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E-Bayesian Estimation of Rayleigh Distribution and Its Evaluation Standards: E-posterior Risks and E-MSEs under Progressive Type-II Censoring

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Abstract

The present study considers the problem of estimating the scale parameter, reliability function, and hazard function of Rayleigh distribution using the E-Bayesian estimation approach when progressively Type-II censored data are available. The evaluation standards of these estimates are accessed through the definition of E-posterior risk (expected posterior risk) and E-MSE (expected mean square error). These estimations are carried out using conjugate prior distributions of the unknown parameters under four different loss functions i.e. quadratic, weighted squared error, Degroot, and entropy loss functions. Further, we perform Monte Carlo simulations to compare the performances of these proposed methods and use a real dataset for illustration purposes.

Keywords: Rayleigh distribution, progressive type-II censoring, loss function, bayesian estimation, prior distribution, posterior risk.

1. Introduction

The Rayleigh distribution is extensively applied in various fields, including survival analysis in medical science, reliability engineering, communication engineering, and life testing of electrovacuum devices. The Rayleigh distribution also has widespread applications in fields such as economics, political science, sociology, cultural studies, and technology. For a more comprehensive understanding, readers may refer to the studies by Wu, Chen, and Chen (2006) and Kumar and Soni (2019). The Rayleigh time-to-failure (TTF) distribution characterizes items undergoing accelerated aging, where failures do not confirm to the conditions of a stationary random process. With a hazard rate that increases linearly, as depicted in Figure 1, it serves as an effective model for the lifetimes of rapidly aging components, as emphasized by Raqab and Madi (2011). As time progresses, the reliability function of the Rayleigh distribution declines at a much faster rate than that observed in the exponential distribution. In rapidly fading communication channels, the amplitude of the signal is described by a Rayleigh distribution (see Cheng and Chen (2021)). The probability density function, reliability function (RF), and hazard function (HF) of the Rayleigh distribution are as follows:

$$f(x;\lambda) = 2\lambda x e^{-\lambda x^2}; \quad x > 0; \lambda > 0, \tag{1}$$

$$R(x) = e^{-\lambda x^2}; \quad x > 0, \tag{2}$$

$$H(x) = 2\lambda x; \quad x > 0; \lambda > 0, \tag{3}$$

In many reliability studies, the primary goal is to minimize both the cost and the time associated with the analysis. In this context, statistical experiments involving censored data are essential, and recent research has increasingly focused on analyzing such experiments. Wu *et al.* (2006) derived Bayesian inference for the Rayleigh distribution under progressive censored samples. Kim and Han (2009) focused on estimating the scale parameter of the Rayleigh distribution in the context of general progressive censoring. Maiti and Kayal (2021) estimated the parameters and reliability characteristics of a generalized Rayleigh distribution under a progressive Type-II censored sample. Rastogi and Tripathi (2014) explored the estimation of the inverted exponentiated Rayleigh distribution under Type-II progressive censoring. Abu-Moussa, Alsadat, and Sharawy (2023) provided estimates for the reliability functions of the extended Rayleigh distribution under the progressive first-failure censoring model.

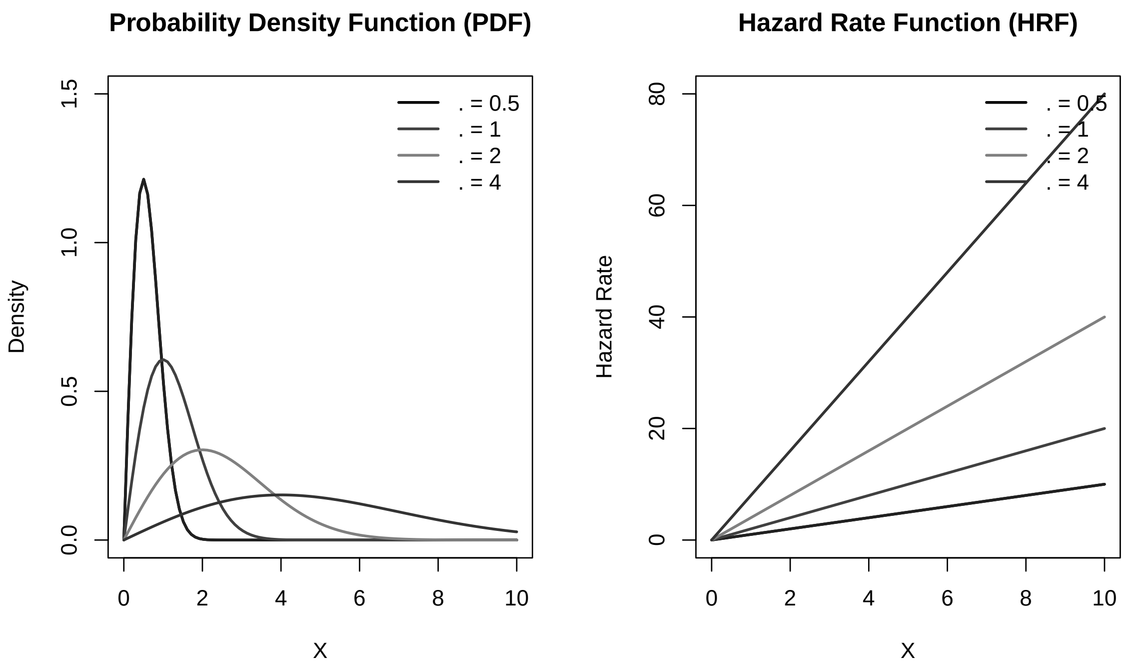

Figure 1: Pdf, and hazard rate function of Rayleigh distribution

Among the various censoring schemes, the conventional Type-I and Type-II censoring methods are the most widely used. However, when a researcher needs to withdraw test items at times other than the scheduled termination time, conventional censoring may not be appropriate. Intermediate withdrawal may be preferred in cases where test items are removed early for other experiments or when a balance is sought between observing extreme lifetimes and minimizing the total testing time. Thus, censoring test items at times other than the termination time can be advantageous, as in scenarios such as loss of connection with individuals under test or accidental breakage of test items. Under the progressive Type-II censoring scheme, n units are placed on test at time zero, and the goal is to observe m failures. The process begins when the first failure is observed. When the first failure is observed, r_1 of the surviving units are randomly selected and removed from the experiment. At the time of the second failure, r_2 of the remaining $n - r_1 - 1$ units are randomly selected and removed from the experiment.

Finally, at the mth failure all remaining surviving units $r_m = n - m - r_1 - r_2 - \ldots - r_{m-1}$ are removed from the experiment. hence, the censoring scheme $R = (r_1, r_2, \ldots, r_m)$ is prefixed.

Bayesian theory integrates prior information into a model, thereby enhancing statistical inference (see Friesl and Hurt (2007)). The quality of the statistical inference in Bayesian theory is contingent on the selection of both the loss function and the prior distribution (see Han (2020); Iqbal and Yousuf Shad (2023), Okasha and Al-bassam (2021), O. Shojaee (2024)). The E-Bayesian and robust Bayesian estimation methods are particularly valuable for addressing the uncertainty associated with prior information by considering a class of priors instead of relying on a single prior distribution. Typically, the E-Bayesian method aims to predict future values of a sequence of random variables or estimate unknown parameters by specifying a prior distribution for the hyperparameters (see Han (2021)). In a recent study, Iqbal and Yousuf Shad (2023) derived E-Bayesian estimates for the rate parameter of the Maxwell distribution under various loss functions, and further computed E-posterior risks and E-mean square errors (MSEs) as performance evaluation standards.

The objective of this paper is to derive Bayesian and E-Bayesian estimates for the scale parameter of the Rayleigh distribution under various loss functions for progressive Type-II censored data. The estimators are calculated using the Quadratic Loss Function (QLF), Weighted Squared Error Loss Function (WSELF), Degroot Loss Function (DLF), and Entropy Loss Function (ELF). The performance of these estimators is evaluated using E-posterior risk and E-mean square error (E-MSE). Numerical simulations and statistical analyses of real datasets are conducted using R-4.2.2. The structure of the article is as follows: Section 2 presents the posterior distribution under the conjugate gamma prior. Section 3 introduces the loss functions, Bayesian estimators, and posterior risks. Section 4 focuses on the computation of E-Bayesian estimates, E-MSE, and E-posterior risk for the given loss functions. Section 5 provides an example using Monte Carlo numerical simulations. Section 6 analyzes a real dataset, and Section 7 concludes the paper.

2. Prior and posterior distribution

Let $X = (x_1, x_2, \ldots, x_n)$ be the sample values observed from the Rayleigh distribution. when we apply progressive Type-II censoring to this sample, the censored sample can be written as $x_{i:m:n}$, where $i = 1, \ldots, m$. Now, the likelihood function can be obtained as:

$$L(\lambda|x) = c\lambda^m e^{-\lambda \sum_{i=1}^{m} x_i^2(1+R_i)} \tag{4}$$

Using Equation (4), we obtained the log-likelihood function as

$$log(L(\lambda|x)) = log(c) + mlog(\lambda) - \lambda \sum_{i=1}^{m} x_i^2(1 + R_i) \tag{5}$$

Now to get the maximum likelihood estimator(MLE) of λ, we will take the derivative of Equation (5) and equate it to 0. Hence, the MLE of λ is

$$\hat{\lambda} = \frac{m}{\sum_{i=1}^{m} x_i^2(1 + R_i)} \tag{6}$$

Let us suppose the conjugate prior density function of λ with hyperparameters a and b, namely $\Gamma(a, b)$, the the pdf is

$$h_1(\lambda; a, b) = \frac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda}; \quad \lambda > 0, a > 0, b > 0 \tag{7}$$

where $\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}$.

The posterior distribution of λ given that X is derived as follows:

$$\begin{aligned}
g(\lambda|x) &= \frac{L(\lambda|x)h_1(\lambda;a,b)}{\int_0^\infty L(\lambda|x)h_1(\lambda;a,b)} \\
&= \frac{c\lambda^m e^{-\lambda\sum_{i=1}^m x_i^2(1+R_i)}\dfrac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda}}{\int_0^\infty c\lambda^m e^{-\lambda\sum_{i=1}^m x_i^2(1+R_i)}\dfrac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda}d\lambda} \\
&= \frac{\lambda^{m+a-1}e^{-\lambda\left(\sum_{i=1}^m x_i^2(1+R_i)+b\right)}}{\int_0^\infty \lambda^{m+a-1}e^{-\lambda\left(\sum_{i=1}^m x_i^2(1+R_i)+b\right)}d\lambda} \\
&= \frac{\left(\sum_{i=1}^m x_i^2(1+R_i)+b\right)^{m+a-1}\lambda^{m+a-1}e^{-\lambda\left(\sum_{i=1}^m x_i^2(1+R_i)+b\right)}}{\Gamma(m+a)}
\end{aligned}$$

Defining $Q = \sum_{i=1}^m x_i^2(1+R_i)$, we have the posterior distribution as

$$g(\lambda|x) = \frac{(Q+b)^{m+a-1}}{\Gamma(m+a)}\lambda^{m+a-1}e^{-\lambda(Q+b)} \tag{8}$$

According to Han (2021), in a conjugate prior distribution of λ given in Equation (7), a and b should be selected to assure that $h_1(\lambda;a,b)$ is a non-increasing function of λ. By differentiating Equation (7) with respect to λ, we get,

$$\begin{aligned}
\frac{d}{d\lambda}[h_1(\lambda;a,b)] &= \frac{d}{d\lambda}\left[\frac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda}\right] \\
&= \frac{b^a}{\Gamma(a)}\lambda^{a-2}e^{-b\lambda}(a-b\lambda-1)
\end{aligned}$$

Note that a, b, and λ are non negative, it proceeds that $0 < a < 1$ and $b > 0$ due to $\frac{d}{d\lambda}[h_1(\lambda;a,b)] < 0$, which means that $[h_1(\lambda;a,b)]$ is a non-increasing function of λ. Given $0 < a < 1$, the tail of gamma distribution will become thinner as value of 'b' increases.

Definition 2.1. According to Han (2021), the E-Bayesian (namely, Expected Bayesian) estimation of $\hat{\lambda}_B$, $\hat{\lambda}_B$ being continuous, is defined as

$$\hat{\lambda}_{EB} = \int\int_S \hat{\lambda}_B g(\Phi;a,b)dadb \tag{9}$$

where S is the precinct of a and b, $\hat{\lambda}_B$ is the Bayesian estimation of λ having the hyper-parameters a and b, whereas $g(\Phi;a,b)$ is the probability function a and b over S.

Definition 2.2. According to Han (2020), the E-Posterior (i.e. Expected Posterior) Risk of $\hat{\lambda}_{EB}$, having $PR(\hat{\lambda}_B)$ being continuous, may be defined as

$$E - PR(\hat{\lambda}_{EB}) = \int\int_S PR(\hat{\lambda}_B)g(\Phi;a,b)dadb \tag{10}$$

where S is the precinct of a and b, $PR(\hat{\lambda}_B)$ is the posterior risk of the Bayesian estimation of λ having the hyper-parameters a and b, whereas $g(\Phi;a,b)$ is the probability function a and b over S.

Definition 2.3. According to Han (2019), the E-MSE (namely, Expected Mean Square Error) of $\hat{\lambda}_{EB}$, having $MSE(\hat{\lambda}_B)$ being continuous, may be defined as

$$E - MSE(\hat{\lambda}_{EB}) = \int\int_S MSE(\hat{\lambda}_B)g(\Phi;a,b)dadb \tag{11}$$

where S is the precinct of a and b, $MSE(\hat{\lambda}_B)$ is the posterior risk of the Bayesian estimation of λ having the hyper-parameters a and b, whereas $g(\Phi;a,b)$ is the probability function a and b over S.

3. The loss functions, Bayesian estimators, and posterior risks

Let $\hat{\lambda}$ represents an estimate of λ. Consider estimating λ, there is always a deviation detected between the parameter and its estimate. When estimating the parameter λ, a loss incurred is usually measured by the loss function. A loss function is also known as a 'measure of error'. According to Mood (1950), the loss function $L = L(\lambda, \hat{\lambda})$ is stated as a real-valued function subject to the following conditions:

(a) $L \geq 0$, $\forall$ possible estimates $\hat{\lambda}$ and $\forall\ \lambda \in \Theta$

(b) $L = 0$, for $\lambda = \hat{\lambda}$

Suppose a random variable X has a pdf $f(x;\lambda)$, for an estimator $\hat{\lambda}$ of parameter λ, the loss functions, respective Bayesian estimators, and posterior Risks of λ for any prior distribution are given in Table 1.

