Austrian Journal of Statistics
2025, Volume 54, 1–21.
http://www.ajs.or.at/
doi:10.17713/ajs.v54i4.2010

Product Autoregressive Models: Review of Properties, Estimation Methods and Applications

Rahul Thekkedath
Reserve Bank of India
Kerala, India

Shiji Kavungal
Sree Kerala Varma College
Kerala, India

Muhammed Anvar P
Kannur University
Kerala, India

Abstract

Analysis of continuous non-negative time series data using multiplicative models is a growing area of research. When the variable of interest is non-negative, often some methodology based on transformation was followed in the literature. Even though a useful class of models known as product autoregressive models was appeared in the literature long back, the further advancements happened only in the last decade. Through subsequent developments, it was shown that the product form of an additive autoregressive model is preferable to its linear counterpart when non-negativity has to be taken care. This paper aims to provide an exhaustive review of theoretical and empirical works conducted on product autoregressive models in the context of non-linear and non-Gaussian time series modelling. The notable properties, estimation methods and applications of these models are discussed followed by a description of some possible future research avenues on this area.

Keywords: autoregressive models, non-Gaussian time series, non-linear time series, product autoregressive models.

1. Introduction

Time series analysis is one of the most widely used methodology in many applied fields of study such as finance, economics, medicine, public health, environmental studies etc. The celebrated Box-Jenkins methodology (Box and Jenkins (1976)) paved the modern theory and methods for analysing time series data. This methodology mainly focused on developing time series models suitable for real valued processes. In particular, it is based on the assumption that such series are realisations of linear Gaussian processes. However, most of the time series that occur in real situations have the tendency to follow non-linearity and the series are generated by non-Gaussian processes. As a result, a large number of non-Gaussian autoregressive (AR) type time series models are introduced in the literature. These models have played a significant role in modelling the dependence structure in the study of Gaussian and non-Gaussian real-valued time series (Balakrishna (2021)). One speciality of these models is that they assume an additive structure of the form $X_t = \mu_t + \varepsilon_t$, where X_t is the time series observed at time point t, μ_t is the conditional mean of X_t conditional on the past values of X_t and ε_t is the error term which satisfies some conditions such as zero mean and uncorrelatedness (Box and

Jenkins (1976)).

Time series of non-negative random variables arise in various applications and various approaches are available to deal with such dependent outcomes. Autoregressive models based on non-negative marginal distributions were developed and studied. For example, several non-Gaussian additive AR models have been introduced by different researchers over the last four decades, including Gaver and Lewis (1980) and Lawrance and Lewis (1982). The properties of other Markov sequences with non-Gaussian positive marginals such as mixed exponential (Lawrance (1980)), gamma (Sim (1990); Adke and Balakrishna (1992)), Mittag-Leffler (Jayakumar and Pillai (1993); Jayakumar (2003)), inverse Gaussian (Abraham and Balakrishna (1999)), approximated beta distribution (Popović (2010); Popović, Pogány, and Nadarajah (2010)), Lindley distribution (Bakouch and Popović (2016)) and Birnbaum-Sunders distribution (Rahul, Balakrishnan, and Balakrishna (2018)) have also been discussed in the literature. For a comprehensive review of non-Gaussian additive autoregressive time series models and their applications, see Balakrishna (2021) and the references cited therein.

As discussed by Engle (2002), the availability of financial data at a very fine time scale such as durations between trades or quotes, number of trades and volumes, number of buys and sells within (possibly fine) intervals, volatility measures derived from ultra–high frequency data etc., motivated researchers to delve with non-negative time series. Moreover, while modelling point processes with dependent inter-arrival times, Markov sequences on non-negative support play an important role. Recently, attention has been devoted to develop models which are able to directly describe the dynamics of the non-negative time series. An important class of models namely multiplicative error models (MEM) of the form $X_t = \mu_t \varepsilon_t$, where X_t and μ_t are as mentioned before, and ε_t is the independent and identically distributed (i.i.d) error term which satisfies $E(\varepsilon_t) = 1$, was introduced and widely used in literature (Brownlees, Cipollini, and Gallo (2012); Cipollini and Gallo (2022)). Different specifications for μ_t results in different multiplicative error models. Even though this type of models encompasses many of the volatility and duration models, they are not specifically intended to be useful for analysing the autocorrelation structure of the non-negative time series.

In the context of exponential smoothing, Akram, Hyndman, and Ord (2009) studied the importance of developing smoothing methods exclusively for non-negative data. They demonstrated the advantages of using the non-Gaussian error model for the positive random variable in a multiplicative model over the methods using the log-transformed model. In a search for direct approach of using ARMA models for non-negative data, Tsai and Chan (2007) discussed the non-negative ARMA process. Recently, Svetunkov and Boylan (2024) demonstrated the need for designing pure multiplicative Exponential Smoothing (ETS) models for non-negative time series. They showed how the appropriate pure multiplicative ETS model with the most fitting non negative distribution can improve the forecasting ability compared to its additive model.

Despite of all these recent developments for modelling non negative time series, an earlier study of Mckenzie (1982) was not gained much attention. He introduced a class of models, referred to as product autoregressive (PAR) models, with a product structure similar to MEM expression that generates a Markov sequence of non-negative random variables with gamma marginal distribution. Specifically, Mckenzie (1982) introduced a MEM model with

$$\mu_t = X_{t-1}^{\alpha}; \quad 0 < \alpha < 1.$$

This study examined the autocorrelation structure of both the linear AR and PAR models with gamma marginal and found that both have the same autocorrelation structure. The novelty of Mckenzie (1982) sets out new avenues to pursuit research on models for non-negative time series data. That is, when the time series variables of interest are non-negative, one can prefer the product form of an autoregressive model to its linear counterpart. Naturally one could follow an approach which is to model the logarithm of the non-negative variable and then revert back by applying exponential transformation. Such transformations sometimes make little difference to the point forecasts but have a large effect on prediction intervals (Hyndman and Athanasopoulos (2018)). Mckenzie (1982) was the first such attempt to develop direct (without using transformation) time series model exclusively for analysing autocorrelation structure of a non-negative time series data.

The novelty of Mckenzie (1982) sets out new problems and lines of research. After a long period, Balakrishna and Shiji (2010) further developed and discussed the properties of a new Markovian Weibull sequence generated using PAR models. Furthermore, Balakrishna and Lawrance (2012) and Abraham and Balakrishna (2012) discussed the theoretical development of general PAR models to describe non-negative random variables with various marginal distributions, including exponential, Weibull, gamma and log-normal. Jose and Thomas (2012) proposed a first-order PAR model having log-Laplace marginal and discussed its correlation structure. Bakouch, Ristić, Sandhya, and Satheesh (2013) discussed the operation of random products of random variables and the notions of infinite divisibility and stability of distributions under this operation. This description enabled Bakouch *et al.* (2013) to discuss first-order PAR models with random coefficients based on this operation. We discuss such models in Section 4.

As a by-product of the research on PAR models, Balakrishna and Shiji (2014) obtained a class of absolutely continuous bivariate exponential distributions using the product form of a first-order AR model. More theoretical results are then appeared on PAR models, for instance, Moriña, Puig, and Valero (2015) characterised the distribution of innovations in an AR(1) model according to the autocorrelation function of its exponentiated series. Balakrishna and Muhammed (2017) proposed the combined estimating function approach to estimate the parameters of PAR(1) models with Weibull and gamma marginal distributions. In a recent study, Muhammed Anvar, Balakrishna, and Abraham (2019) proposed a stationary PAR model with generalised gamma marginal distribution to generate the volatility series in the analysis of stochastic volatility process.

Similar to the multiplicative structure in the PAR models but different from MEM type models, another class of models were also appeared in the literature. We briefly discuss such models. We call this model as Type -II Product autoregressive models. The dynamics in this type of PAR model is given by

$$X_t = \tfrac{X_{t-1}}{\gamma}\varepsilon_t,$$

where this type of models was used to describe the time evolution of the latent rate process in a Poisson process (Soyer, Aktekin, and Kim (2016)). Here, the error term ε_t is assumed to satisfy the condition

$$\varepsilon_t|F_{t-1} \sim \beta\left(\gamma\alpha_{t-1}, \left(1-\gamma\right)\alpha_{t-1}\right); \;\; \alpha_{t-1} > 0, \; 0 < \gamma < 1,$$

where F_{t-1} is the information set available up to the time $t-1$ and $\beta\left(a, b\right)$ denotes the beta distribution with parameters a and b . The statistical properties of this PAR models are discussed in Section 4.

