\lambda = 0.1$		$\lambda = 0.5$		$\lambda = 1.0$		$\lambda = 2.0$	
	Median	BE	Median	BE	Median	BE	Median	BE
Complete	8	2.90271	8	2.90271	9	3.24881	9	3.24881
Group 1	6	2.22091	7	2.55971	8	2.90271	8	2.90271
Group 2	7	2.55971	8	2.90271	8	2.90271	9	3.24881

In the expression of the posterior distribution (9), the model parameter p and the prior parameter λ are known. So, we use $\hat{p}_{mle}$ in place of p and λ is chosen as 0.1, 0.5, 1.0, 1.5 to vary the prior knowledge. Next, we derive the median of the posterior distribution and after substituting it into the expression (11), we will have the corresponding Bayes estimate of the shape parameter. Now, we divide the given data set into two equal groups, each consisting of 10 sub samples. For each of the sub groups, the Bayes estimates are obtained in similar manner. The main idea behind this is to examine the behaviour of the suggested Bayes estimator in real situations. So, the posterior medians and the corresponding Bayes estimates of α for the original as well as the grouped data set have been reported in Table 6.

It is observed that, after splitting the given data set into two groups, the Bayes estimate of

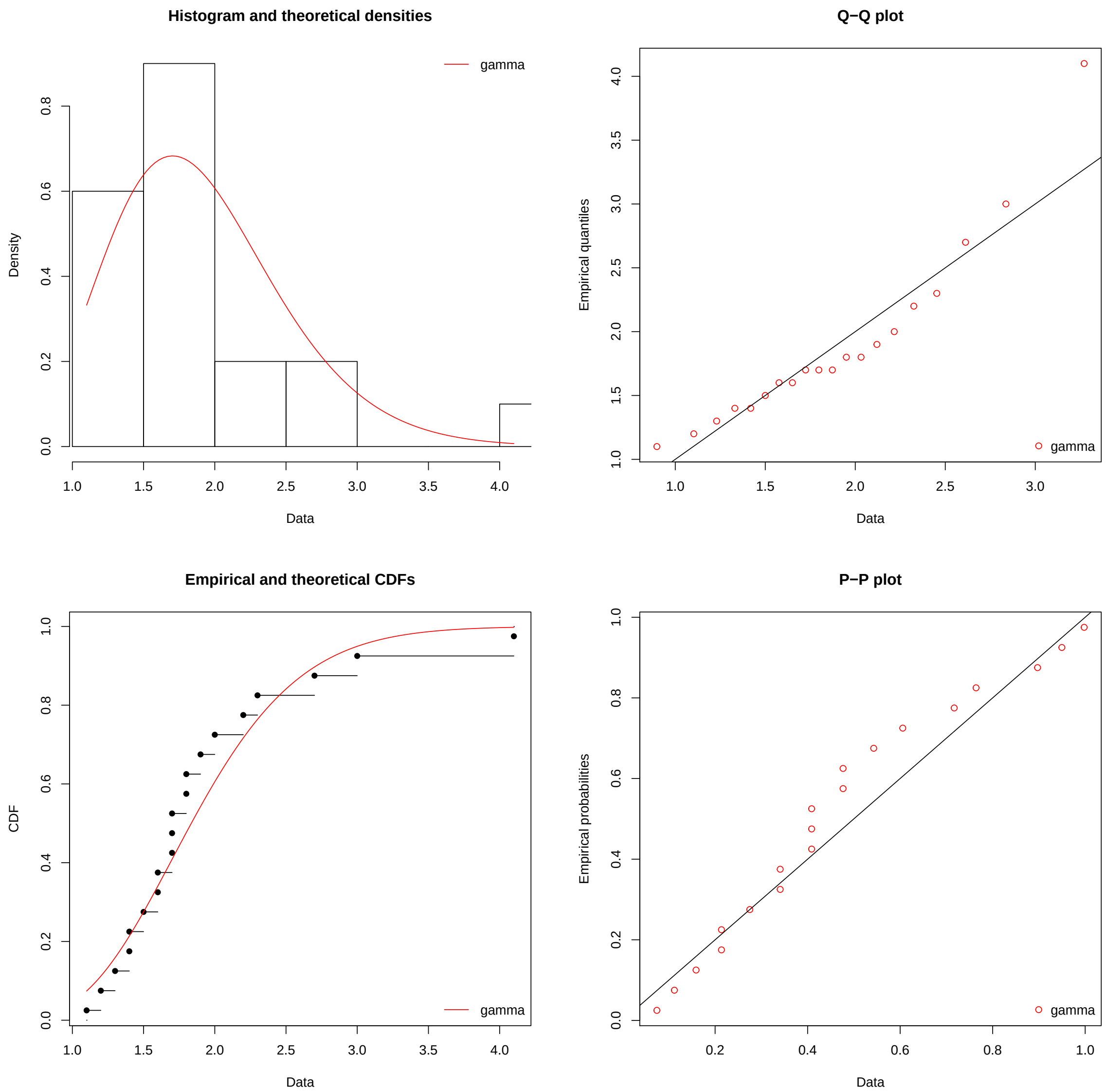

Figure 1: Diagnostic plots of fitted gamma distribution for Relief times data. From upper-left to lower-right: Histogram, Q-Q plot, Empirical VS Theoretical CDF and P-P plot