Table 1: Loss functions, Bayes estimators, and posterior risks

Loss function	**Bayes estimator**	**Posterior Risk**
QLF $= \dfrac{(\lambda-\hat{\lambda})^2}{\lambda^2}$	$\hat{\lambda}_{BQ} = \dfrac{[E_{\lambda\|x}(\lambda^{-1})]}{[E_{\lambda\|x}(\lambda^{-2})]}$	$PR(\hat{\lambda}_{BQ}) = 1 - \dfrac{[E_{\lambda\|x}(\lambda^{-1})]}{[E_{\lambda\|x}(\lambda^{-2})]}$
WSELF $= \dfrac{(\lambda-\hat{\lambda})^2}{\lambda}$	$\hat{\lambda}_{BW} = [E_{\lambda\|x}(\lambda^{-1})]^{-1}$	$PR(\hat{\lambda}_{BW}) = E_{\lambda\|x}(\lambda) - [E_{\lambda\|x}(\lambda^{-1})]^{-1}$
DLF $= \dfrac{(\lambda-\hat{\lambda}^2)}{(\hat{\lambda})^2}$	$\hat{\lambda}_{BD} = \dfrac{[E_{\lambda\|x}(\lambda)^2]}{[E_{\lambda\|x}(\lambda)]}$	$PR(\hat{\lambda}_{BD}) = 1 - \dfrac{[E_{\lambda\|x}(\lambda)^2]}{[E_{\lambda\|x}(\lambda)]}$
ELF $= \dfrac{\hat{\lambda}}{\lambda} - \log\left(\dfrac{\hat{\lambda}}{\lambda}\right) - 1$	$\hat{\lambda}_{BE} = [E_{\lambda\|x}(\lambda^{-1})]^{-1}$	$PR(\hat{\lambda}_{BE}) = [E_{\lambda\|x}\log(\lambda)] + \log[E_{\lambda\|x}(\lambda^{-1})]$

4. E-Bayesian, E-Posterior risk and E-MSE estimations

Theorem 1. *Suppose $\boldsymbol{X} = x_i, (i = 1, 2, \ldots, n)$ be the sample values from Rayleigh distribution mentioned in Equation (1), the likelihood and the prior functions are given in Equations (4) and (7) respectively, we have the following results using QLF:*

(i) *The Bayesian estimator of λ is*

$$\hat{\lambda}_{BQ} = \frac{a+m-2}{Q+b}$$

(ii) *Suppose, for the prior distribution of hyper-parameters a and b given below:*

$$h_2(a,b) = \frac{1}{k}, \qquad 0 < a < 1, 0 < b < k \tag{12}$$

the E-Bayesian estimator of λ is:

$$\hat{\lambda}_{EBQ} = \frac{1}{k}\left(m - \frac{3}{2}\right) ln\left(1 + \frac{k}{Q}\right)$$

(iii) *The E-MSE and E-Posterior risk are given by*

$$E - MSE(\hat{\lambda}_{EBQ}) = \frac{1}{Q(Q+k)}\left(m + \frac{1}{2}\right)$$

and

$$E - PR(\hat{\lambda}_{EBQ}) = ln\left(\frac{m}{m-1}\right)$$

Proof. **(i)** The Bayesian estimator of λ (i.e. $\hat{\lambda}_{BQ}$) under QLF is

$$\hat{\lambda}_{BQ} = \frac{[E_{\lambda|x}(\lambda^{-1}]}{[E_{\lambda|x}(\lambda^{-2})]} = \frac{a+m-2}{Q+b}$$

(ii) The E-Bayesian of λ (i.e. $\hat{\lambda}_{EBQ}$) under QLF is

$$\begin{aligned}\hat{\lambda}_{EBQ} &= \iint_S \hat{\lambda}_{BQ} h_2(a,b) dadb \\ &= \int_0^k \int_0^1 \frac{a+m-2}{Q+b} \frac{1}{k} dadb \\ &= \frac{1}{k}\left(m-\frac{3}{2}\right) ln\left(1+\frac{k}{Q}\right)\end{aligned}$$

(iii) The MSE and E-MSE are given below:

$$MSE[\hat{\lambda}_{BQ}] = E_{\lambda|x}[\lambda - \hat{\lambda}_{BQ}]^2 = \frac{a+m}{(Q+b)^2}$$

$$\begin{aligned}E - MSE[\hat{\lambda}_{EBQ}] &= \iint_S MSE[\hat{\lambda}_{BQ}] h_2(a,b) dada \\ &= \int_0^k \int_0^1 \frac{a+m}{(Q+b)^2} \frac{1}{k} dadb \\ &= \frac{1}{Q(Q+k)}\left(m+\frac{1}{2}\right)\end{aligned}$$

The Posterior Risk and E-Posterior Risk under QLF are computed as follows:

$$PR[\hat{\lambda}_{BQ}] = 1 - \frac{[E_{\lambda|x}(\lambda^{-1}]}{[E_{\lambda|x}(\lambda^{-2})]} = \frac{1}{a+m-1}$$

$$\begin{aligned}E - PR[\hat{\lambda}_{EBQ}] &= \iint_S PR[\hat{\lambda}_{BQ}] h_2(a,b) dada \\ &= \int_0^k \int_0^1 \frac{1}{a+m-1} \frac{1}{k} dadb \\ &= ln\left(\frac{m}{m-1}\right)\end{aligned}$$

□

Theorem 2. *Suppose* $\boldsymbol{X} = x_i, (i = 1, 2, \ldots, n)$ *be the sample values from Rayleigh distribution mentioned in Equation (1), the likelihood and the prior functions are given in Equations (4) and (7) respectively, we have the following results using WSELF:*

(i) *The Bayesian estimator of* λ *is*

$$\hat{\lambda}_{BW} = \frac{a+m-1}{Q+b}$$

(ii) *Suppose, for the prior distribution mentioned in Equation (4.1), the E-Bayesian estimator of* λ *is:*

$$\hat{\lambda}_{EBW} = \frac{1}{k}\left(m-\frac{1}{2}\right) ln\left(1+\frac{k}{Q}\right)$$

(iii) *The E-MSE and E-Posterior risk are given by*

$$E - MSE(\hat{\lambda}_{EBW}) = \frac{m+1}{Q(Q+k)}$$

and

$$E - PR(\hat{\lambda}_{EBW}) = \frac{1}{k} ln \left(1 + \frac{k}{Q}\right)$$

Proof. **(i)** The Bayesian estimator of λ (i.e. $\hat{\lambda}_{BW}$) under WSELF is

$$\hat{\lambda}_{BW} = [E_{\lambda|x}(\lambda^{-1})]^{-1} = \frac{a+m-1}{Q+b}$$

(ii) The E-Bayesian of λ (i.e. $\hat{\lambda}_{EBW}$) under WSELF is

$$\begin{aligned}\hat{\lambda}_{EBW} &= \iint_S \hat{\lambda}_{BW} h_2(a,b) da db \\ &= \int_0^k \int_0^1 \frac{a+m-1}{Q+b} \frac{1}{k} da db \\ &= \frac{1}{k}\left(m - \frac{1}{2}\right) ln \left(1 + \frac{k}{Q}\right)\end{aligned}$$

(iii) The MSE and E-MSE are given below:

$$MSE[\hat{\lambda}_{BW}] = E_{\lambda|x}[\lambda - \hat{\lambda}_{BW}]^2 = \frac{a+m}{(Q+b)^2}$$

$$\begin{aligned}E - MSE[\hat{\lambda}_{EBW}] &= \iint_S MSE[\hat{\lambda}_{BW}] h_2(a,b) da db \\ &= \int_0^k \int_0^1 \frac{a+m}{(Q+b)^2} \frac{1}{k} da db \\ &= \frac{m+1}{Q(Q+k)}\end{aligned}$$

The Posterior Risk and E-Posterior Risk under WSELF are computed as follows:

$$PR[\hat{\lambda}_{Bw}] = E_{\lambda|x}(\lambda) - [E_{\lambda|x}(\lambda^{-1}]^{-1} = \frac{1}{Q+b}$$

$$\begin{aligned}E - PR[\hat{\lambda}_{EBW}] &= \iint_S PR[\hat{\lambda}_{BW}] h_2(a,b) da db \\ &= \int_0^k \int_0^1 \frac{1}{Q+b} \frac{1}{k} da db \\ &= \frac{1}{k} ln \left(1 + \frac{k}{Q}\right)\end{aligned}$$

□

Theorem 3. *Suppose* $\boldsymbol{X} = x_i, (i = 1, 2, \ldots, n)$ *be the sample values from Rayleigh distribution mentioned in Equation (1), the likelihood and the prior functions are given in Equations (4) and (7) respectively, we have the following results using DLF:*

(i) *The Bayesian estimator of* λ *is*

$$\hat{\lambda}_{BD} = \frac{a+m+1}{Q+b}$$

(ii) *Suppose, for the prior distribution mentioned in Equation (4.1), the E-Bayesian estimator of λ is:*

$$\hat{\lambda}_{EBD} = \frac{1}{k}\left(m + \frac{3}{2}\right) ln\left(1 + \frac{k}{Q}\right)$$

(iii) *The E-MSE and E-Posterior risk are given by*

$$E - MSE(\hat{\lambda}_{EBD}) = \frac{1}{Q(Q+k)}\left(m + \frac{3}{2}\right)$$

and

$$E - PR(\hat{\lambda}_{EBD}) = ln\left(\frac{m+2}{m+1}\right)$$

Proof. **(i)** The Bayesian estimator of λ (i.e. $\hat{\lambda}_{BD}$) under DLF is

$$\hat{\lambda}_{BD} = \frac{(\lambda - \hat{\lambda}^2)}{(\hat{\lambda})^2} = \frac{a+m+1}{Q+b}$$

(ii) The E-Bayesian of λ (i.e. $\hat{\lambda}_{EBD}$) under DLF is

$$\begin{aligned}\hat{\lambda}_{EBD} &= \iint_S \hat{\lambda}_{BD} h_2(a,b) dadb \\ &= \int_0^k \int_0^1 \frac{a+m+1}{Q+b}\frac{1}{k} dadb \\ &= \frac{1}{k}\left(m + \frac{3}{2}\right) ln\left(1 + \frac{k}{Q}\right)\end{aligned}$$

(iii) The MSE and E-MSE are given below:

$$MSE[\hat{\lambda}_{BD}] = E_{\lambda|x}[\lambda - \hat{\lambda}_{BD}]^2 = \frac{a+m+1}{(Q+b)^2}$$

$$\begin{aligned}E - MSE[\hat{\lambda}_{EBD}] &= \iint_S MSE[\hat{\lambda}_{BD}] h_2(a,b) dada \\ &= \int_0^k \int_0^1 \frac{a+m+1}{(Q+b)^2}\frac{1}{k} dadb \\ &= \frac{1}{Q(Q+k)}\left(m + \frac{3}{2}\right)\end{aligned}$$

The Posterior Risk and E-Posterior Risk under WSELF are computed as follows:

$$PR[\hat{\lambda}_{BD}] = 1 - \frac{[E_{\lambda|x}(\lambda)^2]}{[E_{\lambda|x}(\lambda)]} = \frac{1}{a+m+1}$$

$$\begin{aligned}E - PR[\hat{\lambda}_{EBD}] &= \iint_S PR[\hat{\lambda}_{BD}] h_2(a,b) dadb \\ &= \int_0^k \int_0^1 \frac{1}{a+m+1}\frac{1}{k} dadb \\ &= ln\left(\frac{m+2}{m+1}\right)\end{aligned}$$

□

Theorem 4. *Suppose $\boldsymbol{X} = x_i, (i = 1, 2, \ldots, n)$ be the sample values from Rayleigh distribution mentioned in Equation (1), the likelihood and the prior functions are given in Equations (4) and (7) respectively, we have the following results using ELF:*

(i) *The Bayesian estimator of* λ *is*

$$\hat{\lambda}_{BE} = \frac{a+m-1}{Q+b}$$

(ii) *Suppose, for the prior distribution mentioned in Equation (4.1), the E-Bayesian estimator of* λ *is:*

$$\hat{\lambda}_{EBE} = \frac{1}{k}\left(m-\frac{1}{2}\right) ln\left(1+\frac{k}{Q}\right)$$

(iii) *The E-MSE and E-Posterior risk are given by*

$$E - MSE(\hat{\lambda}_{EBE}) = \frac{m+1}{Q(Q+k)}$$

and

$$E - PR(\hat{\lambda}_{EBE}) = 1-(1-m)log\left(1-\frac{1}{m}\right)$$

Proof. **(i)** The Bayesian estimator of λ (i.e. $\hat{\lambda}_{BE}$) under ELF is

$$\hat{\lambda}_{BE} = [E_{\lambda|x}(\lambda^{-1}] = \frac{a+m-1}{Q+b}$$

(ii) The E-Bayesian of λ (i.e. $\hat{\lambda}_{EBE}$) under WSELF is

$$\begin{aligned}\hat{\lambda}_{EBE} &= \iint_S \hat{\lambda}_{BE} h_2(a,b)dadb \\ &= \int_0^k \int_0^1 \frac{a+m-1}{Q+b}\frac{1}{k}dadb \\ &= \frac{1}{k}\left(m-\frac{1}{2}\right) ln\left(1+\frac{k}{Q}\right)\end{aligned}$$

(iii) The MSE and E-MSE are given below:

$$MSE[\hat{\lambda}_{BE}] = E_{\lambda|x}[\lambda - \hat{\lambda}_{BE}]^2 = \frac{a+m}{(Q+b)^2}$$

$$\begin{aligned}E - MSE[\hat{\lambda}_{EBE}] &= \iint_S MSE[\hat{\lambda}_{BE}]h_2(a,b)dadb \\ &= \int_0^k \int_0^1 \frac{a+m}{(Q+b)^2}\frac{1}{k}dadb \\ &= \frac{m+1}{Q(Q+k)}\end{aligned}$$

The Posterior Risk and E-Posterior Risk under WSELF are computed as follows:

$$\begin{aligned}PR[\hat{\lambda}_{BE}] &= [E_{\lambda|x}log(\lambda)] + log[E_{\lambda|x}(\lambda^{-1}] \\ &= (a+m)^{(Q+b)} PolyGamma(0, a+m) - log\left(\frac{a+m-1}{Q+b}\right)\end{aligned}$$

$$\begin{aligned}E - PR[\hat{\lambda}_{EBE}] &= \iint_S PR[\hat{\lambda}_{BE}]h_2(a,b)dadb \\ &= \int_0^k \int_0^1 (a+m)^{(Q+b)} PolyGamma(0, a+m) - log\left(\frac{a+m-1}{Q+b}\right)\frac{1}{k}dadb \\ &= 1-(1-m)log\left(1-\frac{1}{m}\right)\end{aligned}$$

□

Let us consider another three conjugate priors, and derive Theorem 1, 2, 3, and 4 for these priors. The three conjugates prior are as follows

$$\left.\begin{aligned} h_3(a,b) &= \frac{1}{k\beta(u,v)}a^{u-1}(1-a)^{v-1}, \quad 0<a<1, 0<b<k \\ h_4(a,b) &= \frac{2}{k^2\beta(u,v)}(k-b)a^{u-1}(1-a)^{v-1}, \quad 0<a<1, 0<b<k \\ h_5(a,b) &= \frac{2b}{k^2\beta(u,v)}a^{u-1}(1-a)^{v-1}, \quad 0<a<1, 0<b<k \end{aligned}\right\} \tag{13}$$

Theorem 5. *Suppose $\boldsymbol{X} = x_i, (i = 1, 2, \ldots, n)$ be the sample values from Rayleigh distribution mentioned in Equation (1), the likelihood and the prior functions are given in Equations (4) and (7) respectively, we have the following results using QLF:*

