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On Approximation of Some Lévy Processes

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Abstract

In this paper, we extend the Asmussen-Rosiński approach for the approximation of Lévy processes. To simulate the value of the process at time t, we introduce a time-dependent truncation of the Lévy measure, which we refer to as *dynamic cutting*, followed by the simulation of the large-jump component. We provide the sufficient condition under which the compensated small-jump part can be replaced by a Gaussian approximation. We also derive weak approximation rates for both approaches. Finally, we run numerical simulations and compare the performance of our method with the Asmussen-Rosiński approach.

Keywords: Lévy process, approximation, weak approximation rate, dynamic cutting.

1. Introduction

Let $Z(t)$, $t \geq 0$, be a Lévy process on $\mathbb{R}$, i.e. a stochastically continuous process with independent and stationary increments. If not stated otherwise, we assume $Z(0) = 0$. Then the characteristic function $\phi_t(\xi) = \mathbb{E}e^{i\xi Z(t)}$, $\xi \in \mathbb{R}$, of Z possesses a specific structure, called the *Lévy-Khinchin* representation:

$$\begin{aligned}\phi_t(\xi) &= \exp\left\{-t\left(-i\ell\xi + \frac{Q^2\xi}{2} + \int_{u\neq 0}\left(1 - e^{i\xi u} + i\xi u \mathbb{1}_{|u|\leq 1}\right)\nu(du)\right)\right\} \\ &=: \exp\{-t\psi(\xi)\},\end{aligned} \tag{1.1}$$

where $\ell, Q \in \mathbb{R}$ and the measure $\nu(\cdot)$ is assumed to satisfy $\int_{u\neq 0}\min(|u|^2, 1)\nu(du) < \infty$; such a measure is called a *Lévy measure*. There is one-to-one correspondence between a given *triplet* $(\ell, Q^2, \nu(\cdot))$ and a Lévy process, i.e. for any Lévy process there exists a triplet $(\ell, Q^2, \nu(\cdot))$ such that (1.1) is satisfied, and the converse is also true.

Starting with the triplet $(\ell, Q^2, \nu(\cdot))$, one can write the *Lévy-Ito* stochastic representation of Z. Denote by $\mathcal{N}(\cdot,\cdot)$ the Poisson random measure, related to Z:

$$\mathcal{N}^Z(du, ds) := \sum_{s>0:\Delta Z(s)\neq 0} \delta_{\Delta Z(s),s}(du, ds). \tag{1.2}$$

For any measurable set $D \subset \mathcal{B}(\mathbb{R})$ denote $\mathcal{N}^Z_t(D) := \int_0^t \mathcal{N}^Z(D, ds)$; $\mathcal{N}^Z_t(D)$ is a Poisson random process and $\mathbb{E}\mathcal{N}_t^Z(D) = t\nu(D)$ is its *compensator*. We use the notation

$$\tilde{\mathcal{N}}^Z(dy, ds) = \mathcal{N}^Z(dy, ds) - \nu(dy)ds, \quad \tilde{\mathcal{N}}_t^Z(dy) = \int_0^t \tilde{\mathcal{N}}^Z(dy, ds) = \mathcal{N}_t^Z(dy) - t\nu(dy)$$

for the compensated Poisson random measure and its time integral. Then (see, for example, (Protter 2004, Th.I.42) or (Böttcher, Schilling, and Wang 2013, Th.2.12)) $Z(t)$ admits the following representation:

$$Z(t) = Z(0) + \ell t + QB(t) + \int_0^t \int_{0<|y|\le 1} y\tilde{\mathcal{N}}^Z(dy, ds) + \sum_{0<s\le t} \Delta Z(s)\mathbb{1}_{|\Delta Z(s)|>1}. \tag{1.3}$$

Here $B(t)$ is the Brownian motion at time $t > 0$. A Lévy process is a semi-martingale: one can rewrite (1.3) as

$$Z(t) = Z(0) + M(t) + A(t), \tag{1.4}$$

where

$$M(t) := QB(t) + \int_{0<|y|\le 1} y\tilde{\mathcal{N}}_t^Z(dy), \quad A(t) = \ell t + \sum_{0<s\le t} \Delta Z_s \mathbb{1}_{|\Delta X_s|>1},$$

are, respectively, the martingale and the process having finite variation on compacts, see (Protter 2004, Th.I.40, 42).

A Lévy process has countably many jumps on any interval $[0, T]$ and finitely many jumps of size bigger than some fixed $\varepsilon > 0$. In order to simulate Z, we need to take finitely many jumps of Z, which gives an adequate description of Z. Apart from some particular cases, e.g. Brownian motion, Gamma process, α-stable process, simulation of the Lévy process with a given triplet is not an easy task. Usually, the distribution function of a Lévy process is unknown or has a rather complicated form, which makes the simulation rather perplex. For the methods of generating infinitely divisible random variables (r.v.'s) and Lévy processes (e.g. methods of Khinchin, Fergusson-Klass, Bondesson, LePage, Rosiński) we refer to Rosiński (2001). We also would like to mention that the Damien-Laud-Smith algorithm from Damien, Laud, and Smith (1995) gives a way to simulate an (approximation of) an arbitrary one-dimensional infinitely divisible r.v., which allows us to simulate a Lévy process. On the other hand, it was observed by Bondesson (1982) and later by Asmussen and Rosiński (2001), that under some conditions small jumps can be substituted by an (arithmetic) Brownian motion. Necessary and sufficient conditions for this substitution were proved in Asmussen and Rosiński (2001). The idea of this method is the following: one can cut away jumps of size less than some fixed level δ and in such a way get the *first order approximation*

$$Z_\delta^{1,AR}(t) := t\left(\ell - \int_{\delta<|u|\le 1} u\nu(du)\right) + QB(t) + \sum_{s\le t} \mathbb{1}_{|\Delta Z(s)|>\delta}\Delta Z(s). \tag{1.5}$$