group 1 does not differ too much from that of the group 2. Moreover, these estimates are also close to the original Bayes estimate considering the total dataset. Additionally, it is clear from the Table 6 that the Bayes estimate does not vary drastically when the prior information is altered with the different choices of λ. Therefore, the Bayes estimate of shape parameter α performs quite well and it seems to be robust in nature.

Data II: The Strength data

The second data set represents the strength data measured in GPA for single carbon fiber and impregnated 1000 carbon fiber tows originally reported by Bader and Priest (1982). It is found that the same data are also used by Ali, Pal, and Woo (2012) to fit the four-parameter generalized gamma Distribution while Raqab, Madi, and Kundu (2008) used it to analyze the goodness-of-fit of the three-parameter generalized exponential distribution. The original data were transformed by Raqab and Kundu (2005) to fit the generalized Rayleigh distribution. Here, the transformed data with the sample size $n = 69$ is used and presented in Table 7.

The sample mean and variance for this strength data set are found as 1.447 and 0.256 respectively. The maximum likelihood estimates of the parameters computed using the data set are

Table 7: Strength measured in GPA for carbon fibers tested under tension at gauge lengths of 20 mm

0.312	0.314	0.479	0.552	0.700	0.803	0.861	0.865	0.944	0.958
0.966	0.977	1.006	1.021	1.027	1.055	1.063	1.098	1.140	1.179
1.224	1.240	1.253	1.270	1.272	1.274	1.301	1.301	1.359	1.382
1.382	1.426	1.434	1.435	1.478	1.490	1.511	1.514	1.535	1.554
1.566	1.570	1.586	1.629	1.633	1.642	1.648	1.684	1.697	1.726
1.770	1.773	1.800	1.809	1.818	1.821	1.848	1.880	1.954	2.012
2.067	2.084	2.090	2.096	2.128	2.233	2.433	2.585	2.585	

