Austrian Journal of Statistics
2025, Volume 54, 117–135.
http://www.ajs.or.at/
doi:10.17713/ajs.v54i4.2049

Inference on Partially Observed Competing Risks Models Using Generalized Type-II Hybrid Censoring Scheme

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Abstract

This article investigates inference in a competing risks model where failure causes are partially observed, assuming latent failure times follow Weibull distributions. Inference is derived under a generalized type-II hybrid censoring scheme. The maximum likelihood estimators for model parameters and their associated confidence intervals are discussed. Also, we compute Bayes estimators under both informative and non-informative priors, along with their credible intervals. The performance of all estimators is evaluated through Monte Carlo simulations. Finally, for illustrative purposes, a real-world case is explored.

Keywords: competing risks, generalized type-II hybrid censoring scheme, Weibull distribution, maximum likelihood estimation, Bayesian estimation.

1. Introduction

It is frequently noted in reliability or medical analysis that an item may fail due to multiple causes. For instance, a medical study may reveal that mortality can result from various diseases, while a reliability experiment may demonstrate that an automobile's failure can be attributed to different factors. In these situations, a researcher is often focused on assessing a specific cause in the presence of other causes. In some manner, these causes (risk factors) compete with each other, leading to the failure of the experimental units. These situations are often represented in literature using the competing risks model. Thus, the competing risk data typically consist of failure times along with an associated indicator variable denoting the causes of failure. The failure causes may be considered either independent or dependent. In most cases, the analysis of competing risk data is based on the assumption that the causes of failure are independent. Competing risks models have garnered significant attention in the literature, with numerous notable contributions. Kundu and Basu (2000) estimates the parameters of the competing risks model when the data may be incomplete using exponential and Weibull distributions. Pareek, Kundu, and Kumar (2009) developed inference methods for competing risk models under a progressive censoring scheme, assuming Weibull distributed latent failure times. Cramer and Schmiedt (2011) discussed progressively Type-II censored competing risks data from Lomax distributions. Dey, Jha, and Dey (2016) explored Bayesian analysis of the modified Weibull distribution under progressively censored competing risk models. Koley, Kundu, and Ganguly (2017) and Koley and Kundu (2017) conducted classical and Bayesian inference of competing risks data, assuming exponential distributions for the competing causes of failure, under Type-II hybrid and generalized progressive hybrid censoring schemes. Chacko and Mohan (2019) analyzed competing risks data under progressive Type-II censoring, assuming the number of units removed at each stage follows a binomial distribution. Almarashi, Algarni, and Abd-Elmougod (2020) conducted statistical analysis of competing risks lifetime data from the Nadarajah and Haghighi distribution under Type-II censoring. Du and Gui (2022) investigated statistical inference of the Burr-XII distribution under adaptive Type-II progressive censored schemes with competing risks. More recently, Zheng, Ye, and Gui (2024) developed a competing risks model under progressive Type-II censored data using the inverted exponentiated half-logistic distribution.

Most of the references mentioned above focus on analyzing models where the causes of failure are observed for each failed experimental unit. However, in real-world scenarios, the causes of failure are often only observed for a portion of the failed units due to limitations in cause detection methods, making it impossible to observe all failure causes. This situation, where failure causes are missing for some units, is referred to as partially observed competing risks (POCR). Conducting lifetime studies in such situations presents significant challenges, as making inferences about failure causes is crucial for improving product quality and addressing safety concerns. For example, an industrial machine may fail due to wear and tear, electrical faults, or overheating, but the exact cause may remain unknown due to lack of diagnostic tools, or incomplete maintenance records. If a machine stops and the technician cannot determine whether the failure was caused by an electrical short circuit or overheating, the cause remains ambiguous. These scenarios of POCR require specialized statistical methods to ensure accurate reliability assessments. Several research articles have explored POCR, including, Almuhayfith, Darwish, Alharbi, and Marin (2022) examines inference methods for partially observed failure modes under the assumption that the data follows a Burr XII distribution. Abushal, Soliman, and Abd-Elmougod (2022) discussed a competing risks model with partially observed failure causes and latent failure times following a Lomax distribution; Dutta, Ng, and Kayal (2023) investigated statistical inference for a POCR model when latent failure times follow a general family of inverted exponentiated distributions and Singh, Kumar Mahto, and Mani Tripathi (2024) examines a POCR model for the Chen distribution.

Censoring is commonly applied when recording the failure times of all units is impractical due to time and cost constraints. Type-I and Type-II censoring methods are well established techniques for collecting lifetime data under resource constraints, where Type-I censoring terminates data collection at a predetermined time point, and Type-II censoring concludes the process after a specified number of failure events have been observed. However, these schemes have drawbacks: a Type-I censoring scheme might result in no failure items, and a Type-II censoring scheme can lead to prolonged experiment durations. To address the drawbacks of Type-I and Type-II censoring schemes, Epstein (1954) introduced a combination of these methods called the Type-I hybrid censoring scheme. In this censoring scheme, if X_m denotes the m^{th} failure time and T is the predetermined test termination time, then the test concludes at the random time $T^* = \min(X_m, T)$. Childs, Chandrasekar, Balakrishnan, and Kundu (2003) proposed the Type-II hybrid censoring scheme, which terminates the experiment at $T^* = \max(X_m, T)$. It was later observed that Type-I hybrid and Type-II hybrid censoring schemes also exhibit similar drawbacks to those of the original Type-I and Type-II censoring schemes. The Type-I hybrid censoring scheme ensures that the total experimental duration will not exceed T, however, it does not guarantee a sufficient number of failures. Conversely, the Type-II hybrid censoring scheme guarantees that the total number of observed failures is m, but it imposes no restriction on the experimental duration.

To overcome these drawbacks, Chandrasekar, Childs, and Balakrishnan (2004) introduced two more comprehensive censoring schemes called generalized hybrid censoring schemes (GHCS).

The experiments under GHCS ensure both proper control within a defined testing period and the occurrence of at least a fixed number of failures. Statistical inference is more efficient under GHCS because it provides a greater number of observed failures. The basic forms of GHCS are categorized as type-I and type-II GHCS, are expressed as follows:
1. Generalized Type-I Hybrid Censoring Scheme - In the generalized Type-I hybrid censoring scheme (GTIHCS), n experimental units are subjected to a life test. The experimenter pre-selects two numbers, s and m, such that $s < m \leq n$, as well as a time T within the range $(0, \infty)$. If the s^{th} failure occurs before the specified time T, the experiment will be terminated at $T^* = \min(X_m, T)$. If the s^{th} failure occurs after T, the experiment will conclude at T_s, where T_s is the failure time of the s^{th} unit.

2. Generalized Type-II Hybrid Censoring Scheme - In a generalized Type-II hybrid censoring scheme (GTIIHCS), a predefined number m (where m $\leq$ n) and thresholds time T_1 and T_2 are set, with $T_1 < T_2$. If the m^{th} failure happens prior to T_1, the experiment will conclude at T_1. If the m^{th} failure happens within the interval T_1 to T_2, then the experiment will conclude at T_m. If the m^{th} failure happens beyond T_2, then the experiment will conclude at T_2. In GTIIHCS, the experiment is ensured to conclude by time T_2. Then, the following cases are identified under GTIIHCS.

$$\text{Case I} : \text{If } 0 < X_{m:n} < T_1 < T_2, \text{ then } T^* = T_1.$$
$$\text{Case II} : \text{If } 0 < T_1 < X_{m:n} < T_2, \text{ then } T^* = X_{m:n}.$$
$$\text{Case III} : \text{If } 0 < T_1 < T_2 < X_{m:n}, \text{ then } T^* = T_2.$$

The experiment is designed to ensure that it concludes within a predefined maximum duration, with T_2 serving as the upper limit for the experiment's timeline. The GTIIHCS is highly advantageous in applications like reliability testing, medical devices, and clinical trials. By combining failure-based and time-based stopping rules, it ensures either a pre-specified number of failures or termination within a fixed duration. This flexibility allows GTIIHCS to overcome the limitations of traditional Type-I and Type-II censoring schemes. Type-I censoring, which terminates experiments at a fixed time, may result in insufficient failure data and low inference precision, while Type-II censoring ensures a fixed number of failures but often requires excessively long testing periods. GTIIHCS resolves these challenges by balancing test efficiency and data reliability. It provides better control over the testing process and improves the accuracy of statistical inferences, making it an ideal choice for real-world applications with strict time or resource constraints. Numerous researchers have investigated the estimation problem for various statistical models, highlighting the importance of the GTIIHCS. Shafay (2016) examined Bayesian estimation and prediction using GTIIHCS. Mahmoud and Ghazal (2017) estimated the unknown parameters of the exponentiated rayleigh distribution using generalized Type-II hybrid censored data. Rabie and Li (2019) derived maximum likelihood, Bayesian, and E-Bayesian estimators for the unknown shape parameter of the Burr-X distribution under GTIIHCS. Abushal and AL-Zaydi (2024) conducted inference on unknown parameters using GTIIHCS for the inverse Nadarajah–Haghighi distribution in the context of competing risks.