Under certain assumption on the Lévy measure one can write a better approximation by substituting compensated small jumps with the (arithmetic) Brownian motion with certain variance:

$$Z_\delta^{2,AR}(t) := t\left(\ell - \int_{\delta<|u|\le 1} u\nu(du)\right) + (Q^2 + \sigma^2(\delta))^{1/2}B(t) + \sum_{s\le t} \mathbb{1}_{|\Delta Z(s)|>\delta}\Delta Z(s), \tag{1.6}$$

where $\sigma^2(\delta) := \int_{|u|\le\delta} u^2\nu(du)$.

These approximations were used in Asmussen and Rosiński (2001) in order to get the error rates in the form of Berry-Esseen bounds; however, the same technique can be used to get the weak approximation rates. Related results on the Wasserstein and total variation distances between the small-jump part of a Lévy process and its Gaussian approximation were obtained in Mariucci and Reiß (2018) and the works cited therein.

In this paper we propose a development of this approach, which we believe to be more flexible and give more accurate numerical results. We cut the Lévy measure in a time-dependent way, which we call the *dynamic cutting* (DC), and then simulate the large-jump part. We state the sufficient condition which allows us to substitute the compensated small jump-part with a Gaussian component. In both cases we provide the weak approximation rates. Our proof of the error bounds uses the Burkholder-Davis-Gundy inequality for semi-martingales, see the survey paper Kühn and Schilling (2023) as well as the Kunita (2004). This approach is different from that in Asmussen and Rosiński (2001). We provide an algorithm for simulation of Z, as well as some numerical results, which we compare with those obtained by the Asmussen and Rosiński (AR) approach.

Various convergence rates for Lévy-driven SDEs, where the AR approach is used to approximate the corresponding Lévy process, are obtained e.g. in Fournier (2011), Kohatsu-Higa and Ngo (2013), Bally and Qin (2022), Bossy and Maurer (2024), see also the results quoted therein. We believe that our approach and the results, e.g. Theorems 2.2 and 2.3, can be applied for investigations of the Euler scheme for Lévy-driven SDEs and can give better accuracy.

The structure of our paper is the following. In Section 2 we give some notions on Lévy processes and formulate our results. Proofs of Theorems 2.1, 2.2 and 2.3 are given in Section 4. Examples are provided in Section 3.

2. Construction and approximation

2.1. Notation and assumptions

We write $f \lesssim g$ if there exists a generic constant $C > 0$ such that $f \le Cg$. The notation $f \asymp g$ means that $f \lesssim g$ and $g \lesssim f$.

Let $\nu^+ := \nu \mathbb{1}_{(0,\infty)}$, $\nu^- := \nu \mathbb{1}_{(-\infty,0)}$,

$$N^+(r) = \nu^+(r,\infty), \quad N^-(r) = \nu^-(-\infty,-r), \quad r > 0. \tag{2.1}$$

Define the generalized inverses of functions $N^\pm$, respectively, at t^{-1}:

$$\tau^\pm(t) = \sup\{r \ge 0 : N^\pm(r) \ge 1/t\}, \quad t > 0. \tag{2.2}$$

Note that we cannot expect to have the relation $N^\pm(\tau^\pm(t)) = \frac{1}{t}$ unless the functions $N^\pm(r)$ are right-continuous, see (Embrechts and Hofert 2013, Prop.2.3). For simplicity, we make the assumption (see (**A1**) below), which guarantees the above relation.

Define the *Pruitt functions*, see Pruitt (1981):

$$\psi^{L,\pm}(\xi) := \int_{u\neq 0, |u\xi|\le 1} |u\xi|^2 \nu^\pm(du), \quad \psi^{U,\pm}(\xi) := \int_{u\neq 0} (|u\xi|^2 \wedge 1)\nu^\pm(du).$$

Assumption.

Functions $N^\pm$ are strictly monotone and continuous. (**A1**)

$N^\pm(r) \asymp \psi^{L,\pm}(1/r), \quad r > 0.$ (**A2**)

Let us discuss the assumption (**A2**) for N^+, $\psi^{L,+}$ and $\psi^{U,+}$, the consideration for N^-, $\psi^{L,-}$ and $\psi^{U,-}$ are the same.

Assumption (**A2**) means that $\psi^{L,+}(1/r)$, which in general is not monotone in r, can be controlled by the monotone function $N^+(r)$. One can construct an examples in which the measure $\nu^+(\cdot)$ is not absolutely continuous with respect to the Lebesgue measure, the function $\psi^{L,+}(1/r)$ is not monotone, but nevertheless (**A2**) is satisfied, see Knopova and Kulik (2013).