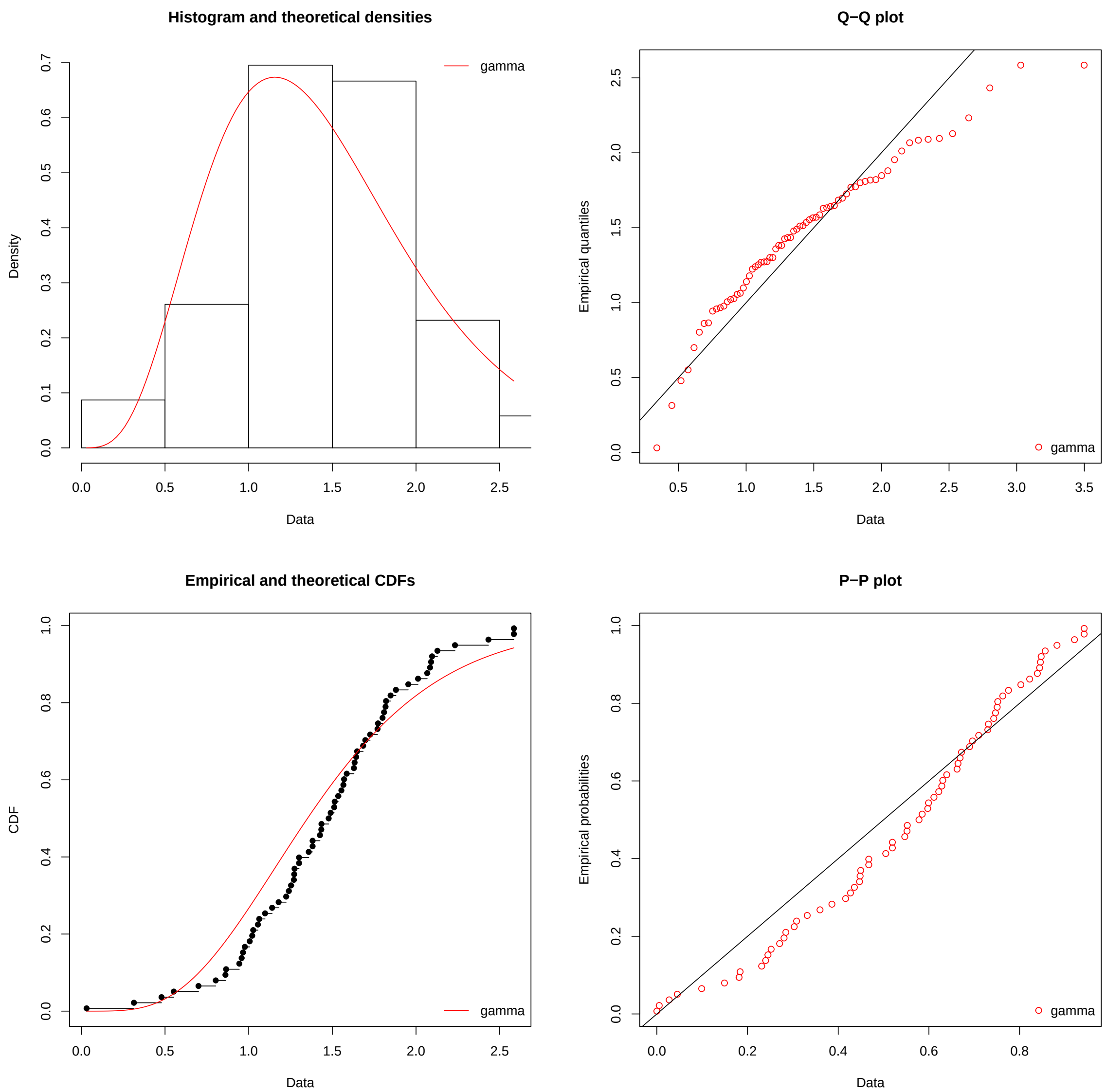

Figure 2: Diagnostic plots of fitted gamma distribution for Strength data. From upper-left to lower-right: Histogram, Q-Q plot, Empirical VS Theoretical CDF and P-P plot

given as $\hat{\alpha}_{mle} = 4.977$ and $\hat{p}_{mle} = 3.440$. Here, also the K-S test statistic and the P-value (given within the bracket) are computed and found as 0.1265 (0.2193), which supports our assumption that the data follows gamma distribution at 5% level of significance. Some diagnostic plots for strength data are provided in Figure 2 to confirm the fitting with the gamma distribution.

To obtain the median of the posterior distribution, we substitute $\hat{p}_{mle}$ in (9) and vary the

Table 8: Posterior median and the Bayes estimate (BE) for Strength data

Data set	$\lambda = 0.1$		$\lambda = 0.5$		$\lambda = 1.0$		$\lambda = 1.5$	
	Median	BE	Median	BE	Median	BE	Median	BE
Complete	5	1.88818	5	1.88818	5	1.88818	5	1.88818
Group 1	3	1.25992	3	1.25992	3	1.25992	3	1.25992
Group 2	5	1.88818	5	1.88818	5	1.88818	5	1.88818
Group 3	6	2.22090	7	2.55971	7	2.55971	7	2.55971

hyperparameter λ as discussed in the earlier data set. After the posterior median is obtained, then by using (11), the Bayes estimate of α has also been computed for the complete data set. Further, we split the data set into three equal subgroups consist of 23 sub samples and obtain the Bayes estimates by considering each of the subgroups. Table 8 provides the posterior median and the corresponding Bayes estimators for the complete data set as well as for the grouped datasets also. The performance of the estimator is quantified on the basis of how the Bayes estimates differ upon changing the datasets and the prior values. The result indicates that, the Bayes estimates for the complete and grouped data set are very close to each other for different choices of prior information λ.