(i) *Now, considering prior $h_3(a,b)$ mentioned in Equation (13), the E-Bayesian estimator of λ is:*

$$\hat{\lambda}_{EBQ} = \frac{1}{k}\left(\frac{u}{u+v} + m - 2\right) ln\left(1 + \frac{k}{Q}\right)$$

(iii) *The E-MSE and E-Posterior risk are given by*

$$E - MSE(\hat{\lambda}_{EBQ}) = \frac{1}{Q(Q+k)}\left(\frac{u}{u+v} + m\right)$$

and

$$E - PR(\hat{\lambda}_{EBQ}) = \frac{1}{\beta(u,v)}\int_0^1 \frac{1}{a+m-1}a^{u-1}(1-a)^{v-1}da$$

Theorem 6. *Suppose $\boldsymbol{X} = x_i, (i = 1, 2, \ldots, n)$ be the sample values from Rayleigh distribution mentioned in Equation (1), the likelihood and the prior functions are given in Equations (4) and (7) respectively, we have the following results using WLF:*

(i) *Now, considering prior $h_3(a,b)$ mentioned in Equation (13), the E-Bayesian estimator of λ is:*

$$\hat{\lambda}_{EBW} = \frac{1}{k}\left(\frac{u}{u+v} + m - 1\right) ln\left(1 + \frac{k}{Q}\right)$$

(iii) *The E-MSE and E-Posterior risk are given by*

$$E - MSE(\hat{\lambda}_{EBW}) = \frac{1}{Q(Q+k)}\left(\frac{u}{u+v} + m\right)$$

and

$$E - PR(\hat{\lambda}_{EBW}) = \frac{1}{k}ln\left(1 + \frac{k}{Q}\right)$$

Theorem 7. *Suppose $\boldsymbol{X} = x_i, (i = 1, 2, \ldots, n)$ be the sample values from Rayleigh distribution mentioned in Equation (1), the likelihood and the prior functions are given in Equations (4) and (7) respectively, we have the following results using DLF:*

(i) *Now, considering prior $h_3(a,b)$ mentioned in Equation (13), the E-Bayesian estimator of λ is:*

$$\hat{\lambda}_{EBD} = \frac{1}{k}\left(\frac{u}{u+v} + m + 1\right) ln\left(1 + \frac{k}{Q}\right)$$

(iii) *The E-MSE and E-Posterior risk are given by*

$$E - MSE(\hat{\lambda}_{EBD}) = \frac{1}{Q(Q+k)}\left(\frac{u}{u+v} + m + 1\right)$$

and

$$E - PR(\hat{\lambda}_{EBD}) = \frac{1}{\beta(u,v)}\int_0^1 \frac{1}{a+m+1}a^{u-1}(1-a)^{v-1}da$$

Theorem 8. *Suppose $\boldsymbol{X} = x_i, (i = 1, 2, \ldots, n)$ be the sample values from Rayleigh distribution mentioned in Equation (1), the likelihood and the prior functions are given in Equations (4) and (7) respectively, we have the following results using ELF:*

(i) *Now, considering prior $h_3(a, b)$ mentioned in Equation (13), the E-Bayesian estimator of λ is:*

$$\hat{\lambda}_{EBE} = \frac{1}{k}\left(\frac{u}{u+v} + m - 1\right) ln\left(1 + \frac{k}{Q}\right)$$

(iii) *The E-MSE and E-Posterior risk are given by*

$$E - MSE(\hat{\lambda}_{EBE}) = \frac{1}{Q(Q+k)}\left(\frac{u}{u+v} + m\right)$$

and

$$E - PR(\hat{\lambda}_{EBE}) = \int_0^k \int_0^1 PR_{BE} h_3(a, b) da db$$

Theorem 9. *Suppose $\boldsymbol{X} = x_i, (i = 1, 2, \ldots, n)$ be the sample values from Rayleigh distribution mentioned in Equation (1), the likelihood and the prior functions are given in Equations (4) and (7) respectively, we have the following results using QLF:*

(i) *Now, considering prior $h_4(a, b)$ mentioned in Equation (13), the E-Bayesian estimator of λ is:*

$$\hat{\lambda}_{EBQ} = \frac{2}{k}\left(\frac{u}{u+v} + m - 2\right)\left[\left(1 + \frac{Q}{k}\right) ln\left(1 + \frac{k}{Q}\right) - 1\right]$$

(iii) *The E-MSE and E-Posterior risk are given by*

$$E - MSE(\hat{\lambda}_{EBQ}) = \frac{2}{k^2}\left(\frac{u}{u+v} + m\right)\left[\frac{k}{Q} - ln\left(1 + \frac{k}{Q}\right)\right]$$

and

$$E - PR(\hat{\lambda}_{EBQ}) = \frac{1}{\beta(u, v)} \int_0^1 \frac{1}{a + m - 1} a^{u-1}(1-a)^{v-1} da$$

Theorem 10. *Suppose $\boldsymbol{X} = x_i, (i = 1, 2, \ldots, n)$ be the sample values from Rayleigh distribution mentioned in Equation (1), the likelihood and the prior functions are given in Equations (4) and (7) respectively, we have the following results using WLF:*

(i) *Now, considering prior $h_4(a, b)$ mentioned in Equation (13), the E-Bayesian estimator of λ is:*

$$\hat{\lambda}_{EBW} = \frac{2}{k}\left(\frac{u}{u+v} + m - 1\right)\left[\left(1 + \frac{Q}{k}\right) ln\left(1 + \frac{k}{Q}\right) - 1\right]$$

(iii) *The E-MSE and E-Posterior risk are given by*

$$E - MSE(\hat{\lambda}_{EBW}) = \frac{2}{k^2}\left(\frac{u}{u+v} + m\right)\left[\frac{k}{Q} - ln\left(1 + \frac{k}{Q}\right)\right]$$

and

$$E - PR(\hat{\lambda}_{EBW}) = \frac{2}{k}\left[\left(1 + \frac{Q}{k}\right) ln\left(1 + \frac{k}{Q}\right) - 1\right]$$

Theorem 11. *Suppose $\boldsymbol{X} = x_i, (i = 1, 2, \ldots, n)$ be the sample values from Rayleigh distribution mentioned in Equation (1), the likelihood and the prior functions are given in Equations (4) and (7) respectively, we have the following results using DLF:*

(i) *Now, considering prior $h_4(a, b)$ mentioned in Equation (13), the E-Bayesian estimator of λ is:*

$$\hat{\lambda}_{EBD} = \frac{2}{k}\left(\frac{u}{u+v} + m + 1\right)\left[\left(1 + \frac{Q}{k}\right) ln\left(1 + \frac{k}{Q}\right) - 1\right]$$

(iii) *The E-MSE and E-Posterior risk are given by*

$$E - MSE(\hat{\lambda}_{EBD}) = \frac{2}{k^2}\left(\frac{u}{u+v} + m + 1\right)\left[\frac{k}{Q} - ln\left(1 + \frac{k}{Q}\right)\right]$$

and

$$E - PR(\hat{\lambda}_{EBD}) = \frac{1}{\beta(u,v)}\int_0^1 \frac{1}{a+m+1} a^{u-1}(1-a)^{v-1} da$$

Theorem 12. *Suppose* $\boldsymbol{X} = x_i, (i = 1, 2, \ldots, n)$ *be the sample values from Rayleigh distribution mentioned in Equation (1), the likelihood and the prior functions are given in Equations (4) and (7) respectively, we have the following results using ELF:*

(i) *Now, considering prior* $h_4(a, b)$ *mentioned in Equation (13), the E-Bayesian estimator of* λ *is:*

$$\hat{\lambda}_{EBE} = \frac{2}{k^2}\left(\frac{u}{u+v} + m - 1\right)\left[\left(1 + \frac{Q}{k}\right) ln\left(1 + \frac{k}{Q}\right) - 1\right]$$

(iii) *The E-MSE and E-Posterior risk are given by*

$$E - MSE(\hat{\lambda}_{EBE}) = \frac{2}{k^2}\left(\frac{u}{u+v} + m\right)\left[\frac{k}{Q} - ln\left(1 + \frac{k}{Q}\right)\right]$$

and

$$E - PR(\hat{\lambda}_{EBE}) = \int_0^k \int_0^1 PR_{BE} h_4(a, b) da db$$

Theorem 13. *Suppose* $\boldsymbol{X} = x_i, (i = 1, 2, \ldots, n)$ *be the sample values from Rayleigh distribution mentioned in Equation (1), the likelihood and the prior functions are given in Equations (4) and (7) respectively, we have the following results using QLF:*

(i) *Now, considering prior* $h_5(a, b)$ *mentioned in Equation (13), the E-Bayesian estimator of* λ *is:*

$$\hat{\lambda}_{EBQ} = \frac{2}{k}\left(\frac{u}{u+v} + m - 2\right)\left[\left(1 - \frac{Q}{k} ln\left(1 + \frac{k}{Q}\right)\right)\right]$$

(iii) *The E-MSE and E-Posterior risk are given by*

$$E - MSE(\hat{\lambda}_{EBQ}) = \frac{2}{k^2}\left(\frac{u}{u+v} + m\right)\left[ln\left(1 + \frac{k}{Q}\right) - \frac{1}{Q+k}\right]$$

and

$$E - PR(\hat{\lambda}_{EBQ}) = \frac{1}{\beta(u,v)}\int_0^1 \frac{1}{a+m-1} a^{u-1}(1-a)^{v-1} da$$

Theorem 14. *Suppose* $\boldsymbol{X} = x_i, (i = 1, 2, \ldots, n)$ *be the sample values from Rayleigh distribution mentioned in Equation (1), the likelihood and the prior functions are given in Equations (4) and (7) respectively, we have the following results using WLF:*

(i) *Now, considering prior* $h_5(a, b)$ *mentioned in Equation (13), the E-Bayesian estimator of* λ *is:*

$$\hat{\lambda}_{EBW} = \frac{2}{k}\left(\frac{u}{u+v} + m - 1\right)\left[\left(1 - \frac{Q}{k} ln\left(1 + \frac{k}{Q}\right)\right)\right]$$

(iii) *The E-MSE and E-Posterior risk are given by*

$$E - MSE(\hat{\lambda}_{EBW}) = \frac{2}{k^2}\left(\frac{u}{u+v} + m\right)\left[ln\left(1 + \frac{k}{Q}\right) - \frac{1}{Q+k}\right]$$

and

$$E - PR(\hat{\lambda}_{EBW}) = \frac{2}{k}\left[1 - \frac{Q}{k} ln\left(1 + \frac{k}{Q}\right)\right]$$

Theorem 15. *Suppose $\boldsymbol{X} = x_i, (i = 1, 2, \ldots, n)$ be the sample values from Rayleigh distribution mentioned in Equation (1), the likelihood and the prior functions are given in Equations (4) and (7) respectively, we have the following results using DLF:*

(i) *Now, considering prior $h_4(a, b)$ mentioned in Equation (13), the E-Bayesian estimator of λ is:*

$$\hat{\lambda}_{EBD} = \frac{2}{k}\left(\frac{u}{u+v} + m + 1\right)\left[1 - \frac{Q}{k} ln\left(1 + \frac{k}{Q}\right)\right]$$

(iii) *The E-MSE and E-Posterior risk are given by*

$$E - MSE(\hat{\lambda}_{EBD}) = \frac{2}{k^2}\left(\frac{u}{u+v} + m + 1\right)\left[ln\left(1 + \frac{k}{Q}\right) - \frac{1}{Q+k}\right]$$

and

$$E - PR(\hat{\lambda}_{EBD}) = \frac{1}{\beta(u, v)}\int_0^1 \frac{1}{a + m + 1} a^{u-1}(1-a)^{v-1} da$$

Theorem 16. *Suppose $\boldsymbol{X} = x_i, (i = 1, 2, \ldots, n)$ be the sample values from Rayleigh distribution mentioned in Equation (1), the likelihood and the prior functions are given in Equations (4) and (7) respectively, we have the following results using ELF:*

(i) *Now, considering prior $h_4(a, b)$ mentioned in Equation (13), the E-Bayesian estimator of λ is:*

$$\hat{\lambda}_{EBE} = \frac{2}{k}\left(\frac{u}{u+v} + m - 1\right)\left[1 - \frac{Q}{k} ln\left(1 + \frac{k}{Q}\right)\right]$$

(iii) *The E-MSE and E-Posterior risk are given by*

$$E - MSE(\hat{\lambda}_{EBE}) = \frac{2}{k^2}\left(\frac{u}{u+v} + m\right)\left[ln\left(1 + \frac{k}{Q}\right) - \frac{1}{Q+k}\right]$$

and

$$E - PR(\hat{\lambda}_{EBE}) = \int_0^k \int_0^1 PR_{BE} h_4(a, b) da db$$

5. E-Bayesian, E-PR and E-MSE estimations of the RF

Suppose $\mathbf{X} = x_i, (i = 1, 2, \ldots, n)$ be the sample values from Rayleigh distribution mentioned in Equation (1), the likelihood and the prior functions are given in Equations (4) and (7) respectively. Then the Bayesian estimates of the Reliability functions under assumed loss functions are as follows:

(i) The Bayes estimator, MSE, and PR of $R(x)$ under QLF is

$$\hat{R}_{BQ} = \left(\frac{Q + b - 2x^2}{Q + b - x^2}\right)^{m+a}$$

$$MSE(\hat{R}_{BQ}) = \left(\frac{Q+b}{Q+b+x^2}\right)^{m+a} - 2\left(\frac{(Q+b)(Q+b-2x^2)}{(Q+b)^2 - 2x^2}\right)^{m+a} + \left(\frac{Q+b-2x^2}{Q+b-x^2}\right)^{m+a}$$

$$PR(\hat{R}_{BQ}) = 1 - \frac{(Q+b)^{m+a}(Q+b-2x^2)^{m+a}}{(Q+b-x^2)^{2(m+a)}}$$

(ii) The Bayes estimator, MSE, and PR of $R(x)$ under DLF is

$$\hat{R}_{BD} = \left(\frac{Q+b+2x^2}{Q+b+x^2}\right)^{m+a}$$

$$MSE(\hat{R}_{BD}) = \left(\frac{Q+b}{Q+b+2x^2}\right)^{m+a} - 2\left(\frac{(Q+b)(Q+b+2x^2)}{(Q+b+x^2)^2}\right)^{m+a} + \left(\frac{Q+b+2x^2}{Q+b+x^2}\right)^{m+a}$$

$$PR(\hat{R}_{BD}) = 1 - \frac{(Q+b)^{m+a}(Q+b+2x^2)^{m+a}}{(Q+b+x^2)^{2(m+a)}}$$

(iii) The Bayes estimator, MSE, and PR of $R(x)$ under WSELF is

$$\hat{R}_{BW} = \left(\frac{Q+b-x^2}{Q+b}\right)^{m+a}$$

$$MSE(\hat{R}_{BW}) = \left(\frac{Q+b}{Q+b+x^2}\right)^{m+a} - 2\left(\frac{Q+b-x^2}{Q+b+x^2}\right)^{m+a} + \left(\frac{Q+b-x^2}{Q+b}\right)^{m+a}$$

$$PR(\hat{R}_{BW}) = \left(\frac{Q+b}{Q+b+x^2}\right)^{m+a} - \left(\frac{Q+b-x^2}{Q+b}\right)^{m+a}$$

(ii) The Bayes estimator, MSE, and PR of $R(x)$ under ELF is

$$\hat{R}_{BE} = \left(\frac{Q+b-x^2}{Q+b}\right)^{m+a}$$

$$MSE(\hat{R}_{BE}) = \left(\frac{Q+b}{Q+b+x^2}\right)^{m+a} - 2\left(\frac{Q+b-x^2}{Q+b+x^2}\right)^{m+a} + \left(\frac{Q+b-x^2}{Q+b}\right)^{m+a}$$

$$PR(\hat{R}_{BE}) = (m+a)log\left(\frac{Q+b}{Q+b-x^2}\right) - x^2\left(\frac{m+a}{Q+b}\right)$$

Now, using the Bayes estimates of the Reliability function, we are going to compute E-Bayes, E-MSE, and E-PR of the assumed loss functions under the 4 priors we have taken for the hyper-parameters i.e. $h_1(a,b)$, $h_2(a,b)$, $h_3(a,b)$, and $h_4(a,b)$ with the help of Equations (9), (10), and (11).