The objective of this study is to briefly introduce PAR models in the context of non-linear and non-Gaussian time series modelling and review the theoretical and empirical work that has been done on PAR models since Mckenzie (1982). This study aims to combine a variety of such PAR models to analyse non-negative time series data. The plan is to present an exhaustive review of PAR models and provide their notable properties, estimation methods and applications. The remainder of this paper is organised as follows: Section 2 discusses the construction of PAR model and its basic properties. Section 3 provides a systematic review of PAR models with specified marginal distributions and functional forms of their innovation random variables. The random coefficient PAR model and Type –II PAR models and their properties are presented in Section 4. The methods for estimating the model parameters are discussed in Section 5. The applications of PAR models are discussed in Section 6. Section

7 provides the numerical illustration of the relevance and applications of the PAR models. Finally, Section 8 concludes the study.

2. Construction and general properties of PAR models

A classical linear autoregressive model of order one (denoted by AR(1)) states that the process $\{X_t\}$ is defined by the difference equation

$$X_t = \varphi X_{t-1} + \varepsilon_t, \tag{1}$$

where $|\varphi| < 1$ and the innovation process $\{\varepsilon_t\}$ are white noise with mean zero and variance σ^2. If $\{\varepsilon_t\}$ is a sequence of uncorrelated normal random variables, then $\{X_t\}$ will be a Gaussian process. Granger and Newbold (1976) studied in detail the transformation $T(x) = e^x$, because a large range of time series of econometric indicators are analysed in the logarithmic scale, although inference on the original series is the main concern. The study of the time series of non-negative random variables of the form $Z_t = e^{X_t}$, where X_t is a general additive AR(1) with non-Gaussian innovations, was first discussed by Mckenzie (1982).

In this section, we study a class of models proposed by Mckenzie (1982) as a multiplicative version of an additive autoregressive model to generate a sequence of non-negative random variables. Let $\{\eta_t\}$ be a sequence of independent and identically distributed positive random variables, and assume that Z_0 is independent of η_t for $t = 1, 2, ...$ For $0 \leq \alpha < 1$, define

$$Z_t = Z_{t-1}^{\alpha}\eta_t\,;\ t = 1, 2, ..., \tag{2}$$

where $\{Z_t\}$ is a stationary Markov sequence, and the model is referred to as the PAR model of order one (PAR(1)). For a detailed analysis of (2), one needs to study the distributional aspects of $\{\eta_t\}$ for a specified marginal distribution of Z_t and vice versa.

We obtain the innovation distributions for the specified marginal of Z_t using the transformation method. The logarithmic transformation of (2) yields the following equation

$$\log Z_t = \alpha \log Z_{t-1} + \log \eta_t\,,\ \ 0 \leq \alpha < 1, \tag{3}$$

which is the linear AR(1) model in $\log Z_t$. In terms of the moment generating function (MGF), we can express equation (3) as

$$\phi_{\log\eta}(s) = \phi_{\log Z}(s)/\phi_{\log Z}(\alpha s), \tag{4}$$

where $\phi_Z(s) = E(\exp(sZ))$ is the MGF of Z, which is assumed to exist. Thus, model (2) defines a stationary sequence $\{Z_t\}$ if the right-hand side of (4) is a proper MGF for every $\alpha \in (0, 1)$. This occurs if $\log Z_t$ is a self-decomposable random variable. The MGF of $\log Z_t$ can be expressed as the Mellin transform (MT) of Z_t, defined by $M_Z(s) = E(Z_t^s),\ s \geq 0$. Thus, we can use MT to identify the innovation distribution for PAR(1) models. Equation (4) now can be written in terms of MT as

$$M_\eta(s) = M_Z(s)/M_Z(\alpha s). \tag{5}$$

If η_t admits a probability density function $f_\eta(.)$, then its one-step transition density of $\{Z_t\}$ can be expressed as

$$f(z_t|z_{t-1}) = \frac{1}{z_{t-1}^{\alpha}} f_\eta\left(z_t/z_{t-1}^{\alpha}\right). \tag{6}$$

Conditional on past observations, the mean and variance of Z_t in (2) are given by

$$E\left(Z_t|Z_{t-1}\right)=\mu_\eta Z_{t-1}^{\alpha}\ ,\ \ V\left(Z_t|Z_{t-1}\right)=\sigma_\eta^2 Z_{t-1}^{2\alpha}, \tag{7}$$

where μ_η and σ_η^2 are respectively denote the mean and variance of η_t.

The autocorrelation function (ACF) of PAR(1) sequence $\{Z_t\}$ is given by (cf; Mckenzie (1982))

$$\rho_{Z_t}\left(k\right)=\frac{E\left(Z_t\right)\left\{E\left(Z_{t-k}^{\alpha^k+1}\right)-E\left(Z_{t-k}^{\alpha^k}\right)E\left(Z_{t-k}\right)\right\}}{E\left(Z_{t-k}^{\alpha^k}\right)V\left(Z_t\right)}. \tag{8}$$

From (8), ACF depends only on the moments of the stationary marginal distribution of Z_t. Moriña *et al.* (2015) further showed that the ACF of PAR(1) model characterizes the innovation random variable of an additive AR(1) model. That is, under some regularity conditions, ACF of the PAR(1) model can be expressed as a function of MGF of innovation of linear AR(1) model and it uniquely determines. Specifically, suppose that the distribution of the innovations ε_t in (1) are such that the MGF of X_t exists and the marginal distribution of its exponentiated series exists, with the ACF $\rho_x\left(k\right)$ satisfying

$$\lim_{k\to\infty} g\left(k+1\right)-g\left(k\right)=c\geq 0,$$

where $g\left(x\right)=\log\left(\frac{\phi_x(2)-\phi_x(1)^2}{\phi_x(1)}\rho_x\left(k\right)+\phi_x\left(1\right)\right)$, and $\phi_x(t)$ is the MGF of X_t, then the distribution of ε_t is the unique one having the ACF structure same as that of X_t. This characterization can be thought of as a generalization of Mckenzie (1982) result. As Mckenzie (1982) has established the unique relationship between ACF of additive and product autoregressions with gamma marginal where as Moriña *et al.* (2015) emphasised the unique connection between the ACF of product form (transform) and distribution of innovations in additive models.

3. PAR(1) models with specified marginal distributions

In this section, we review and obtain the explicit form of the innovations for the PAR (1) models with specified stationary marginal distributions. Mckenzie (1982) introduced the PAR(1) model to define a stationary sequence of gamma random variables and proved a characterisation result that the gamma distribution is the only one, among the self-decomposable distributions, for which the PAR(1) process has the Markovian correlation structure. Mckenzie (1982) assumed that Z_t has a gamma distribution (Gamma(θ,λ)) with a probability density function (pdf)

$$f\left(z\right)=\frac{1}{\Gamma\left(\theta\right)}\exp\left(-\lambda z\right)\lambda^{\theta}z^{\theta-1}\ ,\ \ z\geq 0,\ \ \lambda>0,\ \ \theta>0. \tag{9}$$

Mckenzie (1982) obtains the distribution of innovations for an exponential PAR(1) model ($\theta=1,\ \lambda=1$ in the above discussion) and shows that it is distributed as $S^{-\alpha}$, where S is the positive stable random variable with Laplace transform $\phi_S\left(s\right)=\exp\left(-s^{\alpha}\right)$. However, in the gamma case, an explicit form of the innovation distribution was not available, and later, Balakrishna and Lawrance (2012) obtained an approximate distribution with good accuracy.

Balakrishna and Shiji (2010) discussed the properties of a Markov sequence of Weibull random variables generated using the PAR model. They focused on the PAR(1) model with Weibull marginal distribution

$$f\left(z\right)=\lambda\,\theta\,z^{\theta-1}\exp\left(-\lambda z^{\theta}\right),\ \ z\geq 0,\ \ \lambda>0,\ \ \theta>0. \tag{10}$$

As the innovation random variable η_t does not possess a closed-form density, Balakrishna and Shiji (2010) proposed an approximate approach for studying the statistical properties of the model.

Balakrishna and Lawrance (2012) discussed the PAR(1) model with gamma marginal distribution by approximating the innovation density. They also identified additional PAR(1) models with specified marginal distributions, which admit explicit solutions for $\{\eta_t\}$ in model (2). Balakrishna and Lawrance (2012) proved that the random variable Z_t defined by (2) follows a Weibull distribution denoted by Weibull(θ, λ) with pdf (10) if the distribution of the corresponding innovations η_t is given by that of $\left(\lambda^{-(1-\alpha)} S^{-\alpha}\right)^{1/\theta}$, where S has a positive stable distribution, with Laplace transform $\phi_S(s) = \exp(-s^{\alpha})$.

Balakrishna and Lawrance (2012) discussed a simple PAR(1) model obtained by exponentiating a linear Gaussian AR(1) model. If X_t in (1) follows a normal distribution with mean μ and variance σ^2, then $Z_t = e^{X_t}$ follows a log-normal(μ, σ^2) distribution with pdf

$$f(z) = \frac{1}{z\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln(z) - \mu)^2}{2\sigma^2}\right), \quad z > 0,\ \mu > 0,\ \sigma > 0. \tag{11}$$

Then, the corresponding innovation random variable η_t follows a log-normal distribution with parameters $((1-\alpha)\mu, (1-\alpha^2)\sigma^2)$.