Data III: GAGurine Data

This GAGurine [glycosaminoglycans (GAG)] data are about the concentration of GAG (measured in milligrams per millimole creatinine) in the urine of children up to the aged 17 years was discussed by Venables and Ripley (2013). Pediatricians may find it useful to analyze such data in order to determine whether or not a child's GAG concentration is normal. This data set includes random sample of 40 children between the age 12 to 17 years are provided in Table 9.

Table 9: GAGurine data for children of age between 12 to 17 years

5.8	5.4	5.7	3.1	6.4	7.0	5.7	3.9	9.4	4.4
5.0	15.9	3.7	9.1	4.7	3.6	3.7	4.1	7.9	3.3
6.6	1.9	3.0	5.7	3.2	3.8	5.3	3.2	4.2	6.0
9.7	3.4	3.2	2.5	2.0	4.0	4.3	2.8	2.2	4.7

The respective sample mean and variance for this data are 4.987 and 6.901. The maximum likelihood estimates of the unknown parameters are $\hat{\alpha}_{mle} = 4.886$ and $\hat{p}_{mle} = 0.980$. The K-S statistic and corresponding P-value (reported within the bracket) are derived as 0.0971 (0.8448) and these values support our assumption that the data fits the gamma model at 5% level of significance. The histogram and the estimated density for the GAGurine data along with P-P plot, estimated cdf and Q-Q plot are graphically presented in Figure 3.

Table 10: Posterior median and the Bayes estimate (BE) for GAGurine data

Data set	$\lambda = 0.1$		$\lambda = 0.5$		$\lambda = 1.0$		$\lambda = 1.5$	
	Median	BE	Median	BE	Median	BE	Median	BE
Complete	4	1.56509	5	1.88818	5	1.88818	5	1.88818
Group 1	5	1.88818	5	1.88818	5	1.88818	5	1.88818
Group 2	4	1.56509	4	1.56509	4	1.56509	4	1.56509

As done in the previous two examples, here also to evaluate the Bayes estimate of the shape parameter, we divide the data set into two equal parts which contains 20 sub samples in each group. We compute the posterior median and the respective Bayes estimate by considering the