The Weibull distribution has gained popularity in life-testing analysis due to its ability to accommodate various shapes of the probability density function and its flexibility in modeling the hazard rate function (see Lee and Lee (1978), Yu, Tian, and Tang (2008)). Its versatility makes it highly applicable in fields like engineering, medical research, and life testing. In this study, we assume the presence of two distinct competing risk factors influencing the failure of experimental units. The latent failure times are modeled using Weibull distributions with a common shape parameter α and different scale parameters λ_1 and λ_2, where $\lambda_1 \neq \lambda_2$. Let X_{ji} represent the i^{th} latent failure time under the j^{th} cause, where $i = 1, 2, \ldots, n$ following a Weibull distribution. Then the corresponding probability density function (PDF) and cumulative distribution function (CDF) can be expressed as follows:

$$f_j(x; \alpha, \lambda_j) = \alpha \lambda_j x^{\alpha-1} e^{-\lambda_j x^\alpha} \text{ and } F_j(x; \alpha, \lambda_j) = 1 - e^{-\lambda_j x^\alpha}; \; x > 0, \; j = 1, 2. \quad (1)$$

Competing risk models are widely used in life testing experiments because they effectively handle situations where failures can arise from multiple causes. Drawing conclusions about unknown quantities is essential, especially when some failure causes are partially unobserved. The Weibull distribution, known for its ability to exhibit decreasing, constant, or increasing hazard rates, is highly adaptable for modeling various lifetime data and has become one of the most commonly used distributions in lifetime analysis. In this study, we examine the Weibull model within the POCR framework. Classical and Bayesian estimation methods are used to obtain parameter estimates under the GTIIHC scheme. The performance of the estimators is assessed using both simulated and real datasets, demonstrating their practical applicability and effectiveness.

The structure of this paper is organized as follows: Section 2 outlines the model assumptions for POCR data under the GTIIHCS framework. Section 3 discusses the maximum likelihood estimation (MLE) of unknown parameters, along with approximate confidence intervals. Bayesian estimation techniques using the importance sampling method are explored in Section 4. Section 5 provides a Monte Carlo simulation study, while Section 6 includes a real-world example for illustration. Finally, a concluding remark for the paper is presented in Section 7.

2. Model framework

Consider an experiment where n independent units are tested, with their respective lifetimes denoted by random variables $X_1,\ X_2,\ \ldots,\ X_n$. Under the POCR model, we consider two independent causes of failure. Then, the latent failure times resulting from two independent causes of failure for any unit are defined as,

$$X_i = \min\{X_{1i}, X_{2i}\} \quad i = 1, 2, \ldots, n. \tag{2}$$

where X_{ji} represents the failure time of the i^{th} unit due to j^{th} cause. Under GTIIHC, if T_1, T_2, and m denote the test duration time and the predetermined number of observed failures, then the experiment concludes at time T^*, which can be expressed as follows for the three scenarios:

$$T^* = \begin{cases} T_1, & 0 < X_{m:n} < T_1, \\ X_{m:n}, & T_1 < X_{m:n} < T_2, \\ T_2, & T_1 < T_2 < X_{m:n}, \end{cases} \tag{3}$$

where $X_{1:n}, \ldots, X_{D:n}$ are the generalized Type-II Hybrid censoring sample corresponding to $X_1, X_2,\ \ldots,\ X_n$. Then the data on POCR under GTIIHCS can be described as follows:

$$\begin{aligned} &\text{Case I: } (X_{1:n}, \xi_1), \ldots, (X_{m_1:n}, \xi_{m_1}),\ \text{if } 0 < X_{m:n} < T_1 \text{ and } T^* = T_1, \\ &\text{Case II: } (X_{1:n}, \xi_1), \ldots, (X_{m:n}, \xi_m),\ \text{if } T_1 < X_{m:n} < T_2 \text{ and } T^* = X_{m:n}, \\ &\text{Case III: } (X_{1:n}, \xi_1), \ldots, (X_{m_2:n}, \xi_{m_2}),\ \text{if } T_1 < T_2 < X_{m:n} \text{ and } T^* = T_2, \end{aligned} \tag{4}$$

where T^* represents the termination point of the experiment, while m_1 and m_2 are two positive integers, with $x_{m_1:n} < T_1 < x_{m_1+1:n}$ and $x_{m_2:n} < T_2 < x_{m_2+1:n}$. Assume that the causes of failures are independent. Then, the observed failure times are $(x_{1:n}, \xi_1), (x_{2:n}, \xi_2), \ldots, (x_{D:n}, \xi_D)$ and the corresponding cause of failure is represented by an indicator function ξ_i such that $\xi_i = j$, j = 1, 2, 3, i = 1, 2, . . . , D where,

$$\xi_i = \begin{cases} 1, & \text{if the failure occurs due to cause 1}, \\ 2, & \text{if the failure occurs due to cause 2}, \\ 3, & \text{if the cause of the failure cannot be determined}. \end{cases}$$

Considering the POCR data in (4), then the likelihood function for the observed data $\underline{x} = (x_{1:n}, \xi_1), \ldots, (x_{D:n}, \xi_D)$ can be expressed as,

$$L(\underline{x}) = \frac{n!}{(n-D)!} \prod_{i=1}^{D} [f_1(x_i)S_2(x_i)]^{I(\xi_i=1)} [f_2(x_i)S_1(x_i)]^{I(\xi_i=2)} [f(x_i)]^{I(\xi_i=3)} S(T^*)^{n-D}, \quad (5)$$

where f(.) and S(.) are the density and survival functions of $X_i = \min\{X_{1i}, X_{2i}\}$ and D represents the total number of failures under the following cases,

$$D = \begin{cases} m_1, & \text{Case I,} \\ m, & \text{Cases II ,} \\ m_2, & \text{Cases III .} \end{cases}$$

Also, we define the indicator function

$$I(\xi_i = j) = \begin{cases} 1, & \text{if } \xi_i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $D_1 = \sum_{i=1}^{D} I(\xi_i = 1)$ and $D_2 = \sum_{i=1}^{D} I(\xi_i = 2)$ represent the total number of failures due to cause 1 and cause 2. Also, $D_3 = \sum_{i=1}^{D} I(\xi_i = 3)$ represents the number of failures with unobserved cause. Now, consider that the generalized type-II hybrid censoring competing risks data (4) are derived from Weibull distributions with two independent failure causes such that $X_{1i} \sim$W(α, λ_1) and $X_{2i} \sim$W(α, λ_2) for i = 1,2, ..., n. Consequently, the observations $X_i = \min\{X_{1i}, X_{2i}\}$ for i= 1, 2, ..., n are also independent and follow Weibull distribution with parameters (α, λ_1+λ_2).

Remark 1. *If the failure times X_1 and X_2 are independent and identically distributed random variables following Weibull distributions with parameters (α, λ_1) and (α, λ_2) respectively, then the random variable $X = \min(X_1, X_2)$ follows a Weibull distribution with parameters (α, λ_1+λ_2), where α represents the shape parameter and (λ_1+λ_2) denotes the scale parameter. The reliability function of the random variable X, denoted as $\bar{F}_X(x)$, is provided as follows,*

$$\begin{aligned} \bar{F}_X(x) &= P\left(\min(X_1, X_2) > x\right) \\ &= P\left(X_1 > x\right) P\left(X_2 > x\right) \\ &= e^{-(\lambda_1+\lambda_2)x^{\alpha}}. \end{aligned}$$

Then, the corresponding distribution function $F_X(x)$ and the probability density function $f_X(x)$ are given by,

$$F_X(x) = 1 - e^{-(\lambda_1+\lambda_2)x^{\alpha}} \textit{ and } f_X(x) = \alpha(\lambda_1 + \lambda_2)x^{\alpha-1}e^{-(\lambda_1+\lambda_2)x^{\alpha}}. \quad (6)$$

The likelihood functionof λ_1, λ_2 and α can be formulated based on equation (5) as:

$$L(\underline{x}) \propto \alpha^D \lambda_1^{D_1} \lambda_2^{D_2} (\lambda_1 + \lambda_2)^{D_3} e^{-(\lambda_1+\lambda_2)(T^*)^{\alpha}(n-D)} \prod_{i=1}^{D} x_i^{\alpha-1} e^{-(\lambda_1+\lambda_2)x_i^{\alpha}}. \quad (7)$$

Thus, certain comments made on this model were considered as,

1. The proposed model indicates that the failure time is noted for certain units with an unknown cause of failure. Then, the latent failure time follows a Weibull distribution with the scale parameter $\lambda_1 + \lambda_2$ and shape parameter α.