By the definition of N^+ and $\psi^{U,+}$ we always have $N^+(r) \le \psi^{U,+}(1/r) \lesssim 1/r^2, \quad r \in (0,1]$. However, one can show that $\psi^{U,+}$ has sub-quadratic growth at infinity. Indeed, by definition, $\psi^{L,+}(r) = r^2 \int_{0<|u|\le 1/r} u^2 \nu^+(du)$. Since we exclude 0 in the integral, the integral part tends to 0 as $r \to \infty$, implying the sub-quadratic growth of $\psi^{L,+}$ and, by the relation $\psi^{U,+} \asymp \psi^{L,+}$, the same property holds true for $\psi^{U,\pm}(r)$. Condition (**A2**) implies in fact more, namely, the existence of $0 < \beta_+ < \alpha_+ < 2$, such that

$$r^{\beta_+} \lesssim \psi^{U,+}(r) \lesssim r^{\alpha_+}, \quad r \ge 1. \tag{2.3}$$

Indeed, by (**A2**), there exist $c, C > 0$, such that $c\psi^{L,+}(1/r) \le N^+(r) \le C\psi^{L,+}(1/r)$ for $r \in (0,1]$, implying that

$$\gamma\psi^{L,+}(r) \le \psi^{U,+}(r) \le \beta\psi^{L,+}(r), \quad r \ge 1, \tag{2.4}$$

where $\gamma = c + 1 \le \beta = C + 1$. Estimates (2.4) allow us to derive the estimates $r^{2/\beta} \lesssim \psi^{U,+}(r) \lesssim r^{2/\gamma}$ for $r \ge 1$, see Knopova and Kulik (2013) for the lower bound, also Lemma 4.1 below, where the same argument was used. The same argument can be used to prove the upper bound. Since $c > 0$, we have (2.3) with some $\alpha_+ = 2/\gamma \in (0,2)$.

Note that if (**A2**) fails, we cannot guarantee that $\alpha_+ < 2$. Indeed, take $\nu^+(du) = \frac{du}{u^3|\ln u|^{1+\varepsilon}}$, $u > 0$, where $\varepsilon > 0$. We have

$$N^+(r) \asymp \frac{1}{r^2|\ln r|^{1+\varepsilon}}, \quad \psi^{L,+}\left(\frac{1}{r}\right) \asymp \frac{1}{r^2|\ln r|^{\varepsilon}}, \quad r \in (0,1].$$

In this case we still have $\psi^{U,+}(r) \asymp \psi^{L,+}(r)$, $r \ge 1$, but (**A2**) fails. The right-hand side of (2.3) holds with $\alpha_+ = 2$, but fails for any $\alpha_+ \in (0,2)$.

Thus, under (**A2**), there exists $\alpha_\pm \in (0,2)$, such that

$$N^\pm(r) \lesssim r^{-\alpha_\pm}, \quad r \in (0,1]. \tag{2.5}$$

By definition of $\tau^\pm(t)$, (2.5) implies

$$\tau^\pm(t) \lesssim t^{1/\alpha_\pm}, \quad t \in (0,1]. \tag{2.6}$$

We consider two types of approximation of $Z(t)$. In what follows we fix the endpoint $T > 0$ and the parameter $\varepsilon \in (0,1)$, which is needed for the existence of finite intensity functions $\lambda^\pm_{h,\varepsilon}(t)$, see below. We also fix the "tuning parameter" $h \in (0,1)$ such that $Th \ll 1$.

2.2. First order approximation: construction and simulation algorithm

We delete the small-jump part and its compensator by the following dynamic cutting:

$$Z^{1,DC}_{h,\varepsilon}(t) := Z_0 + m^{DC}_{h,\varepsilon}(t) + Z^{CP,+}_{h,\varepsilon}(t) + Z^{CP,-}_{h,\varepsilon}(t), \quad t > 0, \tag{2.7}$$

where

$$Z^{CP,+}_{h,\varepsilon}(t) := \sum_{s\le t} \Delta Z_s \mathbb{1}_{\Delta Z_s \ge \tau^+((sh)^\varepsilon)}, \quad Z^{CP,-}_{h,\varepsilon}(t) := \sum_{s\le t} \Delta Z_s \mathbb{1}_{\Delta Z_s \le -\tau^-((sh)^\varepsilon)}, \tag{2.8}$$

are the upward and downward jumps, and

$$\begin{aligned} m^{DC}_{h,\varepsilon}(t) :&= \ell t - \int_0^t \left(\int_{-1<u<-\tau^-((sh)^\varepsilon)} u\nu^-(du) + \int_{\tau^+((sh)^\varepsilon)<u<1} u\nu^+(du) \right) ds \\ &= \ell t - \left(\ell^{DC,-}_{h,\varepsilon}(t) + \ell^{DC,+}_{h,\varepsilon}(t) \right) \\ &= \int_0^t \ell_{h,\varepsilon}(s) ds, \end{aligned} \tag{2.9}$$

is the (compensated) drift.

Let us calculate the logarithm of the Laplace transform of $Z_{h,\varepsilon}^{CP,+}(t)$. Using the Fubini theorem and the relation $(\tau^+(u))^{-1} = \frac{1}{N^+(u)}$, we get

$$\begin{aligned}
-\ln \mathbb{E}e^{-rZ_{h,\varepsilon}^{CP,+}(t)} &= \int_0^t \int_{u>\tau^+((sh)^\varepsilon)} (1-e^{-ru})\nu^+(du)ds \\
&= \int_0^\infty \int_0^{\frac{1}{h}((\tau^+)^{-1}(u))^{1/\varepsilon}\wedge t} (1-e^{-ru})ds\nu^+(du) \\
&= \int_0^\infty (1-e^{-ru})\left(\frac{1}{h}(N^+(u))^{-1/\varepsilon}\wedge t\right)\nu^+(du) \\
&= \int_0^\infty (1-e^{-ru})\mu_{h,\varepsilon}^+(t,du),
\end{aligned} \tag{2.10}$$

where

$$\mu_{h,\varepsilon}^+(t,du) := \left(\frac{1}{h}(N^+(u))^{-1/\varepsilon}\wedge t\right)\nu^+(du). \tag{2.11}$$