Theorem 17. *Using the Bayes estimate of the Reliability function under different loss functions and priors in Equations (12) and (13), the E-Bayes estimator,E- MSE, and E-PR of $R(x)$*

$$\hat{R}_{EB} = \int_0^k \int_0^1 \hat{R}_{Bayes} h_i(a,b)dadb, \quad i = 2,3,4,5$$

$$E - MSE(\hat{R}_{EB}) = \int_0^k \int_0^1 MSE(\hat{R}_{Bayes})h_i(a,b)dadb, \quad i = 2,3,4,5$$

$$E - PR(\hat{R}_{EB}) = \int_0^k \int_0^1 PR(\hat{R}_{Bayes})h_i(a,b)dadb, \quad i = 2,3,4,5$$

The E-Bayesian estimate of the Reliability function under different loss functions can not be obtained in a closed form, hence we are going to use Monte Carlo simulation for integration to compute these values.

6. E-Bayesian, E-PR and E-MSE estimations of the HRF

Suppose $\mathbf{X} = x_i, (i = 1,2,\ldots,n)$ be the sample values from Rayleigh distribution mentioned in Equation (1), the likelihood and the prior functions are given in Equations (4) and (7) respectively. Then the Bayesian estimates of the Hazard rate functions under assumed loss functions are as follows:

(i) The Bayes estimator, MSE, and PR of $H(x)$ under QLF is

$$\hat{H}_{BQ} = \frac{2x(m+a-2)}{Q+b}$$

$$MSE(\hat{H}_{BQ}) = \frac{4x^2(m+a+4)}{(Q+b)^2}$$

$$PR(\hat{H}_{BQ}) = \frac{1}{m+a-1}$$

(ii) The Bayes estimator, MSE, and PR of $H(x)$ under DLF is

$$\hat{H}_{BD} = \frac{2x(m+a+1)}{Q+b}$$

$$MSE(\hat{H}_{BD}) = \frac{4x^2(m+a+1)}{(Q+b)^2}$$

$$PR(\hat{H}_{BD}) = \frac{1}{m+a+1}$$

(iii) The Bayes estimator, MSE, and PR of $H(x)$ under WSELF is

$$\hat{H}_{BW} = \frac{2x(m+a-1)}{Q+b}$$

$$MSE(\hat{H}_{BW}) = \frac{4x^2(m+a+1)}{(Q+b)^2}$$

$$PR(\hat{H}_{BW}) = \frac{2x}{Q+b}$$

(ii) The Bayes estimator, MSE, and PR of $H(x)$ under ELF is

$$\hat{H}_{BE} = \frac{2x(m+a-1)}{Q+b}$$

$$MSE(\hat{H}_{BE}) = \frac{4x^2(m+a+1)}{(Q+b)^2}$$

$$PR(\hat{H}_{BE}) = (a+m)^{Q+b} Polygamma(0, a+m) + log\left(\frac{Q+b}{a+m-1}\right)$$

Now, using the Bayes estimates of the Hazard rate function, we are going to compute E-Bayes, E-MSE, and E-PR of the assumed loss functions under the 4 priors we have taken for the hyper-parameters i.e. $h_1(a,b)$, $h_2(a,b)$, $h_3(a,b)$, and $h_4(a,b)$ with the help of Equations (9), (10), and (11).

Theorem 18. *Using the Bayes estimate of the Hazard rate function under different loss functions and priors in Equations (12) and (13), the E-Bayes estimator,E- MSE, and E-PR of $H(x)$*

$$\hat{H}_{EB} = \int_0^k \int_0^1 \hat{H}_{Bayes} h_i(a,b)dadb, \quad i = 2,3,4,5$$

$$E - MSE(\hat{H}_{EB}) = \int_0^k \int_0^1 MSE(\hat{H}_{Bayes}) h_i(a,b)dadb, \quad i = 2,3,4,5$$

$$E - PR(\hat{H}_{EB}) = \int_0^k \int_0^1 PR(\hat{H}_{Bayes}) h_i(a,b)dadb, \quad i = 2,3,4,5$$

Using Theorem 18, we have the following results for prior-I mentioned in Equation (12).

(i) The Bayes estimator, MSE, and PR of $H(x)$ under QLF is

$$\hat{H}_{EBQ} = \frac{2x}{K}\left(m - \frac{3}{2}\right) ln\left(1 + \frac{k}{Q}\right)$$

$$E - MSE(\hat{H}_{BQ}) = \frac{4x^2}{K}\frac{1}{Q(Q+k)}\left(m + \frac{9}{2}\right)$$

$$E - PR(\hat{H}_{BQ}) = ln\left(\frac{m}{m-1}\right)$$

(ii) The Bayes estimator, MSE, and PR of $H(x)$ under DLF is

$$\hat{H}_{EBD} = \frac{2x(a+m+1)}{Q+b}$$

$$E - MSE(\hat{H}_{BD}) = \frac{4x^2}{K}\frac{1}{Q(Q+k)}\left(m + \frac{3}{2}\right)$$

$$E - PR(\hat{H}_{BD}) = \frac{2x}{k} ln\left(1 + \frac{k}{Q}\right)$$

(iii) The Bayes estimator, MSE, and PR of $H(x)$ under WSELF is

$$\hat{H}_{EBW} = \frac{2x}{K}\left(m - \frac{1}{2}\right) ln\left(1 + \frac{k}{Q}\right)$$

$$E - MSE(\hat{H}_{BW}) = \frac{4x^2}{K}\frac{1}{Q(Q+k)}\left(m + \frac{3}{2}\right)$$

$$E - PR(\hat{H}_{BW}) = \frac{2x}{k} ln\left(1 + \frac{k}{Q}\right)$$

(iv) The Bayes estimator, MSE, and PR of $H(x)$ under ELF is

$$\hat{H}_{EBE} = \frac{2x(m+a-1)}{Q+b}$$

$$E - MSE(\hat{H}_{BE}) = \frac{4x^2(m+a+1)}{(Q+b)^2}\left(m + \frac{3}{2}\right)$$

$$E - PR(\hat{H}_{BE}) = \int_0^k \int_0^1 \frac{1}{k}(a+m)^{Q+b} Polygamma(0, a+m) + log\left(\frac{Q+b}{a+m-1}\right) dadb$$

to solve this Equation, we are going to use Monte Carlo simulation for integration.

Using Theorem 18, we have the following results for prior-II mentioned in Equation (13).

(i) The Bayes estimator, MSE, and PR of $H(x)$ under QLF is

$$\hat{H}_{EBQ} = \frac{2x}{K}\left(\frac{u}{u+v} + m - 2\right) ln\left(1 + \frac{k}{Q}\right)$$

$$E - MSE(\hat{H}_{BQ}) = \frac{4x^2}{Q(Q+k)}\left(\frac{u}{u+v} + m + 4\right)$$

$$E - PR(\hat{H}_{BQ}) = \int_0^k \int_0^1 \frac{1}{a+m-1}\frac{1}{k\beta(u,v)} a^{u-1}(1-a)^{v-1} dadb$$

(ii) The Bayes estimator, MSE, and PR of $H(x)$ under DLF is

$$\hat{H}_{EBD} = \frac{2x}{K}\left(\frac{u}{u+v} + m + 1\right) ln\left(1 + \frac{k}{Q}\right)$$

$$E - MSE(\hat{H}_{BD}) = \frac{4x^2}{Q(Q+k)}\left(\frac{u}{u+v} + m + 1\right)$$

$$E - PR(\hat{H}_{BD}) = \int_0^k \int_0^1 \frac{1}{a+m+1}\frac{1}{k\beta(u,v)} a^{u-1}(1-a)^{v-1} dadb$$

(iii) The Bayes estimator, MSE, and PR of $H(x)$ under WSELF is

$$\hat{H}_{EBW} = \frac{2x}{K}\left(\frac{u}{u+v} + m - 1\right) ln\left(1 + \frac{k}{Q}\right)$$

$$E - MSE(\hat{H}_{BW}) = \frac{4x^2}{Q(Q+k)}\left(\frac{u}{u+v} + m + 1\right)$$

$$E - PR(\hat{H}_{BW}) = \frac{2x}{k} ln\left(1 + \frac{k}{Q}\right)$$

(iv) The Bayes estimator, MSE, and PR of $H(x)$ under ELF is

$$\hat{H}_{EBE} = \frac{2x}{K}\left(\frac{u}{u+v} + m - 1\right) ln\left(1 + \frac{k}{Q}\right)$$

$$E - MSE(\hat{H}_{BE}) = \frac{4x^2}{Q(Q+k)}\left(\frac{u}{u+v} + m + 1\right)$$

$$E - PR(\hat{H}_{BE}) = \int_0^k \int_0^1 \frac{1}{k\beta(a,b)} a^{u-1}(1-a)^{v-1}(a+m)^{Q+b}$$
$$Polygamma(0, a+m) + log\left(\frac{Q+b}{a+m-1}\right) dadb$$

to solve this Equation, we are going to use Monte Carlo simulation for integration.

Using Theorem 18, we have the following results for prior-III mentioned in Equation (13).

(i) The Bayes estimator, MSE, and PR of $H(x)$ under QLF is

$$\hat{H}_{EBQ} = \frac{2x}{K}\left(\frac{u}{u+v} + m - 2\right)\left(1 - \frac{Q}{k} ln\left(1 + \frac{k}{Q}\right)\right)$$

$$E - MSE(\hat{H}_{BQ}) = \frac{8x^2}{k^2}\left(\frac{k}{Q} - ln\left(1 + \frac{k}{Q}\right)\right)\left(\frac{u}{u+v} + m + 4\right)$$

$$E - PR(\hat{H}_{BQ}) = \int_0^k \int_0^1 \frac{1}{a+m-1}\frac{1}{k\beta(u,v)} a^{u-1}(1-a)^{v-1} dadb$$

(ii) The Bayes estimator, MSE, and PR of $H(x)$ under DLF is

$$\hat{H}_{EBD} = \frac{4x}{K^2}\left(\frac{u}{u+v} + m + 1\right)\left(1 - \frac{Q}{k} ln\left(1 + \frac{k}{Q}\right)\right)$$

$$E - MSE(\hat{H}_{BD}) = \frac{8x^2}{k^2}\left(\frac{k}{Q} - ln\left(1 + \frac{k}{Q}\right)\right)\left(\frac{u}{u+v} + m + 1\right)$$

$$E - PR(\hat{H}_{BD}) = \int_0^k \int_0^1 \frac{1}{a+m+1}\frac{1}{k\beta(u,v)} a^{u-1}(1-a)^{v-1} dadb$$

(iii) The Bayes estimator, MSE, and PR of $H(x)$ under WSELF is

$$\hat{H}_{EBW} = \frac{4x}{K^2}\left(\frac{u}{u+v}+m-1\right)\left(1-\frac{Q}{k}ln\left(1+\frac{k}{Q}\right)\right)$$

$$E-MSE(\hat{H}_{BW}) = \frac{8x^2}{k^2}\left(\frac{k}{Q}-ln\left(1+\frac{k}{Q}\right)\right)\left(\frac{u}{u+v}+m+1\right)$$

$$E-PR(\hat{H}_{BW}) = \frac{4x}{k^2}\left(1-\frac{Q}{k}ln\left(1+\frac{k}{Q}\right)\right)$$

(iv) The Bayes estimator, MSE, and PR of $H(x)$ under ELF is

$$\hat{H}_{EBE} = \frac{4x}{K^2}\left(\frac{u}{u+v}+m-1\right)\left(1-\frac{Q}{k}ln\left(1+\frac{k}{Q}\right)\right)$$

$$E-MSE(\hat{H}_{BE}) = \frac{8x^2}{k^2}\left(\frac{k}{Q}-ln\left(1+\frac{k}{Q}\right)\right)\left(\frac{u}{u+v}+m+1\right)$$

$$E-PR(\hat{H}_{BE}) = \int_0^k\int_0^1 \frac{2(k-b)}{k^2\beta(a,b)}a^{u-1}(1-a)^{v-1}(a+m)^{Q+b}$$
$$Polygamma(0,a+m)+log\left(\frac{Q+b}{a+m-1}\right)dadb$$

to solve this Equation, we are going to use Monte Carlo simulation for integration.

Using Theorem 18, we have the following results for prior-IV mentioned in Equation (13).

(i) The Bayes estimator, MSE, and PR of $H(x)$ under QLF is

$$\hat{H}_{EBQ} = \frac{4x}{K}\left(\frac{u}{u+v}+m-2\right)\left(1-\frac{Q}{k}ln\left(1+\frac{k}{Q}\right)\right)$$

$$E-MSE(\hat{H}_{BQ}) = \frac{8x^2}{k^2}\left(ln\left(1+\frac{k}{Q}\right)-\frac{1}{Q(Q+k)}\right)\left(\frac{u}{u+v}+m+4\right)$$

$$E-PR(\hat{H}_{BQ}) = \int_0^k\int_0^1 \frac{1}{a+m-1}\frac{1}{k\beta(u,v)}a^{u-1}(1-a)^{v-1}dadb$$

(ii) The Bayes estimator, MSE, and PR of $H(x)$ under DLF is

$$\hat{H}_{EBD} = \frac{4x}{K}\left(\frac{u}{u+v}+m+1\right)\left(1-\frac{Q}{k}ln\left(1+\frac{k}{Q}\right)\right)$$

$$E-MSE(\hat{H}_{BD}) = \frac{8x^2}{k^2}\left(ln\left(1+\frac{k}{Q}\right)-\frac{1}{Q(Q+k)}\right)\left(\frac{u}{u+v}+m+1\right)$$

$$E-PR(\hat{H}_{BD}) = \int_0^k\int_0^1 \frac{1}{a+m+1}\frac{1}{k\beta(u,v)}a^{u-1}(1-a)^{v-1}dadb$$

(iii) The Bayes estimator, MSE, and PR of $H(x)$ under WSELF is

$$\hat{H}_{EBW} = \frac{4x}{k}\left(\frac{u}{u+v}+m-1\right)\left(1-\frac{Q}{k}ln\left(1+\frac{k}{Q}\right)\right)$$

$$E-MSE(\hat{H}_{BW}) = \frac{8x^2}{k^2}\left(ln\left(1+\frac{k}{Q}\right)-\frac{1}{Q(Q+k)}\right)\left(\frac{u}{u+v}+m+1\right)$$

$$E-PR(\hat{H}_{BW}) = \frac{4x}{k^2}ln\left(1+\frac{k}{Q}\right)$$

(iv) The Bayes estimator, MSE, and PR of $H(x)$ under ELF is

$$\hat{H}_{EBE} = \frac{4x}{k^2}\left(\frac{u}{u+v} + m - 1\right)\left(1 - \frac{Q}{k}ln\left(1 + \frac{k}{Q}\right)\right)$$

$$E - MSE(\hat{H}_{BE}) = \frac{8x^2}{k^2}\left(ln\left(1 + \frac{k}{Q}\right) - \frac{1}{Q(Q+k)}\right)\left(\frac{u}{u+v} + m + 1\right)$$

$$E - PR(\hat{H}_{BE}) = \int_0^k \int_0^1 \frac{2b}{k^2\beta(a,b)} a^{u-1}(1-a)^{v-1}(a+m)^{Q+b}$$
$$Polygamma(0, a+m) + log\left(\frac{Q+b}{a+m-1}\right) dadb$$

to solve this Equation, we are going to use Monte Carlo simulation for integration.