2. The observed numbers of failures D_1 and D_2 for the first and second causes follow binomial distributions with sample size (D-D_3) and probability of success $\frac{\lambda_1}{\lambda_1+\lambda_2}$ and $\frac{\lambda_2}{\lambda_1+\lambda_2}$, respectively.

3. The discrete random variable D_3 follows a Bernoulli distribution with a masking probability p, where $0 \leq p \leq 1$. Therefore, the values 0 and 1 represent failures with unknown and known causes, respectively.

3. Maximum likelihood estimation

According to Equation (7), the log-likelihood function is expressed as follows:

$$\begin{aligned} L(\lambda_1, \lambda_2, \alpha \mid \underline{x}) &= D \ln \alpha + D_1 \ln \lambda_1 + D_2 \ln \lambda_2 + D_3 \ln(\lambda_1 + \lambda_2) \\ &+ (\alpha - 1) \sum_{i=1}^{D} \ln x_i - (\lambda_1 + \lambda_2) Z(\alpha), \end{aligned} \tag{8}$$

where Z(α)= $\sum_{i=1}^{D} x_i^{\alpha} + (n - D)(T^*)^{\alpha}$.

Theorem 1. *Given that $D_j \geq 1$, the MLE of λ_j, when α is known, can be represented as:*

$$\hat{\lambda}_j = \frac{D_j}{Z(\alpha)} \left[\frac{D_3}{D_1 + D_2} + 1 \right], j = 1, 2.$$

Proof. By taking the derivatives of $L = L(\lambda_1, \lambda_2, \alpha \mid \underline{x})$ in (8) with respect to λ_1 and λ_2, and then setting these derivatives to zero, we obtain the following equations,

$$\lambda_j = \frac{D_j}{Z(\alpha)} \left[\frac{D_3}{D_1 + D_2} + 1 \right], j = 1, 2.$$

To demonstrate that the MLEs of $\hat{\lambda}_1$ and $\hat{\lambda}_2$ maximize L for a given α, we proceed as follows. Let $H(\lambda_1, \lambda_2)$ represent the Hessian matrix of L at (λ_1, λ_2). Given that,
$H_{jj}(\lambda_1, \lambda_2) = \frac{\partial^2 L}{\partial \lambda_j^2} = -\frac{D_j}{\lambda_j^2} - \frac{D_3}{(\lambda_1 + \lambda_2)^2}, j = 1, 2.$
$H_{12}(\lambda_1, \lambda_2) = -\frac{D_3}{(\lambda_1 + \lambda_2)^2}.$
Thus, the determinant of the Hessian matrix at $(\hat{\lambda}_1, \hat{\lambda}_2)$ is,
$\det(H) = H_{11}(\hat{\lambda}_1, \hat{\lambda}_2) H_{22}(\hat{\lambda}_1, \hat{\lambda}_2) - H_{12}(\hat{\lambda}_1, \hat{\lambda}_2)^2 = \frac{D_1 D_2}{\hat{\lambda}_1^2 \hat{\lambda}_2^2} + \frac{D_1 D_3}{\hat{\lambda}_1^2 (\hat{\lambda}_1 + \hat{\lambda}_2)^2} + \frac{D_2 D_3}{\hat{\lambda}_2^2 (\hat{\lambda}_1 + \hat{\lambda}_2)^2} > 0.$
Therefore, ($\hat{\lambda}_1$, $\hat{\lambda}_2$) represents a local maximum of L for a given α. As there is no singular point of L and it possesses only one critical point, $\hat{\lambda}_1$ and $\hat{\lambda}_2$ emerge as the absolute maximum of L. Thus, the assertion stands proven. □

Remark 2. *As noted in Theorem 1, at least one failure due to the j^{th} cause is required for the MLE of λ_j to be determined, where j = 1,2. If $D_j = 0$, meaning there are no failures from the j^{th} cause or there is no information available about λ_j from the observed competing risks data. Therefore, if $D_j = 0$, the MLE of λ_j does not exist. Also, the ordinary competing risks model is achieved when $D_3 = 0$.*

Theorem 2. *Assume that $D_j \geq 1$ for j = 1,2, the MLE of α exists and is unique, which is the solution of the following equation,*

$$\frac{1}{\alpha} + \frac{\sum_{i=1}^{D} \ln x_i}{D} = \frac{Z^{'}(\alpha)}{Z(\alpha)}, \tag{9}$$

with $Z^{'}(\alpha) = \sum_{i=1}^{D} x_i^{\alpha} \ln(x_i) + (n - D) T^{*^{\alpha}} \ln(T^*)$.

Proof. By substituting $\hat{\lambda}_j$ for λ_j in (8), the profile log-likelihood function of α can be derived as follows.

$$L(\alpha) \propto D \ln \alpha + \alpha \sum_{i=1}^{D} \ln x_i - D \ln Z(\alpha).$$

Then, by differentiating L(α) with respect to α and setting it equal to zero, we obtain the following:

$$\frac{1}{\alpha} + \frac{\sum_{i=1}^{D} \ln x_i}{D} = \frac{Z'(\alpha)}{Z(\alpha)}.$$

Let $G_1(\alpha) = \frac{1}{\alpha} + \frac{\sum_{i=1}^{D} \ln x_i}{D}$ and $G_2(\alpha) = \frac{Z'(\alpha)}{Z(\alpha)}$.
Here $G_1'(\alpha) = -\frac{1}{\alpha^2} < 0$, which implies that $G_1(\alpha)$ is decreasing in the range $\left(\frac{1}{D}\sum_{i=1}^{D} \ln x_i, \infty\right)$.
In the case of $G_2(\alpha)$, we have the following:

$$G_2'(\alpha) = \frac{1}{[Z(\alpha)]^2}\Bigg[\left(\sum_{i=1}^{D} x_i^{\alpha} + (n-D)T^{*^{\alpha}}\right)\left(\sum_{i=1}^{D} x_i^{\alpha}(\ln(x_i))^2 + (n-D)T^{*^{\alpha}}(\ln(T^*))^2\right) - \left(\sum_{i=1}^{D} x_i^{\alpha}\ln(x_i) + (n-D)T^{*^{\alpha}}\ln(T^*)\right)^2\Bigg].$$

Applying the Cauchy–Schwarz inequality, we obtain $G_2'(\alpha) > 0$, indicating that $G_2(\alpha)$ is an increasing function of α with,

$$\lim_{\alpha\to 0} G_2(\alpha) = \frac{1}{n}\left[\sum_{i=1}^{D} \ln x_i + (n-D)\ln T^*\right],$$
$$\lim_{\alpha\to +\infty} G_2(\alpha) = \ln T^*.$$

Also,

$$\lim_{\alpha\to\infty} \frac{G_1(\alpha)}{G_2(\alpha)} = \frac{1}{D}\frac{\sum_{i=1}^{D} \ln x_i}{\ln T^*} < 1.$$

We note that the curves $G_1(\alpha)$ and $G_2(\alpha)$ intersect at a unique point, indicating that the MLE of α exists and is unique. □

Also, it is important to note that the MLE $\hat{\alpha}$ for α does not have any closed-form expression. It can be evaluated using the fixed-point iterative method derived from the non-linear equation (9) as,

$$B(\alpha^{(s)}) = \alpha^{(s+1)},$$

where $\left[B(\alpha) = \frac{Z'(\alpha)}{Z(\alpha)} - \frac{\sum_{i=1}^{D} \ln x_i}{D}\right]^{-1}$. Here, $\alpha^{(s)}$ represents the s^{th} iteration of $\hat{\alpha}$. The process is terminated when the difference $\mid \alpha^{(s)} - \alpha^{(s+1)} \mid$ is sufficiently small. Thus, the MLEs of λ_j denoted as $\hat{\lambda}_j$, j = 1,2 can be derived from Theorem 1 as,

$$\hat{\lambda}_j = \frac{D_j}{Z(\hat{\alpha})}\left[\frac{D_3}{D_1 + D_2} + 1\right], j = 1, 2.$$