Balakrishna and Lawrance (2012) also discussed the development of PAR(1) model with exponential marginal distribution with pdf

$$f(z) = \lambda \exp(-\lambda z), \quad z > 0,\ \lambda > 0. \tag{12}$$

Mckenzie (1982) showed that the distribution of innovations for an exponential PAR(1) model with $\lambda = 1$ is given by that of $\eta_E = S^{-\alpha}$, where S is a positive stable random variable with Laplace transform $\phi_S(s) = \exp(-s^{\alpha})$. If the stationary marginal distribution of the PAR(1) process is exponential with scale parameter λ, the distribution of the associated innovations is given by $\eta_t = \lambda^{-(1-\alpha)}\eta_E$.

Abraham and Balakrishna (2012) obtained an explicit form of the innovation random variable η_t of Mckenzie (1982), which provides gamma marginal distribution for $\{Z_t\}$. Abraham and Balakrishna (2012) also proved that if the PAR(1) sequence defined by model (2) has stationary marginal distribution with pdf given by (9), then the distribution of its innovation random variables η_t is specified by $\eta_t = \lambda^{-(1-\alpha)}[B(\alpha,\theta)]^{\alpha} V_{Ew}$, where $B(\alpha,\theta)$ and V_{Ew} are mutually independent and independent and identically distributed random variables with $B(\alpha,\theta)$ being *beta* $(\alpha\theta, (1-\alpha)\theta)$ with pdf

$$f_B(x) = \tfrac{\Gamma(\theta)}{\Gamma(\alpha\theta)\Gamma((1-\alpha)\theta)} x^{\alpha\theta-1}(1-x)^{(1-\alpha)\theta-1}, \ \ 0 \le x \le 1,$$

and the pdf of V_{Ew} is given by

$$g(x) = \tfrac{\Gamma(\alpha\theta+1)}{\Gamma(\theta+1)} x^{\theta} g_E(x), \ \ x > 0,$$

where $g_E(x) = \frac{1}{\pi}\sum\limits_{k=1}^{\infty} \frac{\Gamma(k\alpha)}{\Gamma(k)} \sin(k\pi\alpha)(-x)^{k-1}, \ \ x > 0.$

In another line of research, Jose and Thomas (2012) developed an additive and product AR structure with log-Laplace marginal distribution with pdf

$$f(z) = \frac{1}{\delta}\frac{\beta\gamma}{\beta+\gamma}\begin{cases} \left(\frac{z}{\delta}\right)^{\gamma-1} & for\ \ 0 < z < \delta \\ \left(\frac{\delta}{z}\right)^{\beta+1} & for\ \ \ \ \ z \ge \delta\,, \end{cases} \tag{13}$$

where $\delta > 0,\ \beta > 0,\ \gamma > 0.$

A more general PAR(1) structure with double Pareto log-normal marginals with pdf

$$f(z)=\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\Big[\exp\left(\lambda_1\nu+\frac{\lambda_1^2\tau^2}{2}\right)z^{-\lambda_1-1}\Phi\left(\frac{\ln z-\nu-\lambda_1\tau^2}{\tau}\right) \\ +z^{\lambda_2-1}\exp\left(-\lambda_2\nu+\frac{\lambda_1^2\tau^2}{2}\right)\Phi^c\left(\frac{\ln z-\nu-\lambda_2\tau^2}{\tau}\right)\Big]\ , \tag{14}$$

where Φ is the cumulative distribution function (cdf) and Φ^c is the complementary cdf of $N(0,1)$. Jose and Thomas (2012) also found that the AR correlation structure is not preserved in the case of log-Laplace and double Pareto log-normal PAR(1) models.

Recently Muhammed Anvar *et al.* (2019) discussed a PAR(1) model with generalised gamma (GG) marginal distribution for $\{Z_t\}$ as given by

$$f(z)=\frac{\lambda\tau}{\Gamma(\theta)}(\lambda z)^{\theta\tau-1}\exp\left(-(\lambda z)^{\tau}\right),\ \ z\geq 0;\ \lambda,\ \tau,\ \theta>0, \tag{15}$$

where $\Gamma(.)$ is the gamma function, τ and θ are shape parameters and λ is the rate parameter. The distribution of innovations $\{\eta_t\}$ is specified by $\eta_t=\lambda^{-(1-\alpha)}[B(\alpha,\theta)]^{\frac{\alpha}{\tau}}V_{Ew}^{\frac{1}{\tau}}$, where $B(\alpha,\theta)$ and V_{Ew} are mutually independent and independent and identically distributed random variables, as described in Abraham and Balakrishna (2012).

Table 1 lists the Mellin transforms $M_Z(s)$, $M_\eta(s)$ and ACF $\rho_Z(k)$ of the PAR(1) models with specified marginal distributions discussed in this section.

4. Other product autoregressive models

As mentioned in Section 1, slightly different forms of PAR models are also studied in the literature. The model introduced by Bakouch *et al.* (2013) have studied a random coefficient product autoregressive models whereas Soyer *et al.* (2016) constructed a product auto regression of the form $X_t=\frac{X_{t-1}}{\gamma}\varepsilon_t$. We describe them in following sub sections.

4.1. Random coefficient PAR models

Bakouch *et al.* (2013) proposed a first-order PAR model with random coefficients based on the operation random products of random variables and the notion of infinite divisibility. The stationary PAR(1) model introduced by Bakouch *et al.* (2013) is given by

$$Z_t=Z_{t-1}^{\alpha_t}\eta_t\ ,\ \ t=1,2,..., \tag{16}$$

where $\{\alpha_t\}$ is a sequence of independent and identically distributed random variables with $P(\alpha_t=0)=1-P(\alpha_t=1)=p$ independent of Z_{t-k} for $k>0$ and $\{\alpha_t\}$ and $\{\eta_t\}$ are two mutually independent sequences.

The unconditional mean and variance of Z_t in (16) is given by

$$E(Z_t)=\frac{p\mu_\eta}{1-(1-p)\mu_\eta}\quad and\quad V(Z_t)=\frac{p(1-p)(1-\mu_\eta)^2\left(\mu_\eta^2+\sigma_\eta^2\right)+p^2\sigma_\eta^2}{\left(1-(1-p)\left(\mu_\eta^2+\sigma_\eta^2\right)\right)(1-(1-p)\mu_\eta)^2}, \tag{17}$$

where μ_η and σ_η^2 denote the mean and variance of η_t, respectively.

ACF of random coefficient PAR(1) sequence $\{Z_t\}$ is obtained as

$$\rho_{Z_t}(k)=(1-p)^k\mu_\eta^k;\ \ k>0. \tag{18}$$

Further, $|(1-p)\mu_\eta|<1$ and hence ACF converges to zero as $k\to\infty$.

Table 1: Mellin transforms $M_Z(s)$, $M_\eta(s)$ and ACF $\rho_Z(k)$ of the PAR(1) models with specified marginal distributions