7. Monte Carlo simulation

In this section, a Monte Carlo simulation study was conducted to compare the evaluation standards of the Bayesian and E-bayesian estimates. We use the following steps for performing the study.

Step 1: Fix $\lambda = \{0.75, 1.5, 3\}$.

Step 2: Consider the following censoring schemes

Scheme-1: $R_1 = R_2 = \cdots = R_{m-1} = 0$ and $R_m = n - m$.
Scheme-2: $R_1 = R_2 = \cdots = R_{m-1} = 1$ and $R_m = n - 2m + 1$.
Scheme-3: $R_1 = R_2 = \cdots = R_{m-1} = R_m = \frac{n-m}{m}$.

Step 3: Set different values of $(n, m) = \{(30, 5), (30, 10), (60, 5), (60, 10)\}$.

Step 4: Generate progressively Type-II censored data set of size m from random numbers of the Rayleigh distribution of size n.

Step 5: Set values for hyper-parameters i.e. $a = 7, b = 2$, and constants i.e. $k = 5$.

Step 6: Evaluate all the values that are derived in Section 3.

Step 7: Repeat, steps 4 to 6, 1000 times for different values of (n, m).

Table 2: Bayesian, MSE, Posterior Risk (PR) estimates using QLF

(n,m)	Scheme	$\lambda = 0.75$			$\lambda = 1.5$			$\lambda = 3$		
		$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR
	I	0.600381	0.10849	0.166667	0.209758	0.014138	0.166667	0.059822	0.001208	0.166667
(30,5)	II	0.577668	0.09992	0.166667	0.292126	0.036227	0.166667	0.091754	0.004147	0.166667
	III	0.589278	0.104378	0.166667	0.227509	0.018708	0.166667	0.066563	0.001764	0.166667
	I	0.635847	0.050800	0.090909	0.185803	0.004454	0.090909	0.049726	0.00032	0.090909
(30,10)	II	0.576358	0.041625	0.090909	0.237529	0.009227	0.090909	0.061859	0.000661	0.090909
	III	0.680138	0.061119	0.090909	0.206038	0.005831	0.090909	0.053952	0.000406	0.090909
	I	0.608705	0.111709	0.166667	0.215182	0.015029	0.166667	0.063709	0.001443	0.166667
(60,5)	II	0.759184	0.196964	0.166667	0.330915	0.049228	0.166667	0.110106	0.007097	0.166667
	III	0.645461	0.132548	0.166667	0.246869	0.021971	0.166667	0.073227	0.002141	0.166667
	I	0.69307	0.060883	0.090909	0.207209	0.005599	0.090909	0.054802	0.000391	0.090909
(60,10)	II	0.931097	0.136913	0.090909	0.314954	0.018393	0.090909	0.090632	0.001685	0.090909
	III	0.742862	0.073587	0.090909	0.232661	0.007730	0.090909	0.061662	0.000546	0.090909

Table 3: E-Bayesian, E-MSE, E-Posterior Risk (E-PR) estimates using QLF for Prior-I

(n,m)	Scheme	$\lambda = 0.75$			$\lambda = 1.5$			$\lambda = 3$		
		$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR
	I	0.471537	0.117407	0.223143	0.151379	0.012043	0.223143	0.042203	0.000968	0.223143
(30,5)	II	0.450119	0.105366	0.223143	0.22029	0.037329	0.223143	0.065524	0.003497	0.223143
	III	0.461118	0.111562	0.223143	0.165856	0.016727	0.223143	0.047099	0.00144	0.223143
	I	0.565099	0.049457	0.10536	0.159675	0.003996	0.10536	0.042384	0.000282	0.10536
(30,10)	II	0.50948	0.039954	0.10536	0.205828	0.008517	0.10536	0.052828	0.000586	0.10536
	III	0.609598	0.061534	0.10536	0.177472	0.005273	0.10536	0.046008	0.000358	0.10536
	I	0.479797	0.123087	0.223143	0.155539	0.012903	0.223143	0.044996	0.001165	0.223143
(60,5)	II	0.681299	0.389024	0.223143	0.256821	0.059596	0.223143	0.079597	0.006402	0.223143
	III	0.524612	0.167305	0.223143	0.180765	0.019929	0.223143	0.051871	0.001746	0.223143
	I	0.619659	0.063187	0.10536	0.178348	0.005044	0.10536	0.046725	0.000344	0.10536
(60,10)	II	0.888825	0.176237	0.10536	0.276482	0.017844	0.10536	0.077698	0.001514	0.10536
	III	0.671112	0.076207	0.10536	0.200958	0.007064	0.10536	0.052614	0.000482	0.10536

Table 4: E-Bayesian, E-MSE, E-Posterior Risk (E-PR) estimates using QLF for Prior-II

(n,m)	Scheme	$\lambda = 0.75$			$\lambda = 1.5$			$\lambda = 3$		
		$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR
	I	0.4645	0.1140	0.1485	0.1510	0.0122	0.1087	0.0425	0.0009	0.0003
(30,5)	II	0.6140	0.2789	0.1484	0.2279	0.0384	0.1083	0.0625	0.0034	0.0003
	III	0.4850	0.1366	0.1484	0.1657	0.0154	0.1083	0.0457	0.0013	0.0003
	I	0.5680	0.0500	0.0986	0.1657	0.0042	0.0095	0.0423	0.0002	0.0002
(30,10)	II	0.6723	0.0878	0.0985	0.2060	0.0087	0.0094	0.0544	0.0006	0.0002
	III	0.6147	0.0634	0.0985	0.1800	0.0055	0.0094	0.0462	0.0003	0.0002
	I	0.4889	0.1283	0.1481	0.1609	0.0137	0.1024	0.0446	0.0011	0.0003
(60,5)	II	0.6555	0.3579	0.1480	0.2286	0.1376	0.1022	0.0782	0.0068	0.0003
	III	0.5320	0.1721	0.1480	0.1852	0.0223	0.1022	0.0514	0.0017	0.0003
	I	0.6268	0.0617	0.0833	0.1765	0.0049	0.0089	0.0471	0.0003	0.0002
(60,10)	II	0.8535	0.1582	0.0832	0.2819	0.0185	0.0108	0.0789	0.0015	0.0002
	III	0.6714	0.0763	0.0832	0.1980	0.0069	0.0088	0.0528	0.0005	0.0002

Table 5: E-Bayesian, E-MSE, E-Posterior Risk (E-PR) estimates using QLF for Prior-III

(n,m)	Scheme	$\lambda = 0.75$			$\lambda = 1.5$			$\lambda = 3$		
		$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR
	I	0.5207	0.1445	0.1368	0.1575	0.0135	0.0856	0.043	0.001	0.0098
(30,5)	II	0.7296	0.4075	0.2024	0.2471	0.0472	0.0952	0.0645	0.0038	0.0104
	III	0.5502	0.1794	0.1422	0.174	0.0179	0.0864	0.0464	0.0013	0.0064
	I	0.6013	0.0562	0.0729	0.1686	0.0044	0.0444	0.0425	0.0002	0.0076
(30,10)	II	0.7291	0.1056	0.0652	0.2118	0.0093	0.0546	0.0549	0.0006	0.0052
	III	0.6564	0.0733	0.0728	0.1837	0.0057	0.0487	0.0465	0.0003	0.0078
	I	0.5515	0.1647	0.1223	0.1681	0.0152	0.0788	0.0452	0.0012	0.006
(60,5)	II	0.7889	0.5271	0.1068	0.2573	0.0617	0.0801	0.0818	0.008	0.0087
	III	0.6115	0.2329	0.1267	0.1964	0.0262	0.0689	0.0523	0.0018	0.0066
	I	0.6677	0.073	0.0689	0.1801	0.0051	0.0435	0.0474	0.0003	0.0045
(60,10)	II	0.5513	0.2034	0.0561	0.2942	0.0206	0.0511	0.0799	0.0016	0.0052
	III	0.7214	0.0893	0.0694	0.2026	0.0073	0.0456	0.0531	0.0005	0.0049

Table 6: E-Bayesian, E-MSE, E-Posterior Risk (E-PR) estimates using QLF for Prior-IV

(n,m)	Scheme	$\lambda = 0.75$			$\lambda = 1.5$			$\lambda = 3$		
		$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR
	I	0.4083	0.2503	0.0233	0.1446	0.0781	0.0082	0.042	0.0215	0.0024
(30,5)	II	0.4985	0.3388	0.0284	0.2089	0.1207	0.0119	0.0609	0.0321	0.0034
	III	0.4198	0.2626	0.0239	0.1573	0.0861	0.0089	0.045	0.0232	0.0025
	I	0.5347	0.2333	0.0125	0.1628	0.0663	0.0038	0.0421	0.0167	0.0009
(30,10)	II	0.6155	0.2797	0.0144	0.2001	0.083	0.0047	0.054	0.0216	0.0012
	III	0.573	0.2537	0.0134	0.1763	0.0722	0.0041	0.0646	0.0183	0.001
	I	0.4236	0.2642	0.0243	0.1537	0.0833	0.0087	0.044	0.0226	0.0025
(60,5)	II	0.5221	0.3634	0.0298	0.2859	0.0825	0.0081	0.0747	0.0405	0.0042
	III	0.4525	0.2897	0.0258	0.1737	0.0967	0.0099	0.0505	0.0262	0.0028
	I	0.5859	0.2584	0.0137	0.1734	0.0708	0.004	0.0.0469	0.0187	0.0011
(60,10)	II	0.7557	0.36	0.177	0.2696	0.1147	0.0063	0.0778	0.0314	0.0018
	III	0.6214	0.278	0.0146	0.1933	0.0795	0.0045	0.0525	0.0209	0.0021

Table 7: Bayesian, MSE, Posterior Risk (PR) estimates using WSELF

(n,m)	Scheme	$\lambda = 0.75$			$\lambda = 1.5$			$\lambda = 3$		
		$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR
	I	0.699432	0.0508	0.063584	0.25171	0.014138	0.041951	0.071787	0.001208	0.011964
(30,5)	II	0.633994	0.041625	0.057635	0.350551	0.036227	0.058425	0.110105	0.004147	0.01835
	III	0.748152	0.061119	0.068013	0.27301	0.018708	0.045501	0.079876	0.001764	0.013312
	I	0.720457	0.10849	0.120076	0.204383	0.004454	0.01858	0.054699	0.00032	0.004972
(30,10)	II	0.693201	0.09992	0.120076	0.261282	0.009227	0.023752	0.068045	0.000661	0.006185
	III	0.707134	0.104378	0.117855	0.226642	0.005831	0.020603	0.059347	0.000406	0.005395
	I	0.730446	0.111709	0.121741	0.258218	0.015029	0.043036	0.076451	0.001443	0.012741
(60,5)	II	0.911021	0.196964	0.151836	0.397099	0.049228	0.066183	0.132127	0.007097	0.022021
	III	0.774553	0.132548	0.129092	0.296243	0.021971	0.049373	0.087872	0.002141	0.014645
	I	0.762377	0.060883	0.069307	0.227929	0.005599	0.02072	0.060282	0.000391	0.00548
(60,10)	II	1.024207	0.136913	0.093109	0.346449	0.018393	0.031495	0.099695	0.001685	0.009063
	III	0.817148	0.073587	0.074286	0.255928	0.00773	0.023266	0.067829	0.000546	0.006166

Table 8: E-Bayesian, E-MSE, E-Posterior Risk (E-PR) estimates using WSELF for Prior-I

(n,m)	Scheme	$\lambda = 0.75$			$\lambda = 1.5$			$\lambda = 3$		
		$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR
	I	0.606262	0.128080	0.134725	0.194630	0.013138	0.043251	0.054261	0.001056	0.012058
(30,5)	II	0.578725	0.114945	0.128605	0.283230	0.040722	0.062940	0.084245	0.003815	0.018721
	III	0.592866	0.121704	0.131748	0.213244	0.018247	0.047387	0.060556	0.001571	0.013456
	I	0.631581	0.051812	0.066482	0.178460	0.004186	0.018785	0.047370	0.000295	0.004986
(30,10)	II	0.569419	0.041857	0.059938	0.230043	0.008922	0.024215	0.059043	0.000614	0.006215
	III	0.681316	0.064464	0.071717	0.198351	0.005524	0.020879	0.051421	0.000375	0.005412
	I	0.616882	0.134277	0.137084	0.199978	0.014076	0.044439	0.057852	0.001271	0.012856
(60,5)	II	0.875955	0.424390	0.194656	0.330199	0.065014	0.073377	0.102339	0.006984	0.022742
	III	0.674501	0.182514	0.149889	0.232413	0.021740	0.051647	0.066692	0.001904	0.014820
	I	0.692560	0.063187	0.072901	0.199330	0.005284	0.020982	0.052222	0.000361	0.005497
(60,10)	II	0.993392	0.184629	0.104567	0.309010	0.018693	0.032527	0.086839	0.001586	0.00914
	III	0.750067	0.079836	0.078954	0.224600	0.007400	0.023642	0.058804	0.000505	0.006189

Table 9: E-Bayesian, E-MSE, E-Posterior Risk (E-PR) estimates using WSELF for Prior-II