It follows from representation (2.11) that

$$\mu_{h,\varepsilon}^+(t,du) = \frac{1}{h}\mu_{1,\varepsilon}^+(th,du) = t\mu_{th,\varepsilon}^+(1,du). \tag{2.12}$$

Fix $t>0$. Then $Z_{h,\varepsilon}^{CP,+}(t)$ is an inhomogeneous compound Poisson (ICP) r.v., i.e.

$$Z_{h,\varepsilon}^{CP,+}(t) = \sum_{k=1}^{N_t^+} Z_k^+(t), \tag{2.13}$$

where N_t^+ is a Poisson r.v. with intensity

$$\lambda_{h,\varepsilon}^+(t) = \int_0^t \int_{u\geq\tau^+((sh)^\varepsilon)} \nu^+(du)ds = \mu_{h,\varepsilon}^+(t,\mathbb{R}_+), \tag{2.14}$$

and $Z_k^+(t)$, $k\geq 1$, are i.i.d. r.v.'s with the distribution function

$$F_{th,\varepsilon}^+(x) := \frac{1}{\lambda_{h,\varepsilon}^+(t)}\int_0^x \mu_{h,\varepsilon}^+(t,du). \tag{2.15}$$

Note that by (2.11) this distribution function depends on th. Let us calculate $\lambda_{h,\varepsilon}^+(t)$ explicitly:

$$\lambda_{h,\varepsilon}^+(t) = \int_0^t \nu^+\{u:\ u\geq\tau^+((sh)^\varepsilon)\}ds = \int_0^t\left(\frac{1}{hs}\right)^\varepsilon ds = \frac{t^{1-\varepsilon}}{1-\varepsilon}\left(\frac{1}{h}\right)^\varepsilon. \tag{2.16}$$

One can get a more user-friendly expression for the distribution function of $Z_k^+(t)$.
After several transformations one gets

$$\int_0^x \mu_{1,\varepsilon}^+(t,du) = \begin{cases} \frac{\varepsilon}{1-\varepsilon}(N^+(x))^{1-1/\varepsilon}, & 0\leq x\leq\tau^+(t^\varepsilon), \\ \frac{t^{1-\varepsilon}}{1-\varepsilon} - tN^+(x), & x>\tau^+(t^\varepsilon). \end{cases}$$

This representation allows us to rewrite $F_{h,\varepsilon}^+(x)$ as

$$F_{h,\varepsilon}^+(x) = \begin{cases} \left(\frac{1}{h}\right)^{1-\varepsilon}\varepsilon\left(N^+(x)\right)^{1-1/\varepsilon}, & 0\leq x\leq\tau^+(h^\varepsilon), \\ 1-(1-\varepsilon)h^\varepsilon N^+(x), & x>\tau^+(h^\varepsilon), \end{cases}$$

or, which is the same but more convenient to use in simulation,

$$F^+_{h,\varepsilon}(x) = \begin{cases} \left(\frac{1}{h}\right)^{1-\varepsilon} \varepsilon \left(N^+(x)\right)^{1-1/\varepsilon}, & N^+(x) \ge h^{-\varepsilon}, \\ 1-(1-\varepsilon)h^\varepsilon N^+(x), & N^+(x) < (h)^{-\varepsilon}. \end{cases}$$

Similarly, one can define

$$\lambda^-_{h,\varepsilon}(t) := \int_0^t \nu^-\{u:\ u \ge \tau^-((sh)^\varepsilon)\}ds, \quad \mu^-_{h,\varepsilon}(t,du) := \left(\frac{1}{h}(N^-(u))^{-1/\varepsilon} \wedge t\right)\nu^-(du)$$

$\tilde F^-_{th,\varepsilon}(x) := \frac{1}{\lambda^-_{h,\varepsilon}(t)}\int_0^x \mu^-_{h,\varepsilon}(t,du)$, and write $Z^{CP,-}_{h,\varepsilon}(t) = \sum_{k=1}^{N_t^-} Z_k^-(t)$, where $N_t^- \sim \mathrm{Pois}(\lambda^-_{h,\varepsilon}(t))$, $Z_k^-(t) \sim \tilde F^-_{th,\varepsilon}$, $1 \le k \le N_t^-$. On the other hand, for simulation purposes it is more handy to simulate r.v.'s with the distribution function

$$F^-_{h,\varepsilon}(x) := \begin{cases} \left(\frac{1}{h}\right)^{1-\varepsilon} \varepsilon \left(N^-(x)\right)^{1-1/\varepsilon}, & 0 \le x \le \tau^-(h^\varepsilon), \\ 1-(1-\varepsilon)h^\varepsilon N^-(x), & x > \tau^-(h^\varepsilon), \end{cases} \tag{2.17}$$

defined on the non-negative half line, but then take it with the negative sign. Thus, we will use the representation

$$Z^{CP,-}_{h,\varepsilon}(t) = -\sum_{k=1}^{N_t^-} Z_k^-(t), \tag{2.18}$$

with $N_t^- \sim \mathrm{Pois}(\lambda^-_{h,\varepsilon}(t))$ and $Z_k^-(t) \sim F^-_{th,\varepsilon}$, $1 \le k \le N_t^-$. Observe also, that by our choice of the cutting levels $\tau^\pm$ we have $\lambda^-_{h,\varepsilon}(t) = \lambda^+_{h,\varepsilon}(t)$.