Z	$M_Z(s)$	$M_\eta(s)$	$\rho_Z(k)$
$Gamma\,(\theta,\lambda)$	$\lambda^{-s}\frac{\Gamma(s+\theta)}{\Gamma(\theta)}$	$\lambda^{-(1-\alpha)s}\frac{\Gamma(s+\theta)}{\Gamma(\alpha s+\theta)}$	$\alpha^k\ ; k=0,1,2,...$
$Weibull\,(\theta,\lambda)$	$\lambda^{-\frac{s}{\theta}}\Gamma\left(1+\frac{s}{\theta}\right)$	$\lambda^{-\frac{(1-\alpha)s}{\theta}}\frac{\Gamma\left(1+\frac{s}{\theta}\right)}{\Gamma\left(1+\frac{\alpha s}{\theta}\right)}$	$\frac{\Gamma\left(\frac{1}{\theta}+1\right)\left\{\Gamma\left(\frac{1+\alpha^k}{\theta}+1\right)-\Gamma\left(\frac{\alpha^k}{\theta}+1\right)\Gamma\left(\frac{1}{\theta}+1\right)\right\}}{\Gamma\left(\frac{\alpha^k}{\theta}+1\right)\left\{\Gamma\left(\frac{2}{\theta}+1\right)-\left(\Gamma\left(\frac{1}{\theta}+1\right)\right)^2\right\}};\ k=1,2,...$
$Log-normal\,(\mu,\sigma^2)$	$\exp\left(s\mu+\frac{s^2\sigma^2}{2}\right)$	$\exp\left(s\,(1-\alpha)\,\mu+\frac{s^2\left(1-\alpha^2\right)\sigma^2}{2}\right)$	$\frac{\exp\left(\alpha^k\sigma^2\right)-1}{\exp(\sigma^2)-1};\ k=1,2,...$
$Exponential\,(\lambda)$	$\lambda^{-s}\Gamma\,(s+1)$	$\lambda^{-(1-\alpha)s}\frac{\Gamma(s+1)}{\Gamma(\alpha s+1)}$	$\alpha^k\ ; k=0,1,2,...$
$Log-Laplace\,(\delta,\beta,\gamma)$	$\delta^s\frac{\beta\gamma}{(\beta-s)(\gamma+s)}$	$\delta^{(1-\alpha)s}\frac{(\beta-\alpha s)(\gamma+\alpha s)}{(\beta-s)(\gamma+s)}$	$\frac{(\beta-2)(\gamma+2)}{(\beta-k-1)(\gamma+k+1)}\left[\frac{(\beta-k)(\gamma+k)(\beta-1)(\gamma+1)-\beta\gamma(\beta-k-1)(\gamma+k+1)}{(\beta-1)^2(\gamma+1)^2-\beta\gamma(\beta-2)(\gamma+2)}\right]\ ;\ \beta>2,\ \ k=1,2,...$
$GG\,(\theta,\tau,\lambda)$	$\lambda^s\frac{\Gamma\left(\frac{s}{\tau}+\theta\right)}{\Gamma(\theta)}$	$\lambda^{(1-\alpha)s}\frac{\Gamma\left(\frac{s}{\tau}+\theta\right)}{\Gamma\left(\frac{\alpha s}{\tau}+\theta\right)}$	$\frac{\Gamma\left(\frac{1}{\tau}+\theta\right)\left\{\Gamma(\theta)\Gamma\left(\frac{\alpha^k+1}{\tau}+\theta\right)-\Gamma\left(\frac{\alpha^k}{\tau}+\theta\right)\Gamma\left(\frac{1}{\tau}+\theta\right)\right\}}{\Gamma\left(\frac{\alpha^k}{\tau}+\theta\right)\left\{\Gamma(\theta)\Gamma\left(\frac{2}{\tau}+\theta\right)-\left(\Gamma\left(\frac{1}{\tau}+\theta\right)\right)^2\right\}};\ k=1,2,...$

4.2. Type II- product autoregressive models

This type of models is widely used in the context of latent variable models and their Bayesian analysis. Soyer *et al.* (2016) constructed a Markovian product autoregressive model given by

$$Z_t = \frac{Z_{t-1}}{\gamma}\varepsilon_t. \tag{19}$$

In this model, the sequence is governed by a discounting factor γ and this results an implied stochastic ordering between two consecutive observations, i.e., $Z_t < \frac{Z_{t-1}}{\gamma}$.

The conditional distributions of consecutive observations are all scaled Beta densities,

$$Z_t|F_{t-1} \sim \beta\left(\gamma\alpha_{t-1}, (1-\gamma)\,\alpha_{t-1}; (0, Z_{t-1}\gamma)\right).$$

That is,

$$f\left(Z_t|\, F_{t-1}\right) = \frac{1}{\beta\left(\gamma\alpha_{t-1}, (1-\gamma)\alpha_{t-1}\right)}\left(\frac{\gamma}{Z_{t-1}}\right)^{\alpha_{t-1}-1} {X_t}^{\gamma\alpha_{t-1}-1}\left(\frac{Z_{t-1}}{\gamma} - Z_t\right)^{(1-\gamma)\alpha_{t-1}-1}. \tag{20}$$

From equation (20) we obtain $E\left(Z_t|\, F_{t-1}\right) = Z_{t-1}$ which results in a random walk type of evolution in the expected Poisson rates. Soyer *et al.* (2016) further demonstrate the usefulness of this PAR model by taking suitable prior distributions and getting analytical expressions for various quantities like predictive likelihood.

In the next section, we list the different methods for statistical estimation of PAR models discussed in the literature.

5. Parameter estimation of PAR models

Most of the estimation methods available in the literature for non-linear and non-Gaussian time series are model-specific. That is, these methods depend on a particular model structure and the type of stationary marginal distribution or the type of innovation distribution. The estimation methods for PAR(1) models is very cumbersome in most cases due to the complex form of their innovation distribution. The maximum likelihood estimation approach has been used by Balakrishna and Shiji (2010) and Balakrishna and Lawrance (2012) for Weibull and gamma PAR(1) models, respectively, based on approximate innovation density. Abraham and Balakrishna (2012) proposed conditional least-squares estimates for gamma PAR(1) model and obtained the asymptotic distribution of estimators. Balakrishna and Muhammed (2017) proposed a method of optimal estimating functions for estimation of Weibull and gamma PAR(1) models and described the asymptotic properties. In this section, we review the methods commonly used for parameter estimation of PAR(1) models.

5.1. Maximum likelihood method

If we have an explicit form for the innovation density function, then the likelihood-based inference is possible for the PAR (1) model in (2). Suppose that $(z_0, z_1, ..., z_n)$ is a realisation from a stationary PAR(1) model with stationary marginal density function, $f\,(.)$, and one-step transition density, $f\left(z_t|z_{t-1}\right)$. Let $\theta = (\theta_1, \theta_2, ..., \theta_p)'$ be the vector of parameters indexing the finite-dimensional distribution of the sequence. Then, the likelihood function of θ based on the observed realisation $(z_0, z_1, ..., z_n)$ can be written as

$$L_n\left(\theta\right) = f\left(z_0\right)\prod_{t=1}^{n} f\left(z_t|z_{t-1}\right) = f\left(z_0\right)\prod_{t=1}^{n}\frac{1}{z_{t-1}^{\alpha}} f_\eta\left(\frac{z_t}{z_{t-1}^{\alpha}}\right), \tag{21}$$

where $f_\eta(.)$ is the innovation density function. Ignoring the term corresponding to the initial density (as its influence on the overall likelihood function diminishes as the sample size n increases), the log-likelihood function is

$$\log L_n(\theta) = \sum_{t=1}^{n} \log f_\eta\left(\frac{z_t}{z_{t-1}^{\alpha}}\right) + \sum_{t=1}^{n} \log\left(\frac{1}{z_{t-1}^{\alpha}}\right). \tag{22}$$

Then, the maximum likelihood estimators of model parameters are given by

$$\hat{\theta} = \underset{\theta\in\Theta}{\arg}\ \max \log L_n(\theta). \tag{23}$$

Under certain regularity conditions of Billingsley (1961), the likelihood equations admit consistent solutions and the maximum likelihood estimates are consistent and asymptotically normal. However, the likelihood-based inference of PAR(1) models is cumbersome in most cases because of the complex structure of innovation random variables. Therefore, alternative estimation methods are discussed.

5.2. Conditional least-squares method

Klimko and Nelson (1978) proposed the conditional least-squares (CLS) estimation method for Markov processes. Let $\{Z_t\}$ be the stationary Markov sequence. The CLS estimators of the PAR(1) parameters are obtained by minimising

$$Q_n(\theta) = \sum_{t=1}^{n} \left[Z_t - g(\theta; Z_{t-1})\right]^2 \tag{24}$$

with respect to the parameter vector $\theta = (\theta_1, \theta_2, ..., \theta_p)'$, where $g(\theta; Z_{t-1}) = E(Z_t|Z_{t-1})$. The CLS estimates are obtained by solving the least squares equations:

$$\frac{\partial Q_n(\theta)}{\partial \theta_i} = 0, \quad i = 1, 2, ..., p. \tag{25}$$

Klimko and Nelson (1978) proved under certain regularity conditions that the CLS estimators are consistent and asymptotically normal.

5.3. Estimating functions method

The CLS method described in Section 5.2. is not capable of handling all parameters that are presented in the conditional mean function of a PAR(1) model. In other words, the so called parameter identifiability problem arises when we use CLS method. That is, in CLS method, we need to minimize an objective function which is expressed as the (conditional) sum of squares of deviations taken from conditional mean function, conditional on the past values of the time series. But in most of the cases, we fail to get estimators of all parameters if we restrict the CLS method based on conditional mean function alone. As a remedy to this problem, Balakrishna and Muhammed (2017) suggested the method of estimating function (EF) which is more general method and a convenient way to manage the parameter identifiability by considering an objective function consisting of more than one moment condition. For a general discussion of theory and applications of EF method of estimation in time series analysis, we refer Godambe (1985) and Ghahramani and Thavaneswaran (2009). As standard linear estimating function methods do not help in estimating the parameters of PAR(1) models, Balakrishna and Muhammed (2017) proposed a combined estimating function method that combines linear and quadratic estimating functions to improve the efficiency of the resulting estimates. In what follows, we briefly describe the EF method applied for PAR models.