(n,m)	Scheme	$\lambda = 0.75$			$\lambda = 1.5$			$\lambda = 3$		
		$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR
	I	0.5973	0.1140	0.1327	0.1942	0.0122	0.0431	0.0547	0.0009	0.0121
(30,5)	II	0.7895	0.2789	0.1754	0.0293	0.0384	0.0651	0.0807	0.0034	0.0179
	III	0.6236	0.1366	0.1358	0.2807	0.0154	0.0479	0.0588	0.0013	0.0130
	I	0.6349	0.0500	0.0668	0.1852	0.0042	0.0194	0.0473	0.0002	0.0049
(30,10)	II	0.7514	0.0878	0.0790	0.2302	0.0087	0.0242	0.0609	0.0006	0.0064
	III	0.6871	0.0634	0.0723	0.2011	0.0055	0.0211	0.0517	0.0003	0.0054
	I	0.6286	0.1283	0.1396	0.2069	0.0137	0.0459	0.0573	0.0011	0.0127
(60,5)	II	0.8429	0.3479	0.1873	0.2940	0.1376	0.013	0.01006	0.0068	0.0223
	III	0.684	0.1721	0.1502	0.2380	0.0223	0.0528	0.0661	0.0017	0.0147
	I	0.7006	0.0617	0.0737	0.1975	0.0049	0.0207	0.0527	0.0003	0.0055
(60,10)	II	0.9504	0.1582	0.1004	0.3151	0.0185	0.0331	0.0882	0.0015	0.0092
	III	0.7504	0.0763	0.0789	0.2213	0.0069	0.0232	0.059	0.0005	0.0062

Table 10: E-Bayesian, E-MSE, E-Posterior Risk (E-PR) estimates using WSELF for Prior-III

(n,m)	Scheme	$\lambda = 0.75$			$\lambda = 1.5$			$\lambda = 3$		
		$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR
(30,5)	I	0.6695	0.1445	0.1487	0.2025	0.0135	0.045	0.0553	0.001	0.0123
	II	0.0938	0.4075	0.2084	0.3177	0.0472	0.0706	0.083	0.0038	0.0184
	III	0.7074	0.1794	0.1572	0.2237	0.0179	0.0497	0.0597	0.0013	0.0132
(30,10)	I	0.6721	0.0562	0.0707	0.1884	0.0044	0.0198	0.0475	0.0002	0.005
	II	0.8149	0.1056	0.0857	0.2367	0.0093	0.0249	0.0613	0.0006	0.0064
	III	0.7337	0.0733	0.0723	0.2053	0.0057	0.0216	0.052	0.0003	0.0054
(60,5)	I	0.7091	0.1647	0.1575	0.2162	0.0152	0.048	0.0581	0.0012	0.0129
	II	1.0144	0.5271	0.2254	0.3308	0.0617	0.0735	0.1051	0.008	0.0233
	III	0.774553	0.132548	0.129092	0.296243	0.021971	0.049373	0.087872	0.002141	0.014645
(60,10)	I	0.7463	0.0703	0.0785	0.2013	0.0051	0.0211	0.0529	0.0003	0.0055
	II	1.0633	0.2034	0.1119	0.3288	0.0206	0.0346	0.0893	0.0016	0.0094
	III	0.8062	0.0893	0.0848	0.2265	0.0073	0.0284	0.0594	0.0005	0.0062

Table 11: E-Bayesian, E-MSE, E-Posterior Risk (E-PR) estimates using WSELF for Prior-IV

(n,m)	Scheme	$\lambda = 0.75$			$\lambda = 1.5$			$\lambda = 3$		
		$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR
(30,5)	I	0.525	0.2503	0.0233	0.1859	0.0781	0.0082	0.054	0.0215	0.0024
	II	0.641	0.3388	0.0284	0.2683	0.1207	0.0119	0.0783	0.0321	0.0034
	III	0.5398	0.2626	0.0239	0.2023	0.0861	0.0089	0.0579	0.0232	0.0025
(30,10)	I	0.5977	0.2333	0.0125	0.1819	0.0663	0.0038	0.0471	0.0167	0.0009
	II	0.6879	2717	0.0144	0.2237	0.083	0.0047	0.0604	0.0216	0.0012
	III	0.6405	0.2537	0.0134	0.197	0.0722	0.0041	0.0514	0.0183	0.001
(60,5)	I	0.5481	0.2642	0.0243	0.1976	0.0833	0.0087	0.0566	0.0226	0.0025
	II	0.6713	0.3634	0.0298	0.3676	0.0825	0.0081	0.0961	0.0405	0.0042
	III	0.5817	0.2897	0.0258	0.2234	0.0967	0.0099	0.0649	0.0262	0.0028
(60,10)	I	0.6549	0.2584	0.0137	0.1938	0.0708	0.004	0.0524	0.0187	0.0011
	II	0.8447	0.36	0.177	0.3031	0.1147	0.0063	0.087	0.0314	0.0018
	III	0.6946	0.2728	0.0146	0.2161	0.0795	0.0045	0.0586	0.0209	0.0021

Table 12: Bayesian, MSE, Posterior Risk (PR) estimates using DLF

(n,m)	Scheme	$\lambda = 0.75$			$\lambda = 1.5$			$\lambda = 3$		
		$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR
	I	0.826601	0.055033	0.125000	0.335614	0.016158	0.125000	0.095716	0.001381	0.125000
(30,5)	II	0.749266	0.045094	0.125000	0.467402	0.041403	0.125000	0.146807	0.00474	0.125000
	III	0.884179	0.066212	0.125000	0.364014	0.021380	0.125000	0.106502	0.002016	0.125000
	I	0.960610	0.123989	0.076923	0.241544	0.004826	0.076923	0.064644	0.000347	0.076923
(30,10)	II	0.924268	0.114194	0.076923	0.308788	0.009996	0.076923	0.080417	0.000717	0.076923
	III	0.942846	0.119282	0.076923	0.267850	0.006317	0.076923	0.070138	0.00044	0.076923
	I	0.900991	0.065956	0.125000	0.344291	0.017176	0.125000	0.101935	0.001649	0.125000
(60,5)	II	1.210426	0.148323	0.125000	0.529465	0.056261	0.125000	0.176169	0.008111	0.125000
	III	0.965720	0.079719	0.125000	0.394991	0.025110	0.125000	0.117163	0.002447	0.125000
	I	0.973928	0.127667	0.076923	0.269371	0.006066	0.076923	0.071243	0.000424	0.076923
(60,10)	II	1.214695	0.225102	0.076923	0.409440	0.019926	0.076923	0.117822	0.001826	0.076923
	III	1.032738	0.151484	0.076923	0.302460	0.008375	0.076923	0.080161	0.000591	0.076923

Table 13: E-Bayesian, E-MSE, E-Posterior Risk (E-PR) estimates using DLF for Prior-I

(n,m)	Scheme	$\lambda = 0.75$			$\lambda = 1.5$			$\lambda = 3$		
		$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR
	I	0.764546	0.051467	0.154151	0.281133	0.014233	0.154151	0.078377	0.001144	0.154151
(30,5)	II	0.689297	0.043759	0.15415	0.40911	0.044116	0.15415	0.121688	0.004133	0.15415
	III	0.824751	0.067394	0.15415	0.308019	0.019768	0.15415	0.08747	0.001702	0.15415
	I	0.875712	0.138754	0.087011	0.216031	0.004377	0.087011	0.057343	0.000309	0.087011
(30,10)	II	0.835936	0.124523	0.087011	0.278474	0.009328	0.087011	0.071474	0.000642	0.087011
	III	0.856362	0.131846	0.087011	0.240109	0.005775	0.087011	0.062246	0.000392	0.087011
	I	0.838362	0.066059	0.15415	0.288858	0.015249	0.15415	0.083565	0.001377	0.15415
(60,5)	II	1.202528	0.193022	0.15415	0.476954	0.070432	0.15415	0.147824	0.007566	0.15415
	III	0.907975	0.083464	0.15415	0.335708	0.023552	0.15415	0.096333	0.002063	0.15415
	I	0.891052	0.145466	0.087011	0.241294	0.005524	0.087011	0.063216	0.000377	0.087011
(60,10)	II	1.26527	0.459756	0.087011	0.374064	0.019543	0.087011	0.105121	0.001658	0.087011
	III	0.974279	0.197724	0.087011	0.271884	0.007737	0.087011	0.071184	0.000528	0.087011

Table 14: E-Bayesian, E-MSE, E-Posterior Risk (E-PR) estimates using DLF for Prior-II

(n,m)	Scheme	$\lambda = 0.75$			$\lambda = 1.5$			$\lambda = 3$		
		$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR
	I	0.8628	0.1348	0.1027	0.2805	0.0145	0.1026	0.7900	0.0011	0.1025
(30,5)	II	1.1404	0.3296	0.1026	0.4232	0.0454	0.1025	0.1165	0.0041	0.1024
	III	0.9007	0.1614	0.1026	0.3067	0.0189	0.1025	0.0849	0.0015	0.1024
	I	0.7685	0.0548	0.0689	0.2242	0.0047	0.0423	0.0573	0.0003	0.0666
(30,10)	II	0.9096	0.0962	0.0688	0.2787	0.0095	0.0422	0.0737	0.0007	0.0665
	III	0.8317	0.0650	0.0688	0.2431	0.0060	0.0422	0.0626	0.0030	0.0665
	I	0.9080	0.1516	0.1008	0.2988	0.0162	0.0897	0.0829	0.0013	0.1001
(60,5)	II	1.2175	0.4112	0.1007	0.1626	0.0975	0.0896	0.1454	0.0081	0.1000
	III	0.9880	0.2034	0.1007	0.3437	0.0263	0.0896	0.0955	0.0020	0.1000
	I	0.8481	0.0676	0.0562	0.2391	0.0054	0.0396	0.0638	0.0003	0.0452
(60,10)	II	1.1548	0.1733	0.0561	0.3814	0.0202	0.0395	0.1067	0.0017	0.0451
	III	0.9084	0.0835	0.0561	0.2679	0.0076	0.0395	0.0715	0.0005	0.0451

Table 15: E-Bayesian, E-MSE, E-Posterior Risk (E-PR) estimates using DLF for Prior-III

(n,m)	Scheme	$\lambda = 0.75$			$\lambda = 1.5$			$\lambda = 3$		
		$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR
	I	0.9671	0.1707	0.1134	0.2925	0.0160	0.0123	0.0800	0.0012	0.0009
(30,5)	II	1.3549	0.4817	0.1423	0.4589	0.0558	0.0212	0.1199	0.0045	0.0010
	III	1.0218	0.2120	0.1258	0.3231	0.0212	0.0142	0.0862	0.0016	0.0009
	I	0.8136	0.0616	0.0875	0.2281	0.0048	0.0089	0.0575	0.0003	0.0002
(30,10)	II	0.9864	0.1157	0.0972	0.2866	0.0102	0.0100	0.0743	0.0007	0.0004
	III	0.8881	0.0803	0.0901	0.2485	0.0063	0.0091	0.0629	0.0004	0.0002
	I	1.0243	0.1947	0.1024	0.3123	0.0179	0.0120	0.0840	0.0014	0.0009
(60,5)	II	1.4652	0.6230	0.1222	0.4778	0.0730	0.0181	0.1519	0.0094	0.0010
	III	1.1357	0.2753	0.1100	0.3648	0.0310	0.0135	0.0972	0.0021	0.0009
	I	0.9034	0.0770	0.0798	0.2436	0.0056	0.0088	0.0641	0.0003	0.0002
(60,10)	II	1.2827	0.2227	0.1013	0.3980	0.0226	0.0094	0.1081	0.0017	0.0004
	III	0.9760	0.0978	0.0814	0.2742	0.0080	0.009	0.0719	0.0005	0.0002

Table 16: E-Bayesian, E-MSE, E-Posterior Risk (E-PR) estimates using DLF for Prior-IV

(n,m)	Scheme	$\lambda = 0.75$			$\lambda = 1.5$			$\lambda = 3$		
		$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR
	I	0.2958	0.1717	0.1134	0.0923	0.016	0.0123	0.0255	0.0012	0.0009
(30,5)	II	0.4004	0.4817	0.1423	0.1424	0.0558	0.0212	0.038	0.0045	0.0010
	III	0.3104	0.2120	0.1258	0.1017	0.212	0.0142	0.0274	0.0016	0.0009
	I	0.2555	0.0616	0.0875	0.0726	0.0048	0.0089	0.0183	0.0003	0.0002
(30,10)	II	0.3061	0.1157	0.0972	0.0909	0.0102	0.0100	0.0237	0.0007	0.0004
	III	0.2778	0.0803	0.0901	0.079	0.0063	0.0091	0.0201	0.0004	0.0002
	I	0.3123	0.1947	0.1024	0.0985	0.0179	0.0120	0.0267	0.0014	0.0009
(60,5)	II	0.4294	0.623	0.1222	0.5309	0.0975	0.0181	0.0478	0.0094	0.001
	III	0.3424	0.2753	0.1100	0.1143	0.0310	0.0135	0.0309	0.0021	0.0009
	I	0.2830	0.077	0.0798	0.0775	0.0056	0.0088	0.0204	0.0003	0.0002
(60,10)	II	0.3943	0.2227	0.1013	0.1256	0.0226	0.0094	0.0344	0.0017	0.0004
	III	0.3045	0.0978	0.0814	0.0871	0.0080	0.009	0.0229	0.0005	0.0002

Table 17: Bayesian, MSE, Posterior Risk (PR) estimates using ELF

(n,m)	Scheme	$\lambda = 0.75$			$\lambda = 1.5$			$\lambda = 3$		
		$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR
	I	0.699432	0.0508	2.062357	0.25171	0.014138	0.424679	0.071787	0.001208	-0.851048
(30,5)	II	0.633994	0.041625	1.965614	0.350551	0.036227	0.571243	0.110105	0.004147	-0.653946
	III	0.748152	0.061119	2.104382	0.27301	0.018708	0.441144	0.079876	0.001764	-0.815633
	I	0.720457	0.10849	1.507652	0.204383	0.004454	0.819732	0.054699	0.000321	-0.500625
(30,10)	II	0.693201	0.09992	0.120076	0.261282	0.009227	0.914548	0.068045	0.000661	-0.452626
	III	0.707134	0.104378	1.489935	0.226642	0.005831	0.892679	0.059347	0.000406	-0.454173
	I	0.730446	0.111709	1.520894	0.258218	0.015029	0.448574	0.076451	0.001443	-0.800764
(60,5)	II	0.911021	0.196964	1.650539	0.397099	0.049228	0.660805	0.132127	0.007097	-0.572091
	III	0.774553	0.132548	1.546587	0.296243	0.021971	0.524001	0.087872	0.002141	-0.732853
	I	0.762377	0.060883	2.144037	0.227929	0.005599	0.923717	0.060282	0.000391	-0.406026
(60,10)	II	1.024207	0.136913	2.291028	0.346449	0.018393	1.113494	0.099695	0.001685	-0.194547
	III	0.817148	0.073587	2.187958	0.255928	0.00773	0.99931	0.067829	0.000546	-0.334499

Table 18: E-Bayesian, E-MSE, E-Posterior Risk (E-PR) estimates using ELF for Prior-I