3.1. Approximate confidence interval

In this section, we establish approximate confidence intervals (CIs) for the parameters by exploiting the asymptotic normality property of MLEs. Let's assume that $\theta = (\lambda_1,\ \lambda_2,\ \alpha)$, where $\theta_1 = \lambda_1$, $\theta_2 = \lambda_2$, and $\theta_3 = \alpha$. In this case, the observed Fisher information matrix can be expressed as:

$$I(\theta) = \left[-\frac{\partial^2 L(\lambda_1, \lambda_2, \alpha \mid \underline{x})}{\partial\theta_j \partial\theta_s}\right]_{\theta_j = \hat{\theta}_j, j,s=1,2,3},$$

with

$$-\frac{\partial^2 L}{\partial \lambda_1^2} = \frac{D_1}{\lambda_1^2} + \frac{D_3}{(\lambda_1+\lambda_2)^2}.$$

$$-\frac{\partial^2 L}{\partial \lambda_2^2} = \frac{D_2}{\lambda_2^2} + \frac{D_3}{(\lambda_1+\lambda_2)^2}.$$

$$-\frac{\partial^2 L}{\partial \alpha^2} = \frac{D}{\alpha^2} + (\lambda_1+\lambda_2)\left[\sum_{i=1}^{D} x_i^{\alpha}(\ln x_i)^2 + (n-D)(T^*)^{\alpha}(\ln(T^*))^2\right].$$

$$-\frac{\partial^2 L}{\partial \lambda_1 \partial \lambda_2} = -\frac{\partial^2 L}{\partial \lambda_2 \partial \lambda_1} = \frac{D_3}{(\lambda_1+\lambda_2)^2}.$$

$$-\frac{\partial^2 L}{\partial \lambda_1 \partial \alpha} = -\frac{\partial^2 L}{\partial \alpha \partial \lambda_1} = \sum_{i=1}^{D} x_i^{\alpha}\ln x_i + (n-D)(T^*)^{\alpha}\ln(T^*).$$

$$-\frac{\partial^2 L}{\partial \lambda_2 \partial \alpha} = -\frac{\partial^2 L}{\partial \alpha \partial \lambda_2} = \sum_{i=1}^{D} x_i^{\alpha}\ln x_i + (n-D)(T^*)^{\alpha}\ln(T^*).$$

As established in Theorem 7.63 [pp. 421] of Schervish (2012), it can be directly verified that the likelihood function (7) meets the regularity conditions, including the existence of continuous second partial derivatives, the interchangeability of differentiation and integration, and a finite mean for the Fisher information matrix, etc. Hence, using the asymptotic theory of MLEs, the approximate distribution of the MLE $\hat{\theta}$ is given by, $\hat{\theta} - \theta \sim N(0, I^{-1}(\theta))$, where $I^{-1}(\theta)$ represents the inverse of the observed information matrix, which is defined as:

$$I^{-1}(\hat{\theta}) = \begin{pmatrix} \mathrm{var}(\hat{\lambda}_1) & \mathrm{cov}(\hat{\lambda}_1,\hat{\lambda}_2) & \mathrm{cov}(\hat{\lambda}_1,\hat{\alpha}) \\ \mathrm{cov}(\hat{\lambda}_2,\hat{\lambda}_1) & \mathrm{var}(\hat{\lambda}_2) & \mathrm{cov}(\hat{\lambda}_2,\hat{\alpha}) \\ \mathrm{cov}(\hat{\alpha},\hat{\lambda}_1) & \mathrm{cov}(\hat{\alpha},\hat{\lambda}_2) & \mathrm{var}(\hat{\alpha}) \end{pmatrix} \tag{10}$$

Therefore, for any $0 < \nu < 1$, a 100(1-ν)% asymptotic confidence interval (ACI) for θ_j is given by,

$$\left(\hat{\theta}_j - Z_{\nu/2}\sqrt{\mathrm{var}(\hat{\theta}_j)}, \hat{\theta}_j + Z_{\nu/2}\sqrt{\mathrm{var}(\hat{\theta}_j)}\right); j = 1, 2, 3,$$

where $Z_{\nu/2}$ is the upper $\frac{\nu}{2}$ quantile of the standard normal distribution.

4. Bayesian estimation

The Bayesian estimation method serves as a strong alternative to the classical estimation approach by treating unknown parameters as random variables and combining prior knowledge with sample data for inference. This section covers the Bayesian approach for obtaining point and interval estimates of the unknown model parameters. In the Bayesian framework, prior information about the unknown parameter is necessary before making any inferences using the likelihood function. This information can be either complete or incomplete, depending on the choice of the prior distribution. In a non-informative prior, little or no information is available about the unknown parameter, whereas an informative prior provides sufficient information to quantify the uncertainty associated with the parameter. Therefore, the prior is chosen in such a way that it does not significantly alter the model estimates or influence the model selection. It is worth noting that there is no well-established method for choosing an appropriate prior in Bayesian analysis. The gamma distribution is highly adaptable, capable of modeling various shapes of the density function. Its density function is log-concave over the interval (0, ∞). Jeffrey's prior can be regarded as a specific case of the gamma prior. The gamma distribution is commonly used as an informative prior in various lifetime models due to its simplicity and ease of computation. Due to the relevance of gamma distributions, several authors have recently employed gamma priors to obtain Bayesian estimates for the

Weibull competing risk model (see Ashour and Nassar (2017), Chacko and Mohan (2019) etc.). In a similar way, we also consider the gamma priors for the parameters λ_1, λ_2 and α. Then, the joint prior density function of λ_1, λ_2, and α can be expressed as follows:

$$\pi(\lambda_1, \lambda_2, \alpha) \propto \lambda_1^{a_1-1}\lambda_2^{a_2-1}\alpha^{a_3-1}\exp(-b_1\lambda_1 - b_2\lambda_2 - b_3\alpha), \text{ for } \lambda_1, \lambda_2, \alpha > 0, a_i, b_i > 0, \quad (11)$$

for i = 1, 2, 3. Then, the corresponding posterior distribution is formulated as follows:

$$\pi^*(\lambda_1, \lambda_2, \alpha \mid \underline{x}) = \frac{\pi(\lambda_1, \lambda_2, \alpha)L(\lambda_1, \lambda_2, \alpha \mid \underline{x})}{\int_0^\infty \int_0^\infty \int_0^\infty \pi(\lambda_1, \lambda_2, \alpha)L(\lambda_1, \lambda_2, \alpha \mid \underline{x})d\lambda_1 d\lambda_2 d\alpha}. \quad (12)$$

Then, the Bayes estimates of any function of λ_1, λ_2 and α, represented as $\eta(\lambda_1, \lambda_2, \alpha)$, under the squared error loss function is given by,

$$\hat{\eta}_{SE} = \frac{\int_0^\infty \int_0^\infty \int_0^\infty \eta(\lambda_1, \lambda_2, \alpha)\pi(\lambda_1, \lambda_2, \alpha)L(\lambda_1, \lambda_2, \alpha \mid \underline{x})d\lambda_1 d\lambda_2 d\alpha}{\int_0^\infty \int_0^\infty \int_0^\infty \pi(\lambda_1, \lambda_2, \alpha)L(\lambda_1, \lambda_2, \alpha \mid \underline{x})d\lambda_1 d\lambda_2 d\alpha}. \quad (13)$$

From equation (13), it is evident that the Bayes estimates of $\eta(\lambda_1, \lambda_2, \alpha)$ with respect to the squared error loss function cannot be obtained explicitly. Therefore, an approximation technique like the MCMC technique is needed to compute the desired Bayes estimates. The MCMC method can generate samples from the posterior density function (12), allowing us to compute the Bayes estimates of the unknown parameters and the corresponding credible intervals. This study employs MCMC with an importance sampling technique to compute Bayes estimators under the squared error loss function. Based on equations (7) and (11), the joint posterior density function of λ_1, λ_2, and α can be expressed as follows:

$$\begin{aligned} \pi^*(\lambda_1, \lambda_2, \alpha \mid \underline{x}) &\propto \alpha^{D+a_3-1}\lambda_1^{D_1+a_1-1}\lambda_2^{D_2+a_2-1}(\lambda_1+\lambda_2)^{D_3} \\ &\times \exp\left(-b_1\lambda_1 - b_2\lambda_2 - b_3\alpha + (\alpha-1)\sum_{i=1}^{D}\ln x_i - (\lambda_1+\lambda_2)Z(\alpha)\right). \end{aligned} \quad (14)$$