Suppose that $\{z_t,\ t=1,2,...\}$ is a realisation of a PAR(1) process whose finite-dimensional distribution is indexed by a vector parameter θ belonging to an open subset Θ of the p-dimensional Euclidean space. Let $(\Omega, \Im, P_\theta)$ denote the underlying probability space and $\Im_t^z$ be the sigma field generated by $\{z_1, z_2, ..., z_t,\ \ t \geq 1\}$.

The following conditional moments are useful for constructing the combined estimating function to estimate the parameters of the PAR(1) models discussed in Section 3:

$$\mu_t(\theta) = E\left(Z_t | \Im_{t-1}^z\right), \quad \sigma_t^2(\theta) = V\left(Z_t | \Im_{t-1}^z\right),$$

$$\gamma_t(\theta) = \frac{E\left[\left(Z_t - \mu_t(\theta)\right)^3 | \Im_{t-1}^z\right]}{\sigma_t^3(\theta)} \ \ and\ \ \kappa_t(\theta) = \frac{E\left[\left(Z_t - \mu_t(\theta)\right)^4 | \Im_{t-1}^z\right]}{\sigma_t^4(\theta)} - 3. \tag{26}$$

That is, we assume that the skewness and the excess kurtosis of the standardised variable Z_t does not contain any additional parameters. In order to estimate the parameter vector θ based on the observations $\{z_1, z_2, ..., z_t,\ \ t \geq 1\}$, we consider two classes of martingale differences $\{m_t(\theta) = z_t - \mu_t(\theta),\ t = 1, 2, ...\}$ and $\{s_t(\theta) = m_t^2(\theta) - \sigma_t^2(\theta),\ t = 1, 2, ...\}$ such that

$$\langle m\rangle_t = E\left(m_t^2 | \Im_{t-1}^z\right) = E\left[\left(z_t - \mu_t(\theta)\right)^2 | \Im_{t-1}^z\right] = \sigma_t^2,$$

$$\langle s\rangle_t = E\left(s_t^2 | \Im_{t-1}^z\right) = \sigma_t^4\left(\kappa_t + 2\right),$$

and

$$\langle m, s\rangle_t = E\left(m_t s_t | \Im_{t-1}^z\right) = \sigma_t^3 \gamma_t.$$

The optimal estimating functions based on the martingale differences and are summarised in the following theorem proved by Liang, Thavaneswaran, and Abraham (2011).

Theorem: Let $\{Z_t\}$ be a discrete parameter stochastic process with conditional moments given by (21), then, in the class of all quadratic estimating functions of the form

$$G_Q = \left\{ g_Q(\theta) :\ g_Q(\theta) = \sum_{t=1}^{n} \left(a_{t-1} m_t + b_{t-1} s_t\right) \right\},$$

where a_{t-1} and b_{t-1} are $\Im_{t-1}^z$ measurable functions; the optimal one is given by

$$g_Q^*(\theta) = \sum_{t=1}^{n} \left(a_{t-1}^* m_t + b_{t-1}^* s_t\right), \tag{27}$$

where

$$a_{t-1}^* = \left(1 - \frac{\langle m,s\rangle_t^2}{\langle m\rangle_t \langle s\rangle_t}\right)^{-1} \left(-\frac{\partial \mu_t}{\partial \theta} \frac{1}{\langle m\rangle_t} + \frac{\partial \sigma_t^2}{\partial \theta} \frac{\langle m,s\rangle_t}{\langle m\rangle_t \langle s\rangle_t}\right)$$

and

$$b_{t-1}^* = \left(1 - \frac{\langle m,s\rangle_t^2}{\langle m\rangle_t \langle s\rangle_t}\right)^{-1} \left(\frac{\partial \mu_t}{\partial \theta} \frac{\langle m,s\rangle_t}{\langle m\rangle_t \langle s\rangle_t} - \frac{\partial \sigma_t^2}{\partial \theta} \frac{1}{\langle s\rangle_t}\right)$$

with the associated Godambe information matrix

$$I_{g_Q^*}(\theta) = \sum_{t=1}^{n} \left\{ \left(1 - \frac{\langle m,s\rangle_t^2}{\langle m\rangle_t \langle s\rangle_t}\right)^{-1} \left(\frac{\partial \mu_t}{\partial \theta} \frac{\partial \mu_t}{\partial \theta^T} \frac{1}{\langle m\rangle_t} + \frac{\partial \sigma_t^2}{\partial \theta} \frac{\partial \sigma_t^2}{\partial \theta^T} \frac{1}{\langle s\rangle_t} - \left(\frac{\partial \mu_t}{\partial \theta} \frac{\partial \sigma_t^2}{\partial \theta^T} + \frac{\partial \sigma_t^2}{\partial \theta} \frac{\partial \mu_t}{\partial \theta^T}\right) \frac{\langle m,s\rangle_t}{\langle m\rangle_t \langle s\rangle_t}\right) \right\}.$$

Thus, the optimal estimators are obtained by solving the equation $g_Q^*(\theta) = 0$. The standard errors of the estimators may be found using the information matrix (possibly by observed Godambe information matrix). Note that the EF method do not assumes the knowledge of complete probability distribution of the process except perhaps first few conditional moments. This flexibility and generality of the method facilitates a robust and semi parametric approach to the estimation problem.

6. Applications of PAR(1) models

The application avenues of the multiplicative version of AR models are vast, as the modelling of non-negative random variables plays a major role in the context of economic and financial time series analysis. In this section, we provide some of the applications of PAR(1) models discussed in the literature.

6.1. Modelling non-negative variables

When the time series of interest is a sequence of non-negative random variables, such as volatility, stock market index or wave heights, the product form of the AR models is desirable compared with their linear counterparts. Another context in which the modelling of non-negative random variables plays a major role is the study of financial time series, where one has to model the evolution of conditional variances, called volatility. The applications of PAR(1) models for modelling non-negative random variables have been studied extensively by Balakrishna and Shiji (2010) and Balakrishna and Lawrance (2012). Balakrishna and Shiji (2010) applied the Weibull PAR(1) model to analyse the daily maximum of the Bombay Stock Exchange index values, and Balakrishna and Lawrance (2012) fitted the gamma PAR(1) model to an hourly wave height series from the Bay of Bengal and found that the model could certainly be used in realistic simulations of non-negative time series such as wave heights.

6.2. Bivariate exponential distribution from PAR(1) models

Balakrishna and Shiji (2014) introduced a class of absolutely continuous bivariate exponential distributions in which the components are linked through a product structure of Balakrishna and Lawrance (2012), with an exponential marginal distribution given in (12).

Let (X, Y) be a non-negative random vector, and $F_X(.)$ and $F_Y(.)$ be the distribution functions of X and Y, respectively. Define

$$Y = X^{\alpha} Z, \quad 0 < \alpha < 1, \tag{28}$$

where Z is a non-negative random variable independent of X, such that the equality in (28) holds in the distribution. Balakrishna and Shiji (2014) identified the distribution of Z, for which (X, Y) has an absolutely continuous bivariate exponential distribution when the marginal random variables are tied together using (28).

Let X be an exponential random variable with pdf given in (12), and Z be defined by $Z = \beta^{-1}\left(\frac{\lambda}{S}\right)^{\alpha}$, where S is a positive stable random variable with Laplace transform $\phi_S(s) = \exp(-s^{\alpha})$, $0 < \alpha < 1$.

Then, for any $\alpha \in (0, 1)$, the distribution of $Y = X^{\alpha} Z$ is obtained as

$$f(y; \beta) = \beta \exp(-\beta y), \quad y > 0.$$

Balakrishna and Shiji (2014) employed this result to construct a bivariate random vector with exponential marginals and obtained the density function of (X, Y) as

$$f(x, y) = \left(\beta \lambda^{1-\alpha} x^{-\alpha}\right) \exp(-\lambda x) f_v\left(\frac{\beta y}{(\lambda x)^{\alpha}}\right),$$

where $f_V(.)$ is the pdf of $V = S^{-\alpha}$, which may be expressed as

$$f_V(\upsilon;\alpha) = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\Gamma(k\alpha)}{\Gamma(k)}(-\upsilon)^{k-1}\sin(k\pi\alpha),\ \ \upsilon > 0.$$

In addition to discussing the properties of this distribution, Balakrishna and Shiji (2014) proposed inference procedures for their proposed model. The authors considered a set of data reported in Hanagal (2011) on bone marrow transplantation for leukemia patients and another dataset reported in Jamalizadeh and Kundu (2013) on game time from American Football League from the matches on three consecutive weekends in the year 1986 for the illustration purpose. The results suggest that the proposed bivariate exponential distribution is a good fit for both datasets.

6.3. Stochastic volatility generated by PAR models

Abraham, Balakrishna, and Sivakumar (2006) proposed a stochastic volatility model in which non-negative volatility sequence is generated from linear gamma AR(1) of Gaver and Lewis (1980) and proved that the model captures the excess kurtosis implied by financial return series. Muhammed Anvar *et al.* (2019) used the PAR(1) structure (2) with generalised gamma marginal distribution given in (15) to generate the unobserved volatilities in the analysis of stochastic volatility of financial return series.