Example 2.1. *Suppose that $\nu(du) = C_+\alpha u^{-1-\alpha}du$, $u > 0$, and $\nu(du) = C_-\alpha|u|^{-1-\alpha}du$, $u < 0$, for some $C_\pm > 0$. Then $N^\pm(x) = C_\pm x^{-\alpha}$, $x > 0$, and for such x we have*

$$F^\pm_{h,\varepsilon}(x) = \begin{cases} \left(\frac{1}{h}\right)^{1-\varepsilon} \varepsilon \left(\frac{x^\alpha}{C_\pm}\right)^{\frac{1-\varepsilon}{\varepsilon}}, & 0 \le x \le (C_\pm h^\varepsilon)^{1/\alpha}, \\ 1-(1-\varepsilon)h^\varepsilon \frac{C_\pm}{x^\alpha}, & x > (C_\pm h^\varepsilon)^{1/\alpha}, \end{cases}$$

Algorithm 1 below allows us to simulate the first order approximation for fixed $t > 0$.

Algorithm 1. *Fix $t > 0$, $Z_0 \in \mathbb{R}$, and put $t_0 = 0$.*

Step 1 *Simulate r.v.'s $N_t^\pm := Pois(\lambda^+_{h,\varepsilon}(t))$.*

Step 2 *Simulate i.i.d. $Z_k^\pm(t) \sim F^\pm_{th,\varepsilon}(\cdot)$, $1 \le k \le N_t^\pm$.*

Step 3 *Calculate $Z^{CP,+}_{h,\varepsilon}(t) = \sum_{k=1}^{N_t^+} Z_k^+(t)$, $Z^{CP,-}_{h,\varepsilon}(t) = -\sum_{k=1}^{N_t^-} Z_k^-(t)$.*

Step 4 *Calculate*

$$Z^{1,DC}_{h,\varepsilon}(t) = Z_0 + m^{DC}_{h,\varepsilon}(t) + Z^{CP,+}_{h,\varepsilon}(t) + Z^{CP,-}_{h,\varepsilon}(t). \tag{2.19}$$

If $\nu(du)$ is symmetric, then $m^{DC}_{h,\varepsilon}(t) = 0$.

2.3. Second order approximation: construction and simulation algorithm

Denote by $R_{h,\varepsilon}(t)$ the "small-jump " part of the process $Z(t)$ up to time $t > 0$:

$$R_{h,\varepsilon}(t) := Z(t) - Z^{1,DC}_{h,\varepsilon}(t) = \int_0^t \int_{D^\pm(s)} u\tilde{\mathbb{N}}^Z(du,ds), \tag{2.20}$$

where $D^\pm(s) := \{u:\ -\tau^-((sh)^\varepsilon) \le u \le \tau^+((sh)^\varepsilon),\ u \ne 0\}$.

For $p \geq 2$ denote

$$V^{(p),+}(h) := \frac{1}{h}\int_0^h \int_{0<u\leq \tau^+(s^\varepsilon)} |u|^p \nu^+(du)ds, \quad V^{(p),-}(h) := \frac{1}{h}\int_0^h \int_{-\tau^-(s^\varepsilon)\leq u<0} |u|^p \nu^-(du)ds, \tag{2.21}$$

$$V^{(p)}(h) := V^{(p),+}(h) + V^{(p),-}(h), \tag{2.22}$$

and let

$$\sigma^2_{h,\varepsilon}(t) := tV^{(2)}(th). \tag{2.23}$$

The second-order approximation of $Z(t)$ is motivated by the following result.

Theorem 2.1. *If*

$$\frac{V^{(3)}(h)}{(V^{(2)}(h))^{3/2}} \longrightarrow 0, \quad h \to 0, \tag{2.24}$$

then for any $t > 0$

$$\frac{R_{h,\varepsilon}(t)}{\sqrt{\sigma^2_{h,\varepsilon}(t)}} \stackrel{d}{\Longrightarrow} \mathcal{N}(0,t), \quad h \to 0. \tag{2.25}$$

One can show that (**A2**) is sufficient for (2.24).

Lemma 2.1. *Suppose that* (**A2**) *holds true. Then* (2.24) *is satisfied.*

The proofs of Theorem 2.1 and Lemma 2.1 are given in Section 4.

The above theorem allows us to substitute $R_{h,\varepsilon}(t)$ by a normal r.v. and write the second order approximation of $Z(t)$ (for fixed t) in the spirit of Asmussen and Rosiński (2001):

$$Z^{2,DC}_{h,\varepsilon}(t) := Z^{1,DC}_{h,\varepsilon}(t) + \sigma_{h,\varepsilon}(t)W, \tag{2.26}$$

where $W \sim \mathcal{N}(0,1)$ is independent of $Z^{1,DC}_{h,\varepsilon}(t)$.

Up to now we discussed the simulation of $Z(t)$ at a fixed point $t > 0$. Note that the distribution functions of $Z^{\pm}_k(t)$ depend on t, which does not allow us to simulate $Z^{CP,\pm}_{h,\varepsilon}(t)$ as "usual" inhomogeneous compound Poisson processes. Instead, we propose the following procedure.

Observe that N^+_t is the inhomogeneous Poisson process (IPP) with intensity function $\lambda^+_{h,\varepsilon}(t)$, and N^-_t is its independent copy. In order to simulate the jump times of the IPP with intensity $\lambda^+_{h,\varepsilon}(t)$, we use the following algorithm, proposed by (Çinlar 1975, Ch.4.7):

- Generate $\Gamma_i \sim \mathbb{E}xp(1)$;
- Put $T_i = (\lambda^+_{h,\varepsilon})^{-1}(\Gamma_i)$.

The algorithm below shows how to simulate the trajectory of the second order approximation (2.26) up to a fixed time $t > 0$.