Then, the marginal posterior distributions of λ_1, λ_2 and α from equation (14) are expressed as follows:

$$\pi_1^*(\lambda_1 \mid \alpha, \underline{x}) \propto Gamma(D_1 + a_1, b_1 + Z(\alpha)), \quad (15)$$

$$\pi_2^*(\lambda_2 \mid \alpha, \underline{x}) \propto Gamma(D_2 + a_2, b_2 + Z(\alpha)), \quad (16)$$

$$\pi_3^*(\alpha \mid \underline{x}) \propto \alpha^{D+a_3-1}\exp\left(-b_3\alpha + \alpha\sum_{i=1}^{D}\ln x_i\right), \quad (17)$$

with the associated weight,

$$W(\lambda_1, \lambda_2, \alpha, \mid \underline{x}) = \frac{(\lambda_1+\lambda_2)^{D_3}}{[b_1 + Z(\alpha)]^{D_1+a_1}[b_2 + Z(\alpha)]^{D_2+a_2}}.$$

Here, we present the importance sampling procedure for generating random samples from posterior distribution and to compute the Bayes estimates. The procedure is outlined as follows:

Step1: Set k = 1 and start with an initial guess for $\Theta^{(0)} = (\hat{\lambda}_1, \hat{\lambda}_2, \hat{\alpha})$.

Step2: Generate $\lambda_1^{(k)}$ and $\lambda_2^{(k)}$ from the gamma distributions specified in equations (15) and (16), respectively.

Step3: Generate $\alpha^{(k)}$ from equation (17) using the Metropolis-Hastings (MH) algorithm with a normal proposal distribution with mean $\alpha^{(k-1)}$ and variance σ^2 , where σ^2 is derived from a variance-covariance matrix, as follows:

(a): Generate α^* from the normal distribution $N(\alpha^{(k-1)}, \sigma^2)$ as the proposal distribution.

(b): Calculate the acceptance probability from equation (17) as
$P(\alpha^*,\, \alpha^{(k-1)}) = \min\left[1, \frac{\pi_3^*(\alpha^*|\lambda_1^{(k)},\lambda_2^{(k)},\underline{x})}{\pi_3^*(\alpha^{(k-1)}|\lambda_1^{(k)},\lambda_2^{(k)},\underline{x})}\right]$.
(c): Generate U_k from uniform(0, 1).
(d): If $U_k \leq P(\alpha^*,\, \alpha^{(k-1)})$, take $\alpha^{(k)} = \alpha^*$; Otherwise $\alpha^{(k)} = \alpha^{(k-1)}$.
Step4: Assign k = k+1.
Step5: Repeat steps 2 - 4 M times.
Step6: Considering that M^* represents the number of MCMC iterations needed to reach the stationary distribution, the Bayes estimate of any function g$(\lambda_1, \lambda_2, \alpha)$ of the model parameters under the squared error loss function is given by,

$$\hat{g}_B = \frac{\frac{1}{M-M^*}\sum_{i=M^*+1}^{M} g(\lambda_1^{(i)}, \lambda_2^{(i)}, \alpha^{(i)})W(\lambda_1^{(i)}, \lambda_2^{(i)}, \alpha^{(i)} \mid \underline{x})}{\frac{1}{M-M^*}\sum_{i=M^*+1}^{M} W(\lambda_1^{(i)}, \lambda_2^{(i)}, \alpha^{(i)} \mid \underline{x})}, \tag{18}$$

with the associated posterior variance is,

$$\hat{V}(\lambda_1, \lambda_2, \alpha) = \frac{\frac{1}{M-M^*}\sum_{i=M^*+1}^{M} \left(g(\lambda_1^{(i)}, \lambda_2^{(i)}, \alpha^{(i)}) - \hat{g}_B\right)^2 W(\lambda_1^{(i)}, \lambda_2^{(i)}, \alpha^{(i)} \mid \underline{x})}{\frac{1}{M-M^*}\sum_{i=M^*+1}^{M} W(\lambda_1^{(i)}, \lambda_2^{(i)}, \alpha^{(i)} \mid \underline{x})}. \tag{19}$$

4.1. Bayesian credible interval

As stated in Chen and Shao (1999), the credible interval or HPD credible intervals for any function g$(\lambda_1,\, \lambda_2,\, \alpha)$ can be constructed as follows:
Step1: Sort $g(\lambda_1^{(i)}, \lambda_2^{(i)}, \alpha^{(i)})$ and the corresponding weighted function
$w^{(i)} = \frac{W(\lambda_1^{(i)},\, \lambda_2^{(i)},\, \alpha^{(i)}|\underline{x})}{\sum_{i=M^*+1}^{M} W(\lambda_1^{(i)},\, \lambda_2^{(i)},\, \alpha^{(i)}|\underline{x})}$ are denoted as $g_{(i)}$ and $w_{(i)}$ for i $= M^*+1, M^*+2, \ldots, M$, obtained from the importance sampling technique.
Step2: The marginal posterior of g given in the ordered pairs (g_i, w_i) can be defined as follows.

$$\hat{g}(\gamma) = \begin{cases} g_{(1)}, & \text{if } \gamma = 0 \\ g_{(k)}, & \text{if } \sum_{i=1}^{k-1} w^{(i)} < \gamma < \sum_{i=1}^{k} w^{(i)} \end{cases}$$

Step3: The estimated $100(1-\gamma)\%$ credible interval for $g(\lambda_1, \lambda_2, \alpha)$ is represented as $\left(g\left(\frac{\gamma}{2}\right), g\left(1-\frac{\gamma}{2}\right)\right)$.
Step4: The estimated $100(1-\gamma)\%$ HPD credible interval for $g(\lambda_1, \lambda_2, \alpha)$ is given by $\left(g\left(\frac{L}{M-M^*}\right), g\left(\frac{L+[(1-\gamma)(M-M^*)]}{M-M^*}\right)\right)$, where L ranges from 1 to $\gamma(M-M^*)$. This interval is chosen for having the smallest width among all credible intervals.

5. Simulation

In this section, we carry out comprehensive Monte Carlo simulations to assess the effectiveness of the proposed estimation methods under GTIIHCS. The estimated mean squared errors (MSEs) are used to compare the point estimators of the parameters in competing risks models. Similarly, interval estimators are examined based on average lengths (AL) and their associated coverage probabilities (CP). The following algorithm illustrates the steps to generate Generalized Type-II hybrid censored competing risks data from a Weibull distribution:

1. Generate Type-II censored data from W(α, λ_1+λ_2).

2. For each Type-II data, allocate the cause of failure as either one or two, with probabilities of $\frac{\lambda_1}{\lambda_1+\lambda_2}$ and $\frac{\lambda_2}{\lambda_1+\lambda_2}$ respectively.

3. By comparing T_1, T_2 and $X_{m:n}$ with Type-II censored data, we obtain the Weibull generalized type-II hybrid censored competing risks data with sample size D.

4. Partially observed competing risks GTIIHC data can be obtained by generating a trial of 0 and 1 of size D using a Bernoulli distribution with parameter p, where $0 < p < 1$ represents the masking probability. If 0 appears, the associated failure cause is treated as unknown, whereas if 1 appears, the failure cause is considered known.