(n,m)	Scheme	$\lambda = 0.75$			$\lambda = 1.5$			$\lambda = 3$		
		$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR
	I	0.606262	0.12808	0.107425	0.19463	0.013138	0.107425	0.054261	0.001056	0.107425
(30,5)	II	0.569419	0.041857	0.051755	0.28323	0.040722	0.107425	0.084245	0.003815	0.107425
	III	0.592866	0.121704	0.107425	0.213244	0.018247	0.107425	0.060556	0.001571	0.107425
	I	0.631581	0.051812	0.051755	0.17846	0.004186	0.051755	0.04737	0.000295	0.051755
(30,10)	II	0.578725	0.114945	0.107425	0.230043	0.008922	0.051755	0.059043	0.000614	0.051755
	III	0.681316	0.064464	0.051755	0.198351	0.005524	0.051755	0.051421	0.000375	0.051755
	I	0.616882	0.134277	0.107425	0.199978	0.014076	0.107425	0.057852	0.001271	0.107425
(60,5)	II	0.875955	0.42439	0.107425	0.330199	0.065014	0.107425	0.102339	0.006984	0.107425
	III	0.674501	0.182514	0.107425	0.232413	0.02174	0.107425	0.066692	0.001904	0.107425
	I	0.69256	0.063187	0.051755	0.19933	0.005284	0.051755	0.052222	0.000361	0.051755
(60,10)	II	0.993392	0.184629	0.051755	0.30901	0.018693	0.051755	0.086839	0.001586	0.051755
	III	0.750067	0.079836	0.051755	0.2246	0.007401	0.051755	0.058804	0.000505	0.051755

Table 19: E-Bayesian, E-MSE, E-Posterior Risk (E-PR) estimates using ELF for Prior-II

(n,m)	Scheme	$\lambda = 0.75$			$\lambda = 1.5$			$\lambda = 3$		
		$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR
	I	0.5973	0.1140	0.0045	0.1942	0.0122	0.0032	0.0547	0.0009	0.0003
(30,5)	II	0.7895	0.2789	0.0045	0.2930	0.0385	0.0032	0.0807	0.0034	0.0003
	III	0.6236	0.1366	0.0045	0.2130	0.0159	0.0032	0.0588	0.0013	0.0003
	I	0.6349	0.0500	0.0023	0.1852	0.0042	0.0019	0.0473	0.0002	0.0002
(30,10)	II	0.7514	0.0878	0.0023	0.2302	0.0087	0.0019	0.0609	0.0006	0.0002
	III	0.6871	0.0634	0.0023	0.2011	0.0055	0.0019	0.0517	0.0003	0.0002
	I	0.6286	0.1283	0.0039	0.2069	0.0137	0.0029	0.0573	0.0011	0.0003
(60,5)	II	0.8429	0.3479	0.0039	0.1376	0.0975	0.0029	0.01006	0.0068	0.0003
	III	0.6840	0.1721	0.0039	0.2380	0.0223	0.0029	0.0661	0.0017	0.0003
	I	0.7006	0.0617	0.0020	0.1975	0.0049	0.0016	0.0527	0.0003	0.0002
(60,10)	II	0.954	0.1582	0.0020	0.3151	0.0185	0.0016	0.0882	0.0015	0.0002
	III	0.7084	0.0763	0.0020	0.2213	0.0069	0.0016	0.0590	0.0005	0.0002

Table 20: E-Bayesian, E-MSE, E-Posterior Risk (E-PR) estimates using ELF for Prior-III

(n,m)	Scheme	$\lambda = 0.75$			$\lambda = 1.5$			$\lambda = 3$		
		$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR
	I	0.6695	0.1445	0.1004	0.2025	0.0135	0.0101	0.0553	0.001	0.0008
(30,5)	II	0.9380	0.4075	0.1236	0.3177	0.0472	0.0202	0.083	0.0038	0.0009
	III	0.7074	0.1794	0.1021	0.2237	0.0179	0.0123	0.0597	0.0013	0.0008
	I	0.6721	0.0562	0.0742	0.1884	0.0044	0.0086	0.0475	0.0002	0.0001
(30,10)	II	0.8149	0.1056	0.0859	0.2367	0.0093	0.0092	0.0613	0.0006	0.0003
	III	0.7337	0.0733	0.0666	0.2053	0.0057	0.0088	0.052	0.0003	0.0002
	I	0.7091	0.1647	0.1012	0.2162	0.0152	0.0149	0.0581	0.0012	0.0009
(60,5)	II	1.0144	0.5271	0.1215	0.3308	0.0617	0.0197	0.1051	0.008	0.0012
	III	0.7862	0.2329	0.1058	0.2526	0.0262	0.0156	0.0673	0.0018	0.0010
	I	0.7463	0.0703	0.0699	0.0213	0.0051	0.0089	0.0529	0.0003	0.0002
(60,10)	II	1.0633	0.2034	0.0996	0.3288	0.0206	0.0102	0.0893	0.0016	0.0004
	III	0.8062	0.0893	0.0704	0.2265	0.0073	0.0091	0.0594	0.0005	0.0003

Table 21: E-Bayesian, E-MSE, E-Posterior Risk (E-PR) estimates using ELF for Prior-IV

(n,m)	Scheme	$\lambda = 0.75$			$\lambda = 1.5$			$\lambda = 3$		
		$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR	$\hat{\lambda}$	MSE	PR
	I	0.2958	0.1717	0.1134	0.0923	0.0160	0.0123	0.0255	0.0012	0.0009
(30,5)	II	0.4004	0.4817	0.1423	0.1424	0.0558	0.0212	0.038	0.0045	0.0010
	III	0.3104	0.2120	0.1258	0.1017	0.2120	0.0142	0.0274	0.0016	0.0009
	I	0.2555	0.0616	0.0875	0.0726	0.0048	0.0089	0.0183	0.0003	0.0002
(30,10)	II	0.3061	0.1157	0.0972	0.0909	0.0102	0.0100	0.0237	0.0007	0.0004
	III	0.2778	0.0803	0.0901	0.079	0.0063	0.0091	0.0201	0.0004	0.0002
	I	0.3123	0.1947	0.1024	0.0985	0.0179	0.0120	0.0267	0.0014	0.0009
(60,5)	II	0.4294	0.6230	0.1222	0.5309	0.0975	0.0181	0.0478	0.0094	0.0010
	III	0.3424	0.2753	0.1100	0.1143	0.031	0.0135	0.0309	0.0021	0.0009
	I	0.2830	0.0770	0.0798	0.0775	0.0056	0.0088	0.0204	0.0003	0.0002
(60,10)	II	0.3943	0.2227	0.1013	0.1256	0.0226	0.0094	0.0344	0.0017	0.0004
	III	0.3045	0.0978	0.0814	0.0871	0.0080	0.0090	0.0229	0.0005	0.0002

Table 22: E-MSEs and E-posterior risks (E-PR) of E-Bayesian estimates of the parameter for different values of m and n

n	m	QLF			WSELF			DLF			ELF		
		λ	EMSE	EPR	λ	EMSE	EPR	λ	EMSE	EPR	λ	EMSE	EPR
$\lambda = 0.75$													
	5	0.5959	0.103	0.2231	0.6833	0.1124	0.1274	0.8281	0.1218	0.1541	0.6833	0.1124	0.1274
	10	0.5726	0.0514	0.1053	0.64	0.0538	0.0673	0.7748	0.0563	0.087	0.64	0.0538	0.0673
30	15	0.5422	0.0261	0.0689	0.5823	0.0269	0.0401	0.6627	0.0278	0.0606	0.5823	0.0269	0.0401
	20	0.5219	0.0109	0.0606	0.5685	0.0125	0.0079	0.6199	0.0189	0.054	0.5685	0.0125	0.0079
	5	0.4752	0.1209	0.2231	0.611	0.1319	0.1357	0.8826	0.1429	0.1541	0.611	0.1319	0.1357
	10	0.6135	0.0594	0.1053	0.6857	0.0622	0.0721	0.83	0.065	0.087	0.6857	0.0622	0.0721
50	15	0.6312	0.0355	0.0689	0.678	0.0366	0.0467	0.7715	0.0378	0.0606	0.678	0.0366	0.0467
	20	0.6114	0.0231	0.0512	0.6444	0.0237	0.033	0.7105	0.0242	0.0465	0.6444	0.0237	0.033
	5	0.4854	0.1262	0.2231	0.6241	0.1377	0.1386	0.9014	0.1491	0.1541	0.6241	0.1377	0.1386
	10	0.626	0.0614	0.1053	0.6997	0.0643	0.0736	0.847	0.0673	0.087	0.6997	0.0643	0.0736
70	15	0.6683	0.0398	0.0689	0.7178	0.0411	0.0495	0.8168	0.0424	0.0606	0.7178	0.0411	0.0495
	20	0.6695	0.0278	0.0512	0.7057	0.0285	0.0361	0.7781	0.0291	0.0465	0.7057	0.0285	0.0361
$\lambda = 1.5$													
	5	0.1671	0.0113	0.2231	0.1891	0.0124	0.042	0.2732	0.0134	0.1541	0.1891	0.0124	0.042
	10	0.1655	0.0042	0.1053	0.185	0.0044	0.0194	0.224	0.0046	0.087	0.185	0.0044	0.0194
30	15	0.1461	0.0018	0.0689	0.1569	0.0019	0.0108	0.1786	0.002	0.0606	0.1569	0.0019	0.0108
	20	0.1428	0.0016	0.0606	0.1511	0.0017	0.0079	0.1658	0.0017	0.054	0.1511	0.0017	0.0079
	5	0.1584	0.0135	0.2231	0.2036	0.0147	0.0452	0.2941	0.016	0.1541	0.2036	0.0147	0.0452
	10	0.6857	0.0622	0.0517	0.1768	0.0049	0.1053	0.1976	0.0051	0.0208	0.1768	0.0049	0.1053
50	15	0.1758	0.0027	0.0689	0.1888	0.0028	0.013	0.2148	0.0029	0.0606	0.1888	0.0028	0.013
	20	0.1645	0.0016	0.0512	0.1734	0.0017	0.0088	0.1912	0.0017	0.0465	0.1734	0.0017	0.0088
	5	0.1626	0.0147	0.2231	0.209	0.0161	0.0464	0.3019	0.0174	0.1541	0.209	0.0161	0.0464
	10	0.1822	0.0052	0.1053	0.2037	0.0055	0.0214	0.2466	0.0057	0.087	0.2037	0.0055	0.0214
70	15	0.185	0.003	0.0689	0.1988	0.0031	0.0137	0.2262	0.0032	0.0606	0.1988	0.0031	0.0137
	20	0.1779	0.0019	0.0512	0.1875	0.002	0.0096	0.2067	0.002	0.0465	0.1875	0.002	0.0096

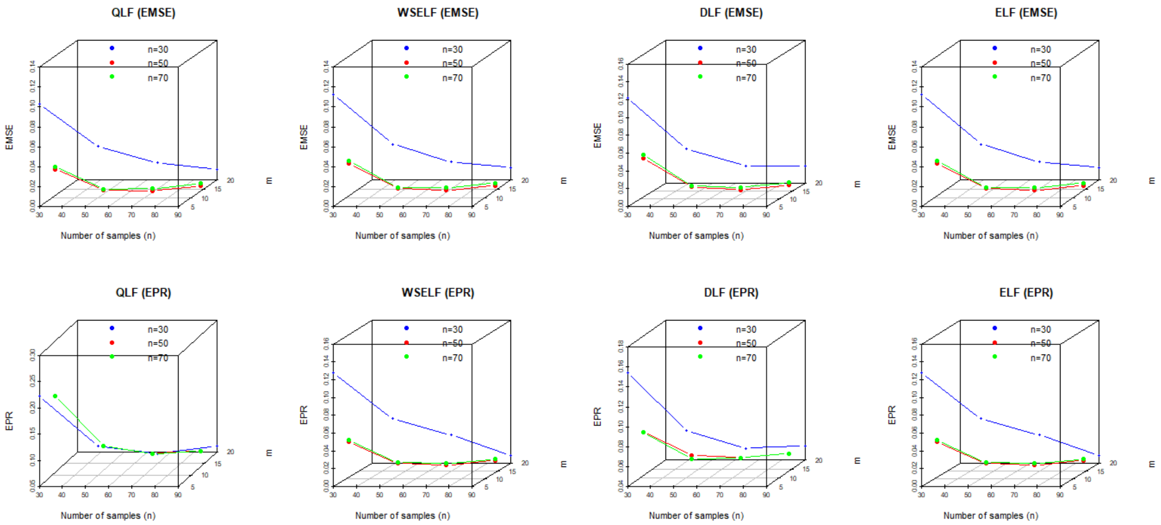

Figure 2: E-MSEs and E-posterior risks (E-PR) of E-Bayesian estimates of the parameter ($\lambda = 0.75$) for different values of m and n

The values of E-MSE and E-PR of E-Bayesian estimates are summarized in Table 22 for different combinations of (n, m) for different loss functions. The values of E-MSE and E-PR for all four loss functions for lambda = 0.75 and 1.5 are displayed in Figure 2 and Figure 3 respectively. It can be instantly observed from Table 22, and Figures 2 − 3 that the E-MSE and E-PR values of E-Bayesian estimates become more precise as the values of (n, m) increase. It is ascertained that both E-MSEs and E-Posterior Risks decrease as the values of (n, m) increase. It can easily observed that DLF has the minimum E-posterior Risk and incurs precise estimate than other assumed loss functions.

8. Real data analysis

(Lawless 2011, page 228) arose in tests on the endurance of deep groove ball bearings. the data, in units of 10^7 revolutions before failure for each of the 23 ball bearings in the life test, are 1.788, 2.892, 3.300, 4.152, 4.212, 4.560, 4.880, 5.184, 5.196, 5.412, 5.556, 6.780, 6.864, 6.864, 6.888, 8.412, 9.312, 9.864, 10.512, 10.584, 12.792, 12.804, and 17.340.

The one parametric Rayleigh and Exponential distributions are fitted by the maximum likelihood method. We consider the following criteria to check the goodness of fit.

(1) Akaike's Information Criterion (AIC)

(2) Bayesian Information Criterion (AIC).