Algorithm 2. *Fix $t > 0$, initial value Z_0, partition size N, dynamic cutting parameters h, ε and the process-specific arguments.*

Step 1: *Divide the interval $[0,t]$ into N subintervals of equal length $\Delta t = t/N$, forming a grid $\pi_N = \{t_k\}_{k=0}^N$, where $t_k = k\Delta t$.*

Step 2: *Independently generate upward jump times $\{T^+_k\}$ and downward jump times $\{T^-_k\}$:*

$$T^{\pm}_k \sim \left(\lambda^{\pm}_{h,\varepsilon}\right)^{-1}(\mathrm{Exp}(1)),$$

and stop when the cumulative sum of upward jump times for $\{T^+_k\}$ exceeds t, and do the same separately for the cumulative sum of downward jump times $\{T^-_k\}$. Denote the upward and downward jump times by the arrays S^+ and S^-, respectively.

Step 3: *Simulate the corresponding jump sizes* Z_k^+ *and* Z_k^- *for the times* T_k^+ *and* T_k^- *from the distribution:*

$$Z_k^\pm \sim F^\pm_{T_k^\pm h,\varepsilon}(\cdot).$$

Step 4: *Merge the grid* π_N *with the upward and downward jump times* S^+ *and* S^-*, creating the array of all times* $S = \{s_k\}$*. Sort* S *in ascending order.*

Step 5: *Simulate the process:*

- *Initialize* $X_0 = Z_0$.
- *For each time step* $\ell = 0, \dots, K-1$*, where* K *is the size of* S*, do the following:*
  - *Simulate* $W_\ell \sim \mathcal{N}(0,1)$.
  - *Compute the time increment* $\Delta s_\ell = s_{\ell+1} - s_\ell$.
  - *Update the process* $X_{\ell+1}$ *according to the following cases:*

$$\begin{aligned}
&\text{If } s_\ell \notin S^+ \cup S^- : && X_{\ell+1} = X_\ell + m^{DC}_{h,\varepsilon}(s_\ell)\Delta s_\ell + \sigma_{h,\varepsilon}(\Delta s_\ell) W_\ell,\\
&\text{If } s_\ell \in S^+ : && X_{\ell+1} = X_\ell + m^{DC}_{h,\varepsilon}(s_\ell)\Delta s_\ell + \sigma_{h,\varepsilon}(\Delta s_\ell) W_\ell + Z_k^+,\\
&\text{If } s_\ell \in S^- : && X_{\ell+1} = X_\ell + m^{DC}_{h,\varepsilon}(s_\ell)\Delta s_\ell + \sigma_{h,\varepsilon}(\Delta s_\ell) W_\ell - Z_k^-.
\end{aligned}$$

By the process-specific arguments we mean e.g. α for the α-stable process.

If one needs to simulate only the first order approximation, one can drop in the above algorithm the term with the normal variable.

In Section 2.1 we illustrate the above results by simulations.

2.4. Weak approximation rates

Now we provide the weak approximation rates for the dynamic cutting schemes proposed above . Denote $\tau(t) = \max(\tau^+(t), \tau^-(t))$. For simplicity we assume that $Z_0 = 0$.

Theorem 2.2. *Let* $f \in C^1(\mathbb{R})$ *with* $\sup_x |f'(x)| \le C_f$*. Suppose that* (**A1**) *and* (**A2**) *hold true. Then*

$$\sup_{t\in[0,T]} \left| \mathbb{E} f(Z(t)) - \mathbb{E} f(Z^{1,DC}_{h,\varepsilon}(t)) \right| \le C^{(1)} \tau((Th)^\varepsilon)(Th)^{-\varepsilon/2}\sqrt{T}, \tag{2.27}$$

$$\sup_{t\in[0,T]} \left| \mathbb{E} f(Z(t)) - \mathbb{E} f(Z^{2,DC}_{h,\varepsilon}(t)) \right| \le C^{(2)} (1 \vee \sqrt{T}) \tau\left((Th)^\varepsilon\right)\left(|\ln (Th)^\varepsilon| + |\ln T|\right), \tag{2.28}$$

where $C^{(i)} > 0$, $i = 1, 2$, *are uniformly bounded in* $h \in (0,1)$ *and* $\varepsilon \in (0, 1/2)$.

Put

$$\alpha := \max(\alpha_+, \alpha_-). \tag{2.29}$$

Note that under (**A2**) we have $\alpha \in (0,2)$. For α close to 2 the convergence to 0 of the right-hand side of (2.27) is slow, which is natural, because the distribution of Z “approaches” the normal distribution, hence the first-order approximation becomes less accurate.

Corollary 2.1. *From* (2.6) *we derive*

$$\tau^\pm(t) \lesssim t^{1/\alpha_\pm} \lesssim t^{1/\alpha}, \quad t \in (0,1]. \tag{2.30}$$

Since we assume that $Th < 1$*, the right-hand sides of* (2.27) *and* (2.28) *can be estimated from above by*

$$\sup_{t\in[0,T]} \left| \mathbb{E} f(Z(t)) - \mathbb{E} f(Z^{1,DC}_{h,\varepsilon}(t)) \right| \lesssim (Th)^{\frac{(2-\alpha)\varepsilon}{2\alpha}} \sqrt{T}, \tag{2.31}$$

$$\sup_{t\in[0,T]} \left| \mathbb{E} f(Z(t)) - \mathbb{E} f(Z^{2,DC}_{h,\varepsilon}(t)) \right| \lesssim (1 \vee \sqrt{T})(Th)^{\frac{\varepsilon}{\alpha}} \left(|\ln (Th)^\varepsilon| + |\ln T|\right). \tag{2.32}$$

In the next theorem we estimate the distance between the distribution functions of $Z(t)$, $Z^{1,DC}_{h,\varepsilon}(t)$ and $Z^{2,DC}_{h,\varepsilon}(t)$, respectively.