The true values of the model parameters for the selected simulation study are $(\lambda_1, \lambda_2, \alpha) = (0.4, 0.7, 1.5)$. Various censoring schemes are examined based on differing values of (n, m, T_1, T_2, p). For each censoring criterion, m values are selected as (15, 20) for sample size $n = 30$, (20, 30) for $n = 40$, and (30, 40) for $n = 60$, with masking probabilities $p = 0.05$ and 0.1. Also, we consider all results for prefixed time constraints $(T_1, T_2) = (0.4, 1)$ and (0.2, 1.3). Here, Bayes estimates are evaluated considering both an informative prior (IP) and a non-informative prior (NIP). The results of simulations for each point and interval estimate, are displayed in Tables 1 - 4. Interval estimates are computed using a nominal significance level of 0.05. Based on Tables 1 - 4, the following conclusions have been drawn:

- The average MSE of both the maximum likelihood estimates (MLEs) and Bayes estimates decline with increasing values of n and m.
- Also, the MSE of the MLE and Bayesian estimates increases as the masking probability p is permitted to increase.
- The average MSE values decrease as the values of the prefixed time increase.
- In terms of MSEs, the Bayes estimates under IP outperform the MLEs and the Bayes estimates under NIP.
- As the values of n amd m increase, the average length of the confidence intervals decreases.
- The average widths (AWs) of the intervals increase as the masking probability p increases, while keeping n and m constant.
- Similarly, the coverage probabilities (CPs) of these intervals closely approximate the nominal significance level in most instances. This behavior persists as p increases.

6. Data analysis

In this section, we explore a real data set for illustrative purposes, derived from an experiment by Dr. H.E. Walburg Jr. at the Oak Ridge National Laboratory (refer to Hoel (1972)). A set of male mice was administered a radiation dose of 300 roentgens at the age of 5-6 weeks. The causes of death were categorized into three groups: (1) Thymic Lymphoma, (2) Reticulum Cell Sarcoma, and (3) Other causes. For our analysis, we have designated Reticulum Cell Sarcoma as Cause-1 and grouped all other causes together as Cause-2. The data is shown in Table 6. For computational simplicity, we analyzed the data after dividing it by 1000. The transformed data sets are assumed to follow a Weibull distribution with a scale parameter $\lambda_1 + \lambda_2$ and a shape parameter α.

A goodness-of-fit test was conducted to evaluate whether the Weibull distribution is a suitable model for the given datasets. Specifically, the Kolmogorov-Smirnov (KS) statistic, along with the corresponding p-value, was used to assess the fit. The results of the goodness-of-fit test were compared with those from the Lognormal, Chen, and inverted exponential distributions. These datasets were previously analyzed in competing risks scenarios using the Chen (Al-Bossly (2022)) and inverted exponential (Farghal, Badr, Abu-Zinadah, and Abd-Elmougod (2023)) distributions under different censoring schemes. Table 5 demonstrates that the proposed model exhibits greater flexibility than the other three distributions. We

Table 1: Average estimates (AEs) and mean squared errors (MSEs) for the parameters (λ_1, λ_2, α) = (0.4, 0.7, 1.5) with (T_1, T_2) = (0.4, 1)

			MLE			IP			NIP		
p	n	m	λ_1	λ_2	α	λ_1	λ_2	α	λ_1	λ_2	α
0.05	30	15	0.5037	0.8989	1.7190	0.3967	0.6898	1.5761	0.3886	0.6849	1.5984
			0.1061	0.3017	0.2092	0.0329	0.0598	0.0824	0.0340	0.0613	0.0898
		20	0.4545	0.7983	1.6518	0.4083	0.7233	1.5168	0.4195	0.7295	1.5053
			0.0389	0.0902	0.1305	0.0269	0.0483	0.0706	0.0287	0.0507	0.0779
0.1	30	15	0.5039	0.8987	1.7190	0.4020	0.7010	1.5717	0.3982	0.6856	1.6184
			0.1079	0.3049	0.2092	0.0335	0.0631	0.0826	0.0345	0.0647	0.0924
		20	0.4620	0.7897	1.6677	0.4163	0.7313	1.5185	0.4203	0.7365	1.5352
			0.0405	0.0908	0.1401	0.0273	0.0493	0.0743	0.0290	0.0516	0.0806
0.05	40	20	0.4660	0.8097	1.6497	0.3823	0.6577	1.5587	0.3830	0.6586	1.5952
			0.0513	0.1046	0.1355	0.0217	0.0396	0.0590	0.0224	0.0432	0.0652
		30	0.4347	0.7618	1.6205	0.4119	0.7244	1.4315	0.4160	0.7163	1.4578
			0.0220	0.0387	0.0865	0.0187	0.0329	0.0441	0.0193	0.0336	0.0449
0.1	40	20	0.4670	0.8087	1.6497	0.3815	0.6733	1.5814	0.3778	0.6728	1.5839
			0.0520	0.1064	0.1355	0.0234	0.0404	0.0615	0.0236	0.0475	0.0669
		30	0.4333	0.7632	1.6205	0.4190	0.7302	1.4375	0.4188	0.7267	1.4551
			0.0226	0.0394	0.0865	0.0194	0.0336	0.0443	0.0198	0.0350	0.0469
0.05	60	30	0.4343	0.7672	1.5909	0.3762	0.6470	1.4011	0.3599	0.6521	1.3809
			0.0229	0.0580	0.0782	0.0132	0.0217	0.0206	0.0139	0.0247	0.0337
		40	0.4197	0.7336	1.5692	0.3964	0.6828	1.5052	0.4061	0.7049	1.5249
			0.0129	0.0246	0.0558	0.0117	0.0199	0.0201	0.0125	0.0216	0.0234
0.1	60	30	0.4343	0.7672	1.5909	0.3691	0.6349	1.3956	0.3603	0.6179	1.3724
			0.0249	0.0587	0.0782	0.0236	0.0230	0.0241	0.0286	0.0250	0.0345
		40	0.4191	0.7343	1.5692	0.3915	0.7031	1.5096	0.3961	0.7062	1.5468
			0.0133	0.0250	0.0558	0.0198	0.0212	0.0340	0.0231	0.0234	0.0357

Table 2: ALs and CPs for the parameters (λ_1, λ_2, α) = (0.4, 0.7, 1.5) with (T_1, T_2) = (0.4, 1)

			ACI			IP HPD			NIP HPD		
p	n	m	λ_1	λ_2	α	λ_1	λ_2	α	λ_1	λ_2	α
0.05	30	15	0.9878	1.4544	1.6142	0.5226	0.5918	1.0826	0.6415	0.8162	1.1727
			0.95	0.97	0.95	0.93	0.93	0.92	0.92	0.93	0.93
		20	0.7069	0.9680	1.3529	0.5113	0.5887	0.9354	0.5011	0.7056	1.0604
			0.96	0.97	0.96	0.93	0.94	0.95	0.94	0.92	0.94
0.1	30	15	0.9995	1.4633	1.6142	0.5740	0.8212	1.1511	0.6548	0.8665	1.3794
			0.94	0.96	0.95	0.93	0.92	0.91	0.93	0.94	0.95
		20	0.7229	0.9706	1.3417	0.5297	0.6991	1.1130	0.5629	0.7929	1.3811
			0.95	0.97	0.96	0.94	0.93	0.94	0.92	0.95	0.95
0.05	40	20	0.7605	1.0698	1.3397	0.4894	0.6586	1.0024	0.4939	0.7059	1.1663
			0.95	0.97	0.96	0.91	0.92	0.93	0.93	0.95	0.95
		30	0.5626	0.7548	1.1052	0.4871	0.6254	0.8388	0.4762	0.6273	1.1475
			0.95	0.96	0.95	0.92	0.93	0.94	0.94	0.96	0.97
0.1	40	20	0.7717	1.0768	1.3397	0.5303	0.5643	1.0686	0.5311	0.6542	1.1822
			0.95	0.97	0.96	0.96	0.95	0.96	0.97	0.95	0.96
		30	0.5711	0.7626	1.1052	0.5284	0.6128	0.8880	0.5301	0.6648	0.8747
			0.95	0.96	0.95	0.95	0.95	0.96	0.96	0.95	0.97
0.05	60	30	0.5683	0.8010	1.0549	0.4178	0.5420	0.8219	0.4235	0.5221	0.8662
			0.95	0.96	0.96	0.94	0.95	0.97	0.93	0.94	0.95
		40	0.4530	0.6071	0.8908	0.3318	0.5027	0.7334	0.3710	0.4938	0.7369
			0.95	0.95	0.95	0.95	0.95	0.96	0.94	0.95	0.95
0.1	60	30	0.5769	0.8075	1.0549	0.4293	0.5689	1.0030	0.4305	0.5737	0.8716
			0.96	0.96	0.96	0.96	0.95	0.94	0.95	0.97	0.93
		40	0.4603	0.6133	0.8908	0.3594	0.5295	0.7362	0.3934	0.5862	0.7818
			0.95	0.96	0.95	0.97	0.96	0.95	0.96	0.96	0.95