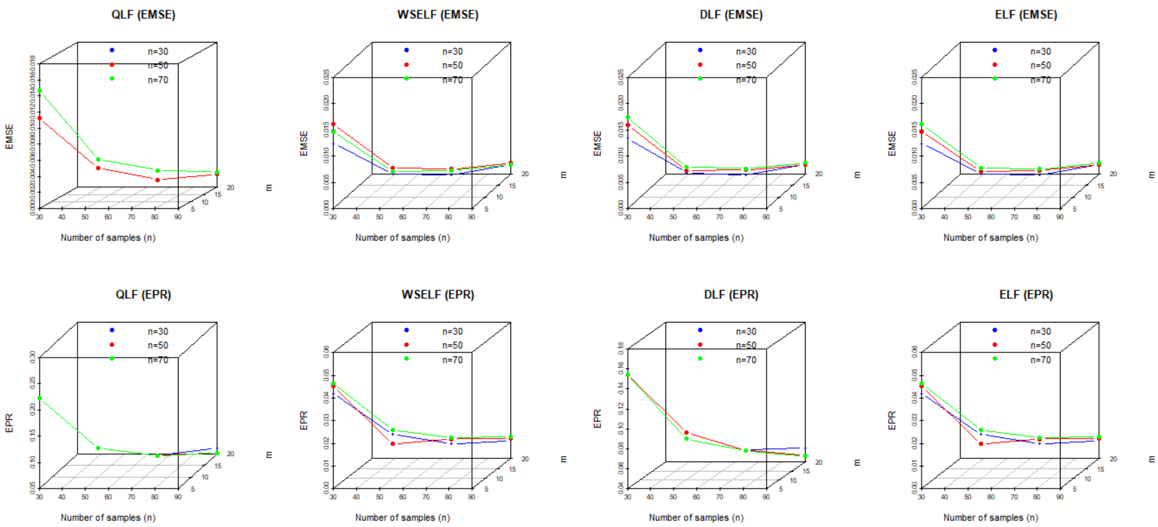

Figure 3: E-MSEs and E-posterior risks (E-PR) of E-Bayesian estimates of the parameter ($\lambda = 1.5$) for different values of m and n

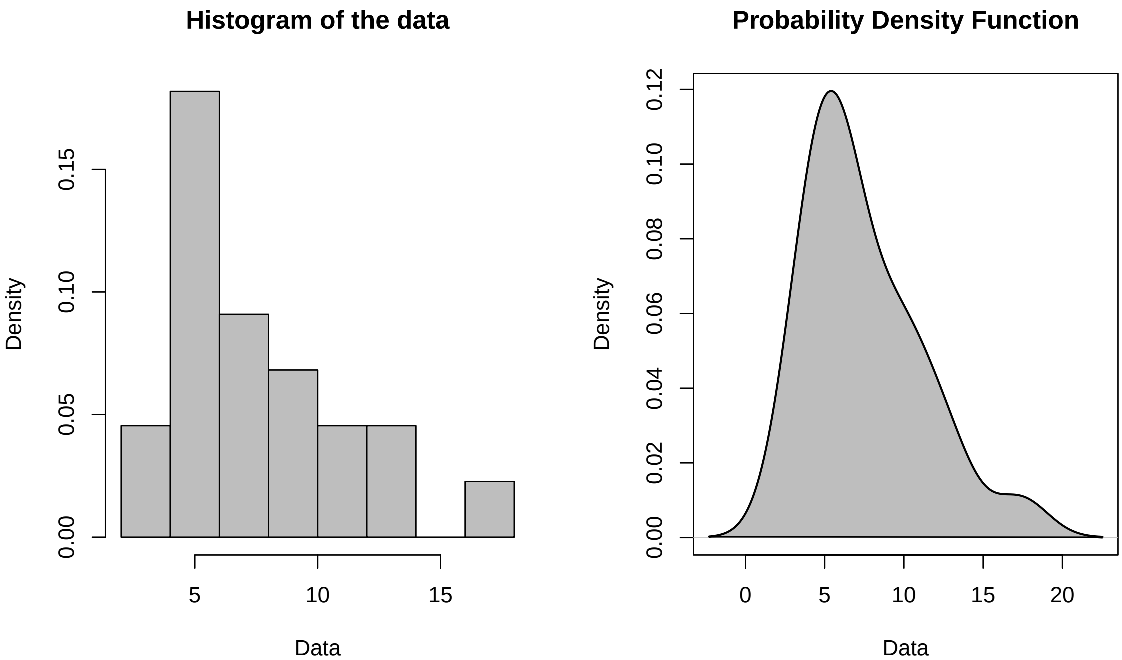

Figure 4: Histogram and PDF of the real dataset

Table 23: Goodness of Fit for Real Data

Distribution	MLE	AIC	BIC	KS	p-value	AD	p-value
Rayleigh	0.000256	308.9675	308.9664	0.1374	0.7776	0.3203	0.9214
Exponential	0.000394	390.8308	391.9218	0.3067	0.0264	2.8143	0.9895

Suppose m and n are the numbers of parameters and observations respectively, then AIC and BIC are stated as

$$AIC = -2 \times log(Likelihood) + 2 \times m$$

$$BIC = -2 \times log(Likelihood) + m \times log(n).$$

Table 23 shows that the Rayleigh distribution provides a better fit to the data compared to the Exponential distribution, as indicated by its higher (less negative) log-likelihood value. Additionally, the AIC and BIC values for the Rayleigh distribution are lower than those for the Exponential distribution, further reinforcing the conclusion that the Rayleigh distribution offers a better fit for this dataset. Initially, we utilize the ball bearings dataset and fit a Rayleigh distribution with the parameter λ. The MLE of the unknown λ is given by $\hat{\lambda} = 0.000394$. Using the Kolmogorov-Smirnov (K-S) test with a significance level of 0.05, the test statistic is found to be 0.1374, with a corresponding p-value of 0.1374. Similarly, the Anderson-Darling (AD) test yields a statistic value of 0.3203, with a p-value of 0.9241. Given the outcomes of the K-S and AD tests, it can be depicted that the Rayleigh distribution is a statistically adequate fit for the data.

Table 24: Bayesian and E-Bayesian Estimates for Real Data and Hazard Rate Function

(n, m)	**L.F.**	**Bayesian Estimate**			**E-Bayesian estimate**											
		λ	**MSE**	**PR**	λ	**E-MSE**	**E-PR**	λ	**E-MSE**	**E-PR**	λ	**E-MSE**	**E-PR**	λ	**E-MSE**	**E-PR**
Scale Parameter																
(23,8)	QLF	0.0097	8.4e-6	0.0714	0.0048	4.7e-6	0.1335	0.0048	4.7e-6	0.1335	0.0048	4.7e-6	0.1335	0.0048	4.7e-6	0.1335
	WSELF	0.0104	8.4e-6	0.0007	0.0056	4.7e-6	0.0007	0.0056	4.7e-6	0.0007	0.0056	4.7e-6	0.0007	0.0056	4.7e-6	0.0007
	DLF	0.0119	9.8e-6	0.0625	0.0071	5.3e-6	0.1053	0.0071	5.3e-6	0.1053	0.0071	5.3e-6	0.1053	0.0071	5.3e-6	0.1053
	ELF	0.0104	8.4e-6	0.0426	0.0056	4.7e-6	0.0652	0.0056	4.7e-6	0.0652	0.0056	4.7e-6	0.0652	0.0056	4.7e-6	0.0652
(23,13)	QLF	0.0074	3.4e-6	0.0526	0.0047	2.3e-6	0.0800	0.0047	2.3e-6	0.0800	0.0047	2.3e-6	0.0800	0.0047	2.3e-6	0.0800
	WSELF	0.0078	3.4e-6	0.0004	0.0051	2.3e-6	0.0004	0.0051	2.3e-6	0.0004	0.0051	2.3e-6	0.0004	0.0051	2.3e-6	0.0004
	DLF	0.0086	3.5e-6	0.0476	0.0059	2.4e-6	0.0690	0.0059	2.4e-6	0.0690	0.0059	2.4e-6	0.0690	0.0059	2.4e-6	0.0690
	ELF	0.0078	3.4e-6	0.0529	0.0051	2.3e-6	0.0394	0.0051	2.3e-6	0.0394	0.0051	2.3e-6	0.0394	0.0051	2.3e-6	0.0394
Reliability Function																
(23,8)	QLF	0.4146	0.0264	0.0476	0.4277	0.0112	0.0297	0.4277	0.0112	0.0297	0.4277	0.0112	0.0297	0.4277	0.0112	0.0297
	WSELF	0.4353	0.0086	0.0194	0.4426	0.0110	0.0082	0.4426	0.0110	0.0082	0.4426	0.0110	0.0082	0.4426	0.0110	0.0082
	DLF	0.7983	0.0099	0.0385	0.8085	0.0126	0.8099	0.8085	0.0126	0.8099	0.8085	0.0126	0.8099	0.8085	0.0126	0.8099
	ELF	0.4353	0.0086	0.0226	0.4776	0.0121	0.0136	0.4776	0.0121	0.0136	0.4776	0.0121	0.0136	0.4776	0.0121	0.0136
(23,13)	QLF	0.4977	0.0177	0.0223	0.4533	0.0121	0.0142	0.4533	0.0121	0.0142	0.4533	0.0121	0.0142	0.4533	0.0121	0.0142
	WSELF	0.5095	0.0057	0.0113	0.4810	0.0092	0.0089	0.4810	0.0092	0.0089	0.4810	0.0092	0.0089	0.4810	0.0092	0.0089
	DLF	0.8085	0.0099	0.0233	0.7984	0.0092	0.4629	0.7984	0.0092	0.4629	0.7984	0.0092	0.4629	0.7984	0.0092	0.4629
	ELF	0.5095	0.0057	0.0112	0.4523	0.0092	0.0085	0.4523	0.0092	0.0085	0.4523	0.0092	0.0085	0.4523	0.0092	0.0085

(n, m)	L.F.	Bayesian Estimate			E-Bayesian estimate											
		λ	MSE	PR	λ	E-MSE	E-PR	λ	E-MSE	E-PR	λ	E-MSE	E-PR	λ	E-MSE	E-PR
Hazard Rate Function																
(23,8)	QLF	0.1331	0.0013	0.0526	0.4076	0.0009	0.5237	0.4076	0.0009	0.5237	0.4076	0.0009	0.5237	0.4076	0.0009	0.5237
	WSELF	0.1405	0.0005	0.0194	0.4251	0.0010	0.5016	0.4251	0.0010	0.5016	0.4251	0.0010	0.5016	0.4251	0.0010	0.5016
	DLF	0.1553	0.0007	0.0476	0.4659	0.0011	0.4929	0.4659	0.0011	0.4929	0.4659	0.0011	0.4929	0.4659	0.0011	0.4929
	ELF	0.1405	0.0003	0.0524	0.5118	0.0109	0.3904	0.5118	0.0109	0.3904	0.5118	0.0109	0.3904	0.5118	0.0109	0.3904
(23,13)	QLF	0.1780	0.0025	0.0534	0.3140	0.0021	0.4982	0.3140	0.0021	0.4982	0.3140	0.0021	0.4982	0.3140	0.0021	0.4982
	WSELF	0.2035	0.0025	0.0625	0.3564	0.0020	0.4999	0.3564	0.0020	0.4999	0.3564	0.0020	0.4999	0.3564	0.0020	0.4999
	DLF	0.2203	0.0003	0.0552	0.4070	0.0025	0.4790	0.4070	0.0025	0.4790	0.4070	0.0025	0.4790	0.4070	0.0025	0.4790
	ELF	0.1780	0.0013	0.0476	0.4657	0.0010	0.4339	0.4657	0.0010	0.4339	0.4657	0.0010	0.4339	0.4657	0.0010	0.4339

Table 24 provides the estimated results of the real data application. It has been concluded after examining the results, that the performance of the loss functions can not determined with the help of MSE and posterior risk as the values are very close in the Bayesian estimations. By taking E-MSE and E-posterior risk (E-PR) of the E-Bayesian estimations as the evaluation standards, we have determined that the Degroot loss function performed best among all four loss functions we have mentioned in this article. Since $\lambda \geq 0$, therefore it has been shown that WSELF is inappropriate because it inflicts an equal penalty on under-estimation as well as over-estimation. Moreover, the efficiency of the estimates and evaluation standards increases with the increase in failure items.

9. Conclusion

The Rayleigh time-to-failure (TTF) distribution is a suitable statistical model for systems undergoing significant aging, where failures do not adhere to the conditions of a stationary process. The present study focuses on deriving the E-Bayesian estimation for the unknown scale parameter of this critical lifetime distribution. Additionally, we evaluate the performance of various loss functions by examining key criteria such as expected mean squared error (E-MSE) and expected posterior risk. Our results demonstrate that both the E-MSE and expected posterior risk of the E-Bayesian estimators decrease as the values of the sample sizes n and m increase. Simulation studies reveal that the DLF minimizes both E-MSE and expected posterior risk compared to the other loss functions. Moreover, the differences in these values among the remaining loss functions are negligible. These findings are further corroborated by statistical analysis of a real dataset, which supports the conclusions drawn from the simulation studies.

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References

Abu-Moussa MH, Alsadat N, Sharawy A (2023). "On Estimation of Reliability Functions for the Extended Rayleigh Distribution under Progressive First-Failure Censoring Model." *Axioms*, **12**(7), 680.

Cheng Y Elsayed EA, Chen X (2021). "Random Multi Hazard Resilience Modeling of Engineered Systems and Critical Infrastructure." *Reliability Engineering & System Safety*, (209), 107–453.

Friesl M, Hurt J (2007). "On Bayesian Estimation in an Exponential Distribution under Random Censorship." *Kybernetika*, **43**(1), 45–60.

Han M (2019). "E-Bayesian Estimation and Its E-MSE under the Scaled Squared Error Loss Function, for Exponential Distribution as Example." *Communications in Statistics-Simulation and Computation*, **48**(6), 1880–1890.

Han M (2020). "E-Bayesian Estimation and Its E-Posterior Risk of the Exponential Distribution Parameter Based on Complete and Type I Censored Samples." *Communications in Statistics-Theory and Methods*, **49**(8), 1858–1872.

Han M (2021). "E-Bayesian Estimations of Parameter and Its Evaluation Standard: E-MSE (Expected Mean Square Error) under Different Loss Functions." *Communications in Statistics-Simulation and Computation*, **50**(7), 1971–1988.

Iqbal A, Yousuf Shad M (2023). "E-Bayesian Estimation of Maxwell Distribution and Its Evaluation Standards: E-Posterior Risks and E-MSES (Expected Mean Square Errors)." *Journal of Statistical Computation and Simulation*, **93**(7), 1192–1206.

Kim C, Han K (2009). "E-Bayesian Estimation of Maxwell Distribution and Its Evaluation Standards: E-Posterior Risks and E-MSES (Expected Mean Square Errors)." *Journal of the Korean Statistical Society*, **38**(3), 239–246.

Kumar M Pathak A, Soni S (2019). "Bayesian Inference for Rayleigh Distribution under Step-stress Partially Accelerated Test With Progressive Type-II Censoring with Binomial Removal." *Annals of Data Science*, **6**(1), 117–152.

Lawless JF (2011). *Statistical Models and Methods for Lifetime Data.* John Wiley & Sons.

Maiti K, Kayal S (2021). "Estimation of Parameters and Reliability Characteristics for a Generalized Rayleigh Distribution under Progressive Type-II Censored Sample." *Communications in Statistics-Simulation and Computation*, **50**(11), 3669–3698.

Mood AM (1950). "Introduction to the Theory of Statistics."

O Shojaee H Zarei FN (2024). "E-Bayesian Estimation and the Corresponding E-MSE under Progressive Type-II Censored Data for Some Characteristics of Weibull Distribution Statistics." *Optimization & Information Computing*, **12**(4), 962–981.

Okasha H Lio Y, Al-bassam M (2021). "On Reliability Estimation of Lomax Distribution under Adaptive Type-I Progressive Hybrid Censoring Scheme." *Journal of Mathematics*, (9), 2903.

Raqab MZ, Madi MT (2011). "Generalized Rayleigh Distribution." *International Encyclopedia of Statistical Science*, **5**(2), 599–603.

Rastogi MK, Tripathi YM (2014). "Estimation for an Inverted Exponentiated Rayleigh Distribution under Type II Progressive Censoring." *Journal of Applied statistics*, **41**(11), 2375–2405.

Wu SJ, Chen DH, Chen ST (2006). "Bayesian Inference for Rayleigh Distribution under Progressive Censored Sample." *Applied Stochastic Models in Business and Industry*, **22**(3), 269–279.

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