Let $\{Y_t\}$ be the sequence of returns of certain financial asset, and the volatilities are generated by a Markov sequence $\{Z_t\}$ of non-negative random variables. Then, the SV model proposed by Muhammed Anvar *et al.* (2019) is expressed as

$$Y_t = \sqrt{Z_t}\nu_t,\ \ Z_t = Z_{t-1}^{\alpha}\eta_t\,;\ t = 1, 2, \ldots, \tag{29}$$

where $\{\nu_t\}$ is a sequence of independent and identically distributed standard normal variables with mean zero and variance one. The sequence of conditional variance $\{Z_t\}$ is generated by a PAR(1) model with the generalised gamma marginal distribution given in (15). Muhammed Anvar *et al.* (2019) derived the properties of volatility model (29) generated by PAR models and illustrated its applications using simulated and real datasets. In the data analysis, the authors utilised the proposed stochastic volatility sequence generated by the generalised gamma PAR(1) model to analyse the exchange rate volatility of Indian Rupee to British Pound for the period January 01, 2012 to December 31, 2017 and found that the model captures the stylised features of the exchange rate return series.

7. Numerical illustration

To illustrate the applications of the PAR(1) models and the associated inferential results, we first consider a simulation study for examining the sample path properties. Then the proposed PAR(1) model is applied to a real data set. All the computations in this section were performed using the **R** package.

7.1. Simulation of PAR(1) sequences

In this section, we describe an algorithm for generating the realizations from PAR(1) models. As already mentioned in Section 3, the innovation distribution of the exponential PAR(1) model plays the key role in generating the realizations from various PAR(1) processes with exponential, Weibull, gamma and generalized gamma marginals. A basic requirement for generating data from these models is an appropriate method for simulating observations from the innovation random variable of a unit exponential PAR(1) model. The random numbers from this distribution can be generated using the formula (see, Balakrishna (2021))

$$\eta_E = E^{1-\alpha} \sin U . (\sin(\alpha U))^{-\alpha} . (\sin((1-\alpha) U))^{-(1-\alpha)}, \tag{30}$$

where U is a uniform random variable over $(0, \pi)$ and E is an unit exponential random variable independent of U. Once we have realizations from i.i.d. random variables $\{\eta_E\}$, the innovation sequences of the PAR(1) models with specified marginal distributions can be obtained. Thus the innovations for exponential and Weibull PAR(1) models can be generated using the following steps:

1. Specify the values for the parameter α and generate a random sample of size N from η_E using the formula (30). Let $\{\eta_E(t),\ t = 1, 2..., N\}$ denote the resulting sample.
2. The innovation sequence $\{\eta_t\}$ for the exponential PAR(1) model is then obtained as $\eta_t = \lambda^{-(1-\alpha)} \eta_E(t),\ t = 1, 2, ..., N$, for specified values of λ.
3. For specified values of α, λ and θ, the innovations of Weibull PAR(1) model can be obtained as $\eta_t = \left(\lambda^{-(1-\alpha)} \eta_E(t)\right)^{1/\theta},\ t = 1, 2, ..., N$.
4. Finally, obtain the PAR(1) sequence using (2).

Now, we generated a sample of size 10000 from Weibull PAR(1) model using the Steps 1–4 by fixing $\alpha = 0.95$, $\lambda = 5$ and $\theta = 3$. The sample path, ACF and histogram of the simulated sample are given in Figure 1. Figure 1(a) gives the sample path properties of simulated Weibull PAR(1) process. Figure 1(b) is the ACF of the simulated series which is geometrically decreasing, a characterizing property of Weibull PAR(1) sequence. Figure 1(c) is a histogram of the realization generated from Weibull PAR(1) model which shows good agreement between the simulated and theoretical Weibull density curve.

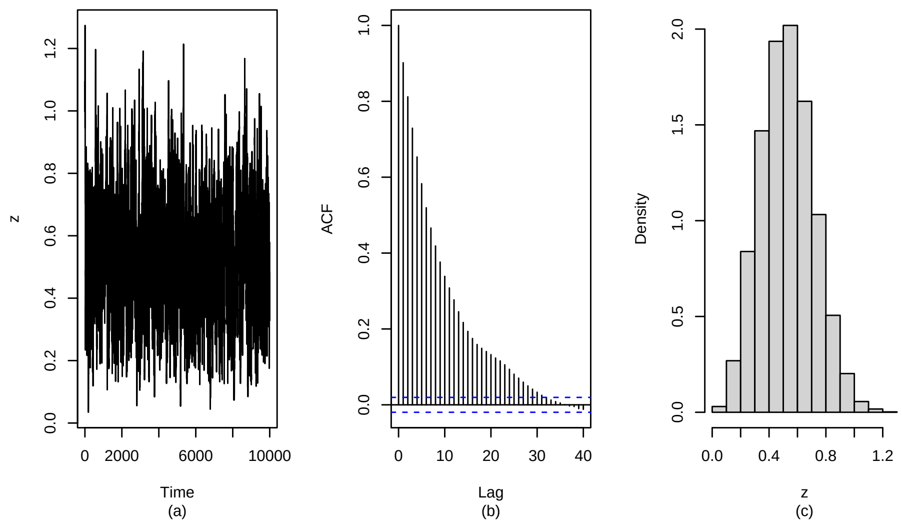

Figure 1: The sample path, ACF and histogram of the simulated sample from Weibull PAR(1) model

7.2. Real data analysis

In this Section, we illustrate the better performance of PAR models compared to its linear additive counter parts when applied to a non-negative series. For the sake of comparison, we now apply the Weibull PAR(1) model of Balakrishna and Shiji (2010) described in Section 3 to a real data set. The data consists of 122 observations of the annual maximum Temperature (°C) in India from the year 1901 to 2022. The data is downloaded from the website of Ministry of Statistics and Programme Implementation, Government of India (https://www.mospi.gov.in). Figure 2 exhibits the basic characteristics of the data. Figure 2(a) provides the time series plot of the original data $\{X_t\}$ and indicates that the series is not stationary. In order to make the data to stationary, we transform the original series by taking the absolute values of first differences $Z_t = |X_t - X_{t-1}|$ and analyse this series by Weibull PAR(1) model. We use the absolute values of the stationary data to retain the non-negativity. Figure 2(b) is the plot of the transformed series, which seems to be stationary.

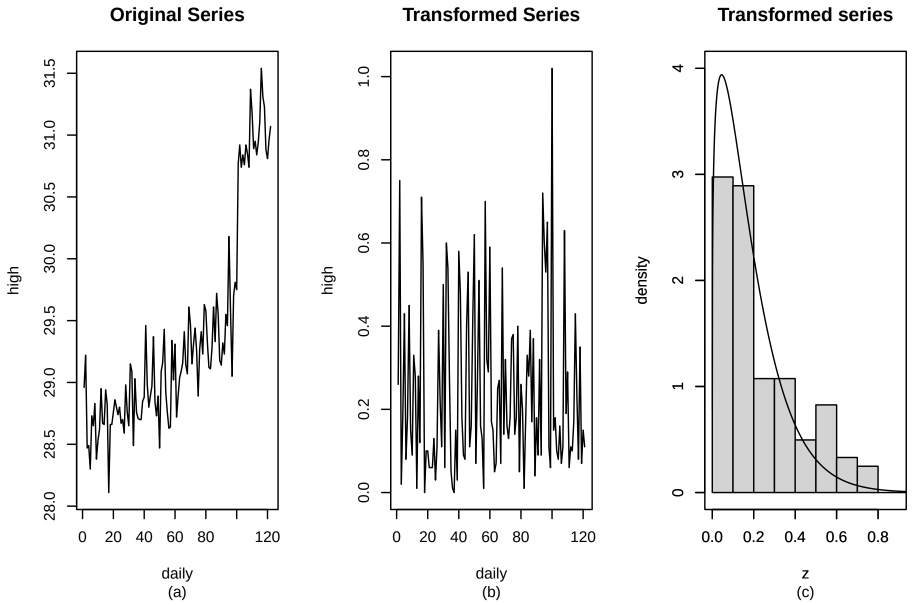

Figure 2: Time series plot of the (a) original series (b) stationary series and (c) Histogram of the stationary series superimposed with Weibull density curve

Table 2 summarizes the descriptive statistics of the original and stationary series, including mean, median, standard deviation, skewness and kurtosis. These statistics suggest the data generating process must be positively skewed and having excess kurtosis than a Gaussian process. The objective is to fit a Weibull PAR(1) model and an additive exponential AR(1) model and then compare their ability to describe the properties of the data. Further, We judiciously fit a Gaussian linear AR(1) model in order to understand the effect of miss-specification.