Table 3: Average estimates (AEs) and mean squared errors (MSEs) for the parameters (λ_1, λ_2, α) = (0.4, 0.7, 1.5) with (T_1, T_2) = (0.2, 1.3)

			MLE			IP			NIP		
p	n	m	λ_1	λ_2	α	λ_1	λ_2	α	λ_1	λ_2	α
0.05	30	15	0.4983	0.8736	1.7104	0.3904	0.7101	1.5092	0.4003	0.7283	1.4997
			0.1027	0.2583	0.2196	0.0325	0.0632	0.0715	0.0334	0.0652	0.0734
		20	0.4435	0.787	1.6461	0.3994	0.7292	1.5747	0.3999	0.7563	1.6326
			0.0399	0.0968	0.1296	0.0248	0.0449	0.0725	0.0271	0.0535	0.0758
0.1	30	15	0.4988	0.8732	1.7100	0.3802	0.6944	1.5136	0.3841	0.7134	1.5559
			0.1043	0.2597	0.2196	0.0338	0.0647	0.0754	0.0364	0.0694	0.0856
		20	0.4447	0.7858	1.6461	0.4102	0.7428	1.5689	0.4169	0.7453	1.6126
			0.0413	0.0979	0.1296	0.0249	0.0488	0.0817	0.0275	0.0542	0.0951
0.05	40	20	0.4644	0.8112	1.6494	0.3823	0.6577	1.5495	0.3923	0.6702	1.5645
			0.0514	0.1060	0.1356	0.0217	0.0396	0.0550	0.0222	0.0397	0.0574
		30	0.4180	0.7465	1.5922	0.3986	0.6969	1.4519	0.3953	0.7124	1.4813
			0.0198	0.0351	0.0708	0.0160	0.0264	0.0420	0.0161	0.0300	0.0478
0.1	40	20	0.4632	0.8123	1.6494	0.3971	0.6859	1.5448	0.3909	0.6567	1.5974
			0.0516	0.1094	0.1356	0.0230	0.0410	0.0561	0.0231	0.0417	0.0586
		30	0.4173	0.7472	1.5922	0.4138	0.7199	1.4315	0.4155	0.7542	1.4961
			0.0204	0.0357	0.0708	0.0170	0.0291	0.0441	0.0185	0.0334	0.0509
0.05	60	30	0.4377	0.7636	1.5906	0.3606	0.6597	1.3831	0.3535	0.6470	1.4107
			0.0295	0.0489	0.0779	0.0131	0.0246	0.0332	0.0232	0.0480	0.0225
		40	0.4172	0.7270	1.5594	0.4136	0.6957	1.5234	0.3971	0.7174	1.5086
			0.0147	0.0251	0.0533	0.0115	0.0194	0.0203	0.0119	0.0216	0.0331
0.1	60	30	0.4376	0.7638	1.5906	0.3696	0.6582	1.3761	0.3555	0.6349	1.3516
			0.0299	0.0495	0.0779	0.0136	0.0248	0.0346	0.0249	0.0507	0.0890
		40	0.4176	0.7267	1.5594	0.4214	0.6873	1.5694	0.3987	0.7128	1.5251
			0.0152	0.0255	0.0533	0.0130	0.0197	0.0206	0.0226	0.0379	0.0371

Table 4: ALs and CPs for the parameters (λ_1, λ_2, α) = (0.4, 0.7, 1.5) with (T_1, T_2) = (0.2, 1.3)

			ACI			IP HPD			NIP HPD		
p	n	m	λ_1	λ_2	α	λ_1	λ_2	α	λ_1	λ_2	α
0.05	30	15	0.8914	1.3991	1.6049	0.5019	0.7000	1.0809	0.5610	0.7090	1.0840
			0.94	0.97	0.95	0.94	0.93	0.95	0.94	0.92	0.93
		20	0.6772	0.9301	1.2922	0.4756	0.6384	0.9512	0.4963	0.7028	1.0228
			0.95	0.96	0.95	0.93	0.95	0.93	0.94	0.93	0.94
0.1	30	15	0.9703	1.3037	1.6049	0.7011	0.8945	1.7862	0.6236	0.9388	1.9115
			0.93	0.98	0.97	0.92	0.93	0.94	0.93	0.92	0.95
		20	0.6869	0.9323	1.2922	0.6682	0.8506	1.0655	0.5016	0.7262	1.0829
			0.93	0.97	0.94	0.93	0.94	0.95	0.94	0.94	0.96
0.05	40	20	0.7582	1.0711	1.3391	0.4309	0.6003	0.7913	0.4401	0.6036	0.9484
			0.95	0.96	0.96	0.96	0.94	0.95	0.94	0.93	0.96
		30	0.5115	0.6877	1.0030	0.4200	0.5833	0.9790	0.4318	0.5874	0.8318
			0.94	0.96	0.95	0.97	0.95	0.96	0.95	0.94	0.95
0.1	40	20	0.7670	1.0800	1.3391	0.4423	0.6061	0.8004	0.4434	0.6085	0.9761
			0.94	0.96	0.96	0.93	0.95	0.95	0.94	0.96	0.94
		30	0.5197	0.6946	1.0030	0.4323	0.5813	0.9286	0.4348	0.5939	0.8458
			0.96	0.96	0.96	0.94	0.96	0.94	0.95	0.97	0.92
0.05	60	30	0.5725	0.7965	1.0546	0.4178	0.5027	0.8662	0.4200	0.4902	0.8668
			0.94	0.97	0.97	0.95	0.94	0.96	0.94	0.93	0.92
		40	0.4414	0.5898	0.8641	0.3576	0.4576	0.7681	0.3381	0.4933	0.7287
			0.93	0.96	0.95	0.93	0.98	0.97	0.94	0.93	0.97
0.1	60	30	0.5810	0.8031	1.0546	0.4305	0.5731	1.0038	0.4499	0.5860	0.8721
			0.95	0.97	0.97	0.96	0.95	0.96	0.93	0.96	0.93
		40	0.4491	0.5953	0.8641	0.4067	0.5295	0.8327	0.4076	0.5352	0.8649
			0.93	0.96	0.95	0.95	0.94	0.95	0.97	0.95	0.96

Table 5: Goodness of fit of datsets

	Dataset 1		Dataset 2	
	KS	p value	KS	p value
Weibull	**0.0733**	**0.9773**	**0.0817**	**0.8095**
Lognormal	0.1701	0.1974	0.1086	0.4675
Inverted exponetial	0.1006	0.7993	0.1507	0.1248
Chen	0.0759	0.9691	0.0848	0.7716