Theorem 2.3. *Suppose that* (**A1**), (**A2**) *are satisfied.*

1. *Suppose that the Lévy measure $\nu(du)$ is such that $\mu_{h,\varepsilon}(t,du)$ satisfies for some $\kappa \in (0,1]$*

$$\mu_{h,\varepsilon}(t, B(z,r)) \leq Cr^{\kappa}, \quad r \in (0,1), \tag{2.33}$$

uniformly in $t \in (0,T]$ and $|z| \geq c$, where c is such that $z \notin \operatorname{supp} \mu_{h,\varepsilon}(t,\cdot)$. Then for such z we have

$$\sup_{|z|>c} \sup_{t\in[0,T]} \left| \mathbb{P}(Z(t) \leq z) - \mathbb{P}(Z^{1,DC}_{h,\varepsilon}(t) \leq z) \right| \lesssim \left(1 \vee T^{1+\frac{\varepsilon(2-\alpha)}{\alpha}}\right) h^{\frac{\kappa\varepsilon(2-\alpha)}{(2+\kappa)\alpha}}. \tag{2.34}$$

2. *For $t \in (0,T]$ we have*

$$\sup_{z\in\mathbb{R}} \left| \mathbb{P}(Z(t) \leq z) - \mathbb{P}(Z^{2,DC}_{h,\varepsilon}(t) \leq z) \right| \lesssim \left(t^{-1/2} \vee 1\right) (th)^{\varepsilon/2}. \tag{2.35}$$

Note (2.35) is non-uniform in t.

Remark 2.1. *Note that* (2.33) *is satisfied with $\kappa = 1$ for any absolutely continuous Lévy measure, in particular, for $\mu_{h,\varepsilon}$ as in Example 2.1.*

3. Examples

In the examples below we assume that the Lévy measure is absolutely continuous with respect to the Lebesgue measure and write its density in the form

$$n(u) = n^{+}(u)\mathbb{1}_{u>0} + n^{-}(u)\mathbb{1}_{u<0}.$$

For simplicity, from now on we fix the time $t = 1$ and omit the time parameter in the calculations below. We calculate the values $Z^{1,DC}_{h,\varepsilon}$ and $Z^{2,DC}_{h,\varepsilon}$ using the following algorithms:

(I) Algorithm 1: Implementation in *C#* for $Z^{1,DC}_{h,\varepsilon}$ and $Z^{2,DC}_{h,\varepsilon}$ is provided here:
`https://gitlab.com/dmytro_ivanenko/levyprocesssimulation/-/tree/master?ref_type=heads`

(II) Algorithm 2: Implementation in Python code for $Z^{1,DC}_{h,\varepsilon}$ and $Z^{2,DC}_{h,\varepsilon}$ is provided here:
`https://github.com/d-platonov/levy-processes-dynamic-cutting`

For (I), we run Monte Carlo simulations with the number of simulations $N = 10^4$, which means that the error is of order $1/\sqrt{N} = 10^{-2}$. In the tables below we provide the simulation results in order to illustrate the difference between the DC and AR approaches; the simulations may vary in the range of the error.

For (II), we take the average and standard deviation of 10^2 Monte Carlo simulations, each with $N = 10^3$ to estimate moments of both α-stable and tempered-stable processes.

In all tables we take $h = 0.05$ and $\varepsilon = 0.01$. Note that Python implementation includes seed parameter for consistent and reproducible results if one decides to check the outputs.

Example 3.1. *Consider a symmetric α-stable r.v. with the characteristic function $\phi^Z(\xi) = e^{-|\xi|^\alpha}$. Denote $M := C_\pm$ (cf. Example 2.1); from*

$$|\xi|^\alpha = M\alpha \int_{\mathbb{R}} \frac{(1-\cos \xi u)}{|u|^{1+\alpha}} du$$

we find that $M = M(\alpha) = \left(2\alpha \int_0^\infty \frac{(1-\cos u)}{u^{1+\alpha}} du\right)^{-1}$. We use the notation $Z \sim S(\alpha,\gamma)$ for a r.v. with such a characteristic function, where $\gamma = \alpha M$. We have

$$n^\pm(u) = \frac{\alpha M}{|u|^{1+\alpha}}, \quad N^\pm(u) = \frac{M}{|u|^\alpha}, \quad \tau^\pm(t) = (tM)^{1/\alpha}.$$

Direct calculation yields

$$V^{(k)}(h) = \frac{2\alpha^2 M^{k/\alpha} h^{\frac{(k-\alpha)\varepsilon}{\alpha}}}{((k-\alpha)\varepsilon + \alpha)(k-\alpha)},$$

and

$$\sigma^2_{h,\varepsilon}(1) = \frac{2\alpha^2 (M)^{2/\alpha}}{((2-\alpha)\varepsilon+\alpha)(2-\alpha)} (h)^{\frac{(2-\alpha)\varepsilon}{\alpha}} \tag{3.1}$$

Let us compare the results obtained by the DC and AR approaches. We simulate $Z^{1,AR}_\delta$ and $Z^{2,AR}_\delta$ with $m^{AR}_\delta = 0$ because $n(u)$ is symmetric, and calculate

$$(\sigma^{AR}_\delta)^2 = \int_{|u|\le\delta} |u|^2 n(u) du = 2\int_0^\delta \frac{\alpha M u^2}{u^{1+\alpha}} \nu(du) = \frac{2\alpha M \delta^{2-\alpha}}{2-\alpha}$$

In order to have the variance of the same order as in the DC case, take $\delta = h^{\varepsilon/\alpha}$.