Table 2: Descriptive statistics of original and stationary versions of annual maximum temperature series

Statistics	Original series	Stationary series
Mean	29.3803	0.2445
Median	29.1300	0.1700
Minimum	28.1100	0.0000
Maximum	31.5400	1.0200
Standard Deviation	0.8341	0.2045
Skewness	1.2443	1.3391
Kurtosis	3.0612	3.8858
Sample size	122	121

The parameters of the proposed Weibull PAR(1) model are estimated using the maximum likelihood method explained in Section 5.1. To compare the performance of PAR(1) model with linear AR models, we have also fitted classical Gaussian AR(1) model and Exponential AR(1) model of Gaver and Lewis (1980) to the stationary data. The estimation results for Weibull PAR(1) and linear AR(1) models are presented in Table 3. Figure 2(c) shows the histogram of the series $\{Z_t\}$ super-imposed by the Weibull probability density function evaluated with these estimated parameters. This plot shows that the marginal distribution of the stationary data follows Weibull distribution as assumed by Balakrishna and Shiji (2010) in the construction of the model.

Table 3: ML estimation results for the real data

Parameter	Normal AR(1)	Exponential AR(1)	Weibull PAR(1)
$\hat{\varphi}$	0.2231	0.2373	-
$\hat{\sigma}$	0.2035	-	-
$\hat{\alpha}$	-	-	0.1490
$\hat{\lambda}$	-	0.8953	0.1904
$\hat{\theta}$	-	-	1.2123
AIC	411.1277	360.1800	316.9335
BIC	411.2932	360.3456	317.1819
$MAPE$	21.0477	15.7203	8.3018

For model selection, we computed the Akaike information criterion (AIC) and Bayesian information criterion (BIC) values for both the fitted models. Table 3 indicates that the Weibull PAR(1) model reported the smallest AIC and BIC values than the fitted linear AR(1) models for the selected data. Note that a substantial reduction in AIC and BIC values is observed for the PAR(1) model. To perform diagnostic checks, the residuals, histogram of residuals and their ACF from the fitted PAR(1) model were plotted in Figures 3 and 4 respectively. The ACF of the resulting residuals and squared residuals in Figure 4 are negligible, indicating the absence of significant serial correlations in the residuals.

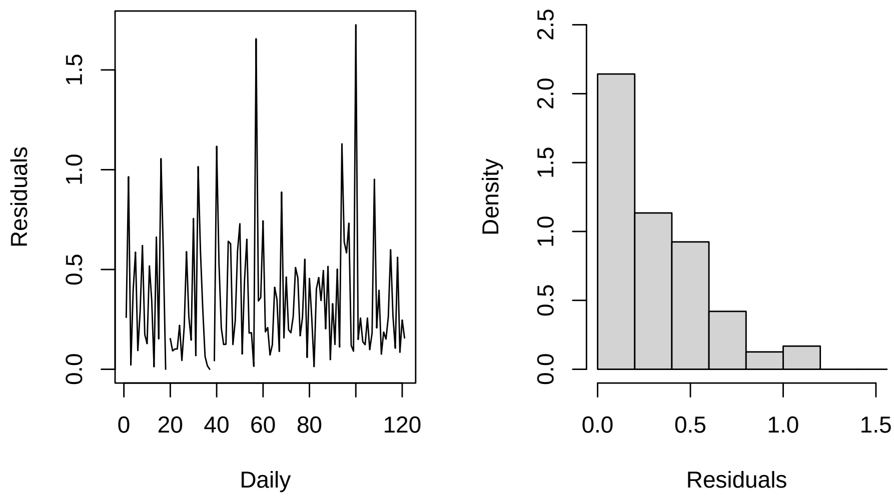

Figure 3: Time series plot and histogram of residuals from the fitted Weibull PAR(1) model

Finally, we evaluated these models based on their forecasting performance. We performed

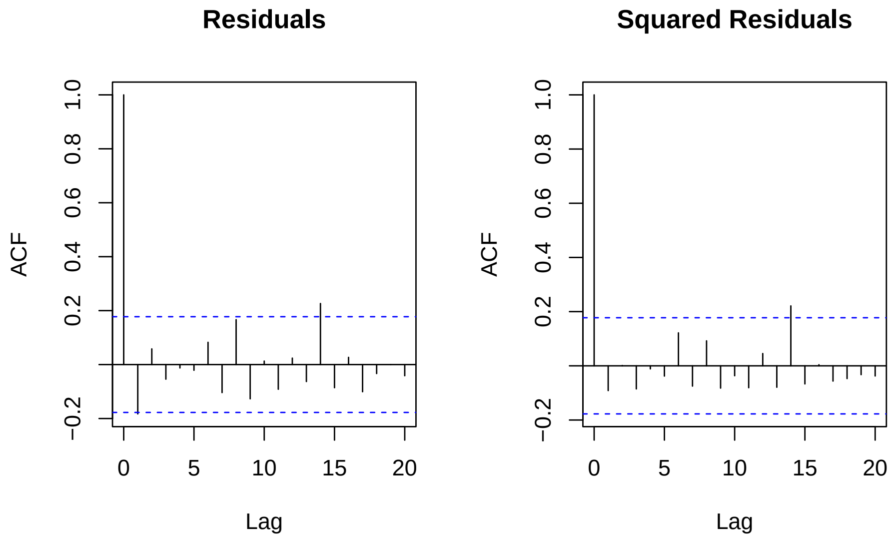

Figure 4: ACF of residuals and squared residuals from fitted Weibull PAR(1) model

an out-of-sample forecast exercise for the selected data using Weibull PAR(1), normal AR(1) and exponential AR(1) models. We divided the data set into training data (first 110 observations; around 90%) and validation data (last 12 observations) for the forecast exercise. Next, we evaluated the one-step-ahead forecast using fitted models and repeated this rolling forecast until we obtained 12 forecast values. The minimum mean squared error forecasts were computed by plugging the estimates of parameters in the conditional mean function of each models. Then the forecast performance measure, viz, mean absolute percentage error (MAPE), was computed using the forecasts from each of the proposed models and the actual validation data. The results are listed in Table 3. The proposed Weibull PAR(1) model clearly outperformed linear AR(1) models with normal and exponential marginals in terms of MAPE. The same pattern in results was also obtained when the data splitting ratio changed to 95:5 percent. However, we didn't include the results due to space restriction. Hence, we verified the adequacy and performance of the proposed PAR(1) model. These illustrations established the relevance and advantage of the proposed PAR models for non-negative time series data in comparison with conventional AR models.

8. Concluding remarks

The modelling of non-negative (continuous valued) time series data gained special attention in many studies, where one has to model non-negative variables such as volatility, trade durations etc. Recently, the necessity of developing pure multiplicative models for non-negative time series has been identified and discussed by various researchers. This lead to apply the multiplicative structure to model a non-negative time series such as MEM, pure multiplicative ETS model, etc. However, a suitable class of models namely product autoregressive that can handle the continuous non negative time series in a direct way and possessing certain auto-correlation structures observed in many empirical non-negative time series data was unnoticed for a long period. PAR type models can appropriately model the auto-correlation structures often present in such data and it is useful for analysing the data in the original scale or in some non-negative transform suggested by theory. Further developments of this type of model were discussed only in the past decade. In view of this, in this paper, we reviewed the developments in the PAR model literature starting from the work of Mckenzie (1982). Since then, several PAR models have been framed to analyse the time series of non-negative random variables. This paper provides an exhaustive review of the construction, properties, estimation methods and applications of PAR models. In best of our knowledge, we have enlisted the works related to the product form of the auto-regressive models discussed in the literature. The PAR models are promising candidates for stochastic volatility analysis and range-based volatility modelling in parameter-driven setups.

Future studies are obviously required for this topic and will be studied vigorously. Several possible extensions to the basic PAR(1) models are in order. First, one can develop the product auto regressive model of order more than one, say p (>1), possibly combining with a moving average component. Identifying the distribution of the innovation random variable will be the challenge as it is not clear how to define infinite divisibility in this case. Further, it is necessary to identify the innovation random variable to simulate observations from these models. Second, a direct multivariate extension is not trivial. Multiple non-negative time series occur frequently in practice. In light of the usefulness of PAR models in univariate case, developing a multivariate PAR models is expected to be helpful. Third, like in the case of classical time series, PAR models can also be viewed in a Bayesian perspective. The main hindrance to the maximum likelihood estimation in PAR models is the lack of manageable likelihood function. Bayesian estimation may be a suitable solution to this problem. Four, as the most cases considered in PAR models are restricted to continuous non-negative variates, some situation may demand the models for discrete valued variates. Even though (linear additive) integer valued autoregressive models (INAR) are available in the literature, it will be instructive to search for a product autoregressive model for discretely valued time series.

References

Abraham B, Balakrishna N (1999). "Inverse Gaussian Autoregressive Models." *Journal of Time Series Analysis*, **20**(6), 605–618. doi:10.1111/1467-9892.00161.