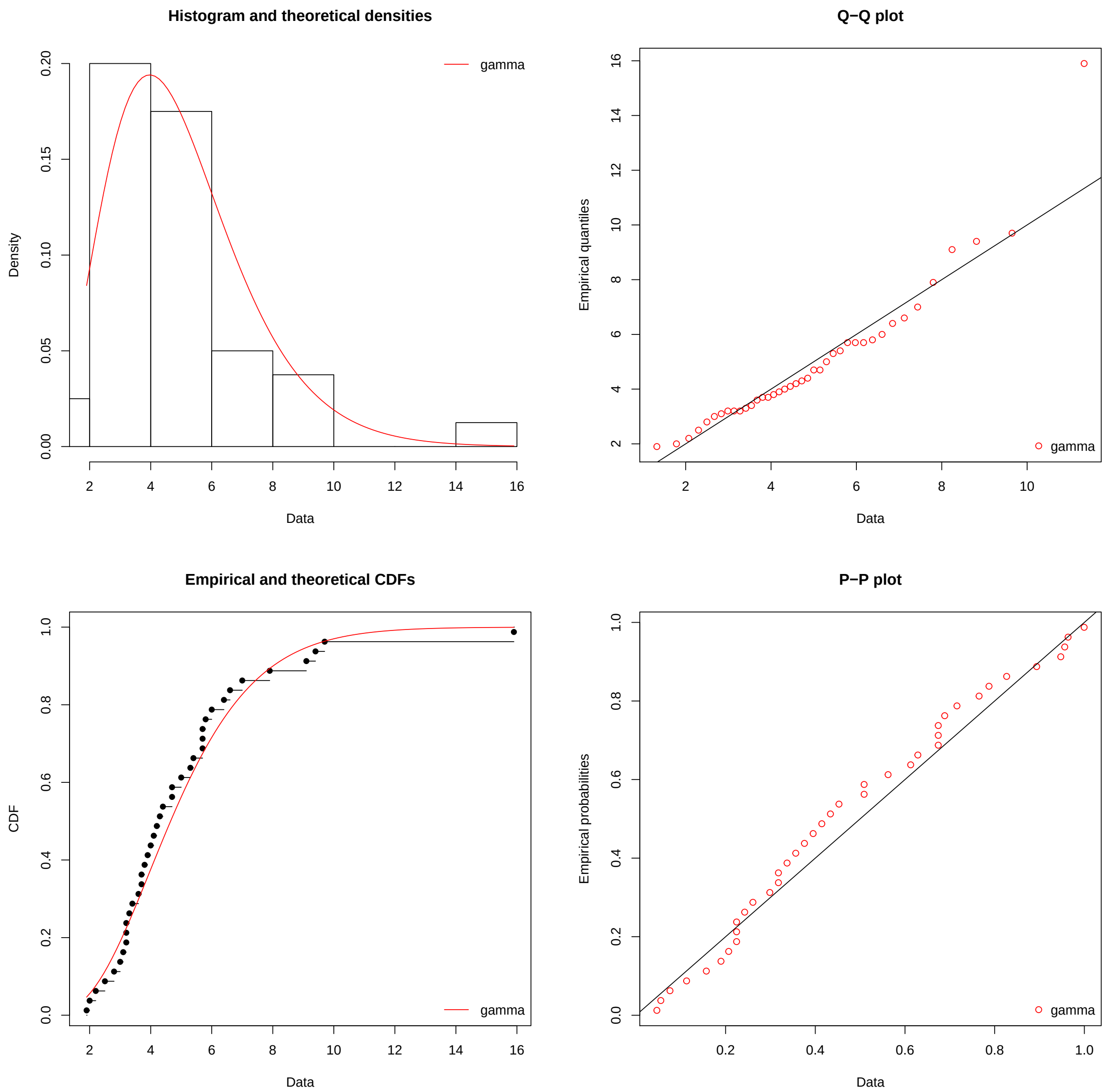

Figure 3: Diagnostic plots of fitted gamma distribution for GAGurine data. From upper-left to lower-right: Histogram, Q-Q plot, Empirical VS Theoretical CDF and P-P plot

complete as well as the grouped datasets. Results are provided in Table 10 and it depicts that the Bayes estimates of both the groups do not differ too much from one another irrespective of the variation of the prior information.

6. Conclusion

A wide variety of loss functions have been developed in the literature to represent different kind of loss structures. In the present work, we have described a procedure to estimate the shape parameter of the gamma distribution by using distance type loss function and studied its performance. Prior knowledge about the parameter of interest is also important in Bayesian inference and truncated Poisson distribution is chosen as the prior information. For gamma distribution, the Kullback-Leibler divergence measure is used to derive the loss function and we used this loss to find the Bayes estimate of the shape parameter α. In particular, the risk function of the estimator is evaluated through an extensive simulation technique.

It is observed from the numerical results provided in Tables 1-5 that the estimator has smaller risk values for different choices of prior parameters. In addition, risk values are also seen to decline as sample sizes increase. To support the theoretical idea in real life situation, we consider three real datasets and demonstrate the applicability of the proposed Bayes estimator for the shape parameter α. The Bayes estimator has proven to be effective for moderate samples and does not rely excessively on the prior information. But, when the size of the data are very small it has been found that the effectiveness of the estimator flawed. Therefore, according to the simulation and real data analysis, the Bayes estimator obtained based on the suggested technique is a good estimator and we may use it in various field of applied studies including survival and reliability studies, forecasting, medical data analysis. In future, by incorporating continuous prior distribution and taking other distance measures, this work may eventually be expanded.

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