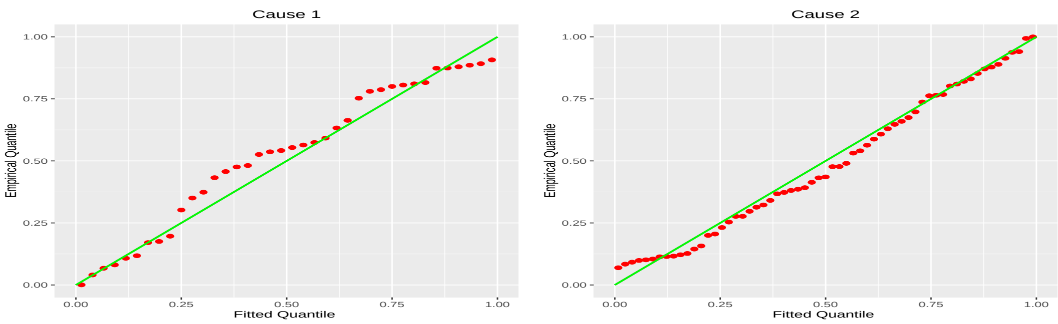

Figure 1: Q-Q plots for the dataset with two different causes

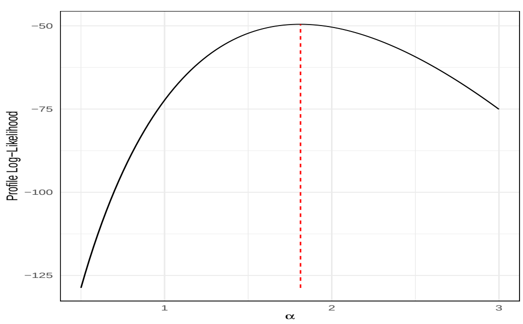

Figure 2: Plots of profile log-likelihood of α for the real data

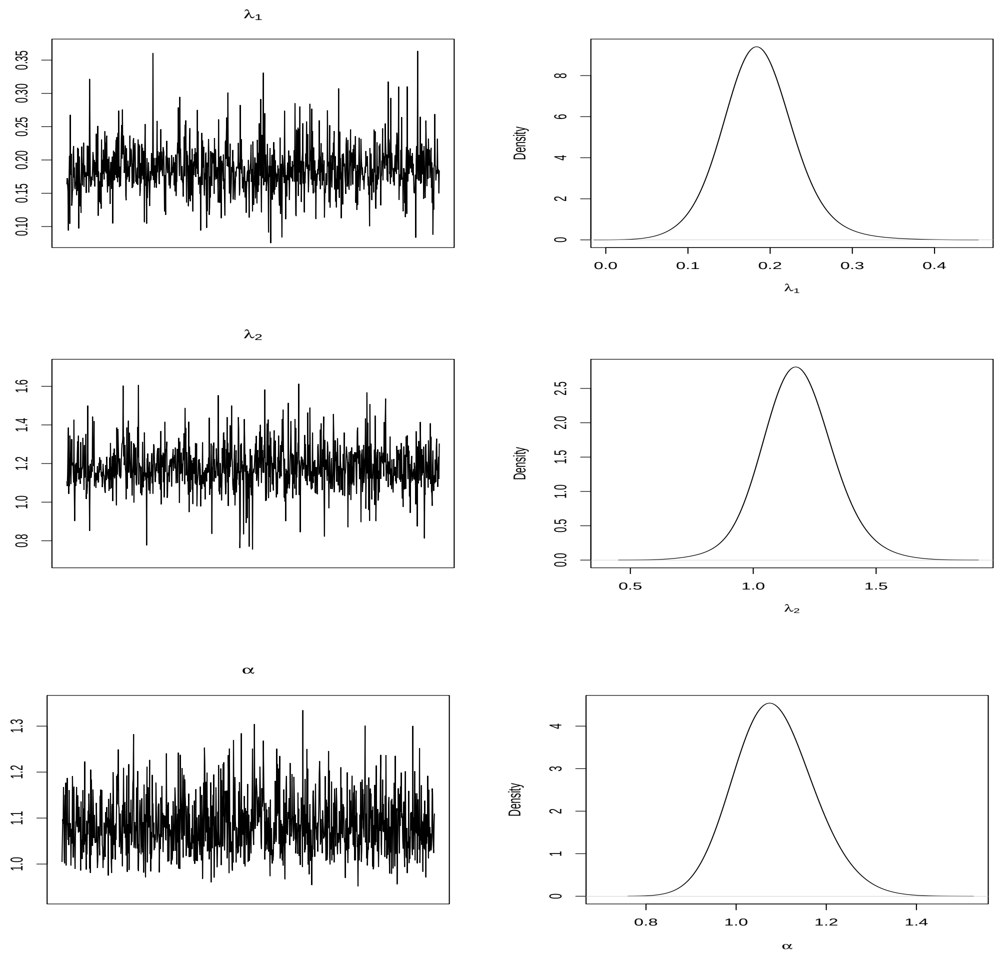

Figure 3: Trace and density plots of MCMC samples for the parameters under the partially observed competing risks in real life data

also provide quantile-quantile (Q-Q) plots in Figure 1. These graphs demonstrate that the considered model offers a relatively good fit for the two data sets under examination.

With n = 99, m = 55, T_1 = 0.45, T_2 = 0.6 and p = 0.1 given, the competing risks for GTIIHCS are summarized in Table 7. Due to insufficient data, some randomly chosen failure causes are marked as missing, indicated by 0. In this context, 1 represents the first cause, 2 the second, and 0 an unobserved cause. Based on the data, we note the values (D_1, D_2, D_3, D) as (7, 44, 4, 55). For the data provided, we compute the maximum likelihood estimates for the unknown parameters, along with the corresponding 95% confidence intervals. Figure 2 presents the profile log-likelihood of α. Upon visual examination, these plots reveal that the profile likelihood is unimodal, suggesting a unique maximum. In the absence of prior information for the unknown parameters, we utilize non-informative priors to compute Bayes estimates and determine the associated Highest Posterior Density (HPD) credible intervals. The results of the maximum likelihood estimates and approximate confidence intervals, as well as the Bayes estimates and HPD intervals, are shown in Table 8. Moreover, we observe that the Bayesian credible intervals for parameters, based on non-informative priors, have shorter lengths compared to MLE based asymptotic confidence intervals. In order to assess the convergence of the MCMC technique for the given datasets, the trace and density graphs for the three parameters are provided in Figures 3.

Table 6: Autopsy Results for 99 RFM Conventional Male Mice Exposed to a radiation dose of 300 Roentgens Dose at 5-6 Weeks of Age

Cause of Death	**Mice ID Numbers**
Thymic Lymphoma	159, 189, 191, 198, 200, 207, 220, 235, 245, 250, 256, 261, 265, 266, 280, 343, 356, 383, 403, 414, 428, 432
Reticulum Cell Sarcoma	317, 318, 399, 495, 525, 536, 549, 552, 554, 557, 558, 571, 586, 594, 596, 605, 612, 621, 628, 631, 636, 643, 647, 648, 649, 661, 663, 666, 670, 695, 697, 700, 705, 712, 713, 738, 748, 753
Other Causes	40, 42, 51, 62, 163, 179, 206, 222, 228, 252, 249, 282, 324, 333, 341, 366, 385, 407, 420, 431, 441, 461, 462, 482, 517, 517, 524, 564, 567, 586, 619, 620, 621, 622, 647, 651, 686, 761, 763

Table 7: Partial competing risk censored data with failure time and cause of failure for n = 99 and m = 55

(0.04, 2)	(0.2, 2)	(0.252, 0)	(0.333, 2)	(0.414, 2)	(0.517, 2)
(0.042, 2)	(0.206, 2)	(0.256, 2)	(0.341, 2)	(0.42, 2)	(0.517, 2)
(0.051, 2)	(0.207, 2)	(0.261, 2)	(0.343, 2)	(0.428, 2)	(0.524, 2)
(0.062, 2)	(0.22, 2)	(0.265, 2)	(0.356, 2)	(0.431, 2)	(0.525, 1)
(0.159, 1)	(0.222, 0)	(0.266, 2)	(0.366, 2)	(0.432, 2)	(0.536, 1)
(0.163, 2)	(0.228, 2)	(0.28, 2)	(0.383, 2)	(0.441, 2)	
(0.179, 0)	(0.235, 2)	(0.282, 2)	(0.385, 2)	(0.461, 0)	
(0.189, 2)	(0.245, 2)	(0.317, 1)	(0.399, 1)	(0.462, 2)	
(0.191, 2)	(0.249, 2)	(0.318, 1)	(0.403, 2)	(0.482, 2)	
(0.198, 2)	(0.25, 2)	(0.324, 2)	(0.407, 2)	(0.495, 1)	

Table 8: Point and interval estimates (with interval lengths in brackets) of the unknown parameters of the Weibull distribution for a real dataset under POCR data

	λ_1	λ_2	α
MLE Estimates	0.3466	2.1790	1.8097
Bayesian Estimates	0.1862	1.1809	1.0851
ACI	(0.0673, 0.6259)[0.5586]	(1.2357, 3.1222)[1.8864]	(1.3729, 2.2466)[0.8737]
HPD	(0.1112, 0.2548)[0.1436]	(0.9787,1.4297)[0.4510]	(0.9797, 1.2113)[0.2316]

7. Conclusion

This paper investigates a generalization of the competing risks problem, known as partially observed competing risks, where data are obtained using generalized type-II hybrid censoring. Various inferences for the unknown parameters of Weibull distributions are obtained using both classical and Bayesian approaches. A comprehensive simulation study is conducted to evaluate the proposed model, and an illustrative real data analysis is also provided. The simulation results suggest that Bayesian methods outperform classical methods. A key observation is that higher masking probabilities degrade the performance of the estimators. This implies that the presence or absence of failure causes significantly impacts the lifetime analysis of units where multiple causes of failure are possible.

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Austrian Journal of Statistics http://www.ajs.or.at/
published by the Austrian Society of Statistics http://www.osg.or.at/
Volume 54 *Submitted:* 2024-09-13
2025 *Accepted:* 2025-01-17