Abraham B, Balakrishna N (2012). "Product Autoregressive Models for Non-negative Variables." *Statistics & Probability Letters*, **82**(8), 1530–1537. doi:10.1016/j.spl.2012.04.022.

Abraham B, Balakrishna N, Sivakumar R (2006). "Gamma Stochastic Volatility Models." *Journal of Forecasting*, **25**(3), 153–171. doi:10.1002/for.982.

Adke SR, Balakrishna N (1992). "Markovian Chi-square and Gamma Processes." *Statistics & Probability Letters*, **15**(5), 349–356. doi:10.1016/0167-7152(92)90152-U.

Akram M, Hyndman RJ, Ord JK (2009). "Exponential Smoothing and Non-negative Data." *Australian & New Zealand Journal of Statistics*, **51**(4), 415–432. doi:10.1111/j.1467-842X.2009.00555.x.

Bakouch HS, Popović BV (2016). "Lindley First-order Autoregressive Model with Applications." *Communications in Statistics-Theory and Methods*, **45**(17), 4988–5006. doi:10.1080/03610926.2014.935429.

Bakouch HS, Ristić MM, Sandhya E, Satheesh S (2013). "Random Products and Product Auto-regression." *Filomat*, **27**(7), 1197–1203. doi:10.2298/FIL1307197B.

Balakrishna N (2021). *Non-Gaussian Autoregressive-type Time Series*. Springer, Singapore. doi:10.1007/978-981-16-8162-2.

Balakrishna N, Lawrance A (2012). "Development of Product Autoregressive Models." *Journal of Indian Statistical Association. The Golden Jubilee Issue*, **50**, 1–20.

Balakrishna N, Muhammed PA (2017). "Estimating Function Method for Product Autoregressive Models." *Communications in Statistics-Simulation and Computation*, **46**(5), 3962–3979. doi:10.1080/03610918.2015.1076468.

Balakrishna N, Shiji K (2010). "A Markovian Weibull Sequence Generated by Product Autoregressive Model and Its Statistical Analysis." *Journal of Indian Society for Probability and Statistics*, **12**, 53–67.

Balakrishna N, Shiji K (2014). "On a Class of Bivariate Exponential Distributions." *Statistics & Probability Letters*, **85**, 153–160. doi:10.1016/j.spl.2013.11.009.

Billingsley P (1961). *Statistical Inference for Markov Processes*. University of Chicago Press, Chicago.

Box GEP, Jenkins GM (1976). *Time Series Analysis: Forecasting and Control*. San Fransisco: Holden Day.

Brownlees CT, Cipollini F, Gallo GM (2012). "Multiplicative Error Models." *In Bauwens L., Hafner C., Laurent S. (Eds.), Volatility Models and Their Applications*, pp. 223–247. doi:10.1002/9781118272039.ch9.

Cipollini F, Gallo GM (2022). "Multiplicative Error Models: 20 Years On." *Econometrics and Statistics*. doi:10.1016/j.ecosta.2022.05.005.

Engle R (2002). "New Frontiers for ARCH Models." *Journal of Applied Econometrics*, **17**(5), 425–446. doi:10.1002/jae.683.

Gaver DP, Lewis PA (1980). "First-order Autoregressive Gamma Sequences and Point Processes." *Advances in Applied Probability*, **12**(3), 727–745. doi:10.2307/1426429.

Ghahramani M, Thavaneswaran A (2009). "Combining Estimating Functions for Volatility." *Journal of Statistical Planning and Inference*, **139**(4), 1449–1461. doi:10.1016/j.jspi.2008.07.014.

Godambe V (1985). "The Foundations of Finite Sample Estimation in Stochastic Processes." *Biometrika*, **72**(2), 419–428. doi:10.1093/biomet/72.2.419.

Granger CW, Newbold P (1976). "Forecasting Transformed Series." *Journal of the Royal Statistical Society: Series B (Methodological)*, **38**(2), 189–203. doi:10.1111/j.2517-6161.1976.tb01585.x.

Hanagal DD (2011). *Modeling Survival Data Using Frailty Models*. Springer. doi:10.1201/b10510.

Hyndman RJ, Athanasopoulos G (2018). *Forecasting: Principles and Practice*. OTexts.

Jamalizadeh A, Kundu D (2013). "Weighted Marshall–Olkin Bivariate Exponential Distribution." *Statistics*, **47**(5), 917–928. doi:10.1080/02331888.2012.670640.

Jayakumar K (2003). "Mittag-Leffler Process." *Mathematical and Computer Modelling*, **37**(12-13), 1427–1434. doi:10.1016/S0895-7177(03)90050-1.

Jayakumar K, Pillai R (1993). "The First-order Autoregressive Mittag–Leffler Process." *Journal of Applied Probability*, **30**(2), 462–466. doi:10.2307/3214855.

Jose KK, Thomas MM (2012). "A Product Autoregressive Model with log-Laplace Marginal Distribution." *Statistica*, **72**(3), 317–336. doi:10.6092/issn.1973-2201/3650.

Klimko LA, Nelson PI (1978). "On Conditional Least Squares Estimation for Stochastic Processes." *The Annals of Statistics*, pp. 629–642. doi:10.1214/aos/1176344207.

Lawrance A (1980). "The Mixed Exponential Solution to the First-order Autoregressive Model." *Journal of Applied Probability*, **17**(2), 546–552. doi:10.2307/3213044.

Lawrance A, Lewis P (1982). "A Mixed Exponential Time Series Model." *Management Science*, **28**(9), 1045–1053. doi:10.1287/mnsc.28.9.1045.

Liang Y, Thavaneswaran A, Abraham B (2011). "Joint Estimation using Quadratic Estimating Function." *Journal of Probability and Statistics*, **2011**(1), Article Id: 372512. doi:10.1155/2011/372512.

Mckenzie E (1982). "Product Autoregression: A Time-Series Characterization of the Gamma Distribution." *Journal of Applied Probability*, **19**(2), 463–468. doi:10.2307/3213502.

Moriña D, Puig P, Valero J (2015). "A Characterization of the Innovations of First Order Autoregressive Models." *Metrika*, **78**, 219–225. doi:10.1007/s00184-014-0497-5.

Muhammed Anvar P, Balakrishna N, Abraham B (2019). "Stochastic Volatility Generated by Product Autoregressive Models." *Stat*, **8**(1), e232. doi:10.1002/sta4.232.

Popović BV (2010). "AR(1) Time Series with Approximated Beta Marginal." *Publications de l'Institut Mathematique*, **88**(102), 87–98. doi:10.2298/PIM1002087P.

Popović BV, Pogány TK, Nadarajah S (2010). "On Mixed AR(1) Time Series Model with Approximated Beta Marginal." *Statistics & Probability Letters*, **80**(19-20), 1551–1558. doi:10.1016/j.spl.2010.06.009.

Rahul T, Balakrishnan N, Balakrishna N (2018). "Time Series with Birnbaum-Saunders Marginal Distributions." *Applied Stochastic Models in Business and Industry*, **34**(4), 562–581. doi:10.1002/asmb.2324.

Sim CH (1990). "First-order Autoregressive Models for Gamma and Exponential Processes." *Journal of Applied Probability*, **27**(2), 325–332. doi:10.2307/3214651.

Soyer R, Aktekin T, Kim B (2016). "Bayesian Modeling of Time Series of Counts with Business Applications." *Handbook of Discrete-Valued Time Series*, pp. 245–266. doi:10.1201/b19485.

Svetunkov I, Boylan JE (2024). "Staying Positive: Challenges and Solutions in Using Pure Multiplicative ETS Models." *IMA Journal of Management Mathematics*, **35**(3), 403–425. doi:10.1093/imaman/dpad028.

Tsai H, Chan KS (2007). "A Note on Non-negative ARMA Processes." *Journal of Time Series Analysis*, **28**(3), 350–360. doi:10.1111/j.1467-9892.2006.00513.x.

Affiliation:

Dr. Rahul Thekkedath
Assistant Adviser
Department of Statistics and Information Management
Reserve Bank of India
Thiruvananthapuram, Kerala - 695003, INDIA
E-mail: rahult@rbi.org.in

Dr. Shiji Kavungal
Assistant Professor
Department of Statistics
Sree Kerala Varma College
Thrissur, Kerala - 680011, INDIA
E-mail: shijik@keralavarma.ac.in

Dr. Muhammed Anvar P
Assistant Professor
Department of Statistical Sciences
Kannur University
Kannur, Kerala - 670567, INDIA
E-mail: anvarp@kannuruniv.ac.in

Austrian Journal of Statistics http://www.ajs.or.at/
published by the Austrian Society of Statistics http://www.osg.or.at/
Volume 54 *Submitted:* 2024-07-25
2025 *Accepted:* 2025-01-03