In Table 1 we collect the results of computation of the absolute p-moment obtained using (I). Note that the exact value of the absolute p-moment of a symmetric α-stable variable Z can be calculated by the following formula (see Shanbhag and Sreehari (1977), Zolotarev (1957)):

$$m_p := \mathbb{E}|Z|^p = \frac{2^p \Gamma((1+p)/2)\Gamma(1-p/\alpha)}{\Gamma(1-p/2)\Gamma(1/2)}, \quad -1 < p < \alpha < 2. \tag{3.2}$$

The empirical absolute moments $\hat{m}^Y_p := \frac{1}{N}\sum_{k=1}^N |Y_k|^p$ are calculated for each

$$Y \in \{Z^{1,DC}_{h,\varepsilon}, Z^{2,DC}_{h,\varepsilon}, Z^{1,AR}_\delta, Z^{2,AR}_\delta\}.$$

In Table 2 we compare the L_2-distance between empirical and theoretical characteristic functions $\hat{\phi}^Y$ and ϕ^Z, respectively:

$$d^2(\hat{\phi}^Y, \phi^Z) := \int_{\mathbb{R}} \left|\hat{\phi}^Y(\xi) - \phi^Z(\xi)\right|^2 \frac{e^{-\xi^2/2}}{\sqrt{2\pi}} d\xi; \tag{3.3}$$

The results of simulations demonstrate that the DC method shows better accuracy than the AR method, which is expectable, since the DC method handles the small-jump part in a more careful way. Approach (I) allows the reader to vary the argument $\alpha \in (0,2)$ and the tuning parameters $h,\varepsilon > 0$. On the other hand, one can try the open code (II); some results for empirical moments and their standard deviations are listed in Table 3.

Example 3.2. *Next we consider the tempered stable (TS) case (see also CGMY distribution, Schoutens (2003), Jacob (2005)):*

$$n^\pm(u) = \frac{\gamma^\pm e^{-\beta^\pm |u|}}{|u|^{1+\alpha^\pm}} du, \quad u \in \mathbb{R}, \tag{3.4}$$

Table 1: p-moments of a symmetric α-stable r.v., $p = 0.3$; (I)

α	m_p	$\hat{m}_p^{(1,DC)}$	$\hat{m}_p^{(2,DC)}$	$\hat{m}_p^{(1,AR)}$	$\hat{m}_p^{(2,AR)}$
0.4	3.134779	2.637734	2.686604	1.759550	2.235229
0.6	1.532501	1.507219	1.569990	0.798177	1.246614
0.8	1.240316	1.169783	1,247960	0.767477	1.185772
1.0	1.122326	1.037327	1.126822	0.747823	1.136584
1.2	1.059521	0.950333	1.053445	0,717630	1.116851
1.4	1.020833	0.897030	1.025016	0.648963	1.075473
1.6	0.994726	0.824191	0.995633	0.542420	1.045514
1.8	0.975973	0.748651	0.974830	0.364412	1.014050

Table 2: Distance (3.3) for theoretical and empirical characteristic functions for a symmetric α-stable r.v.; (I)

α	$d^2(\hat{\phi}^{1,DC}, \phi^Z)$	$d^2(\hat{\phi}^{2,DC}, \phi^Z)$	$d^2(\hat{\phi}^{1,AR}, \phi^Z)$	$d^2(\hat{\phi}^{2,AR}, \phi^Z)$
0.4	$2.8636 * 10^{-5}$	$3.2609 * 10^{-5}$	$8.1892 * 10^{-2}$	$4.9653 * 10^{-2}$
0.6	$9.8685 * 10^{-5}$	$3.1334 * 10^{-5}$	$4.1973 * 10^{-2}$	$1.5020 * 10^{-2}$
0.8	$4.1907 * 10^{-4}$	$4.6133 * 10^{-5}$	$2.0308 * 10^{-2}$	$1.4894 * 10^{-3}$
1.0	$1.0968 * 10^{-3}$	$1.6380 * 10^{-5}$	$1.4672 * 10^{-2}$	$1.5818 * 10^{-4}$
1.2	$3.7066 * 10^{-3}$	$1.8312 * 10^{-5}$	$1.3460 * 10^{-2}$	$1.8001 * 10^{-3}$
1.4	$8.7726 * 10^{-3}$	$1.5414 * 10^{-5}$	$1.8454 * 10^{-2}$	$3.4445 * 10^{-3}$
1.6	$1.9749 * 10^{-2}$	$2.7285 * 10^{-5}$	$3.3820 * 10^{-2}$	$3.7019 * 10^{-3}$
1.8	$6.3345 * 10^{-2}$	$3.4925 * 10^{-6}$	$8.7105 * 10^{-2}$	$1.6689 * 10^{-3}$

where $\alpha^{\pm} \in (0,2)$ and $\beta^{\pm} > 0$, $\gamma^{\pm} > 0$ are some constants (to be chosen). Clearly, after appropriate choice of constants we arrive at the previous example. The characteristic function of a tempered stable r.v. is known (cf. Asmussen (2022), (Küchler and Tappe 2013, Lem.2.6, Rem.2.8), see also the expressions for CGMY distribution in Schoutens (2003), Jacob (2005)). Here we consider the easier for numeric implementation case $\alpha^{\pm} \in (0,1)$; for such $\alpha^{\pm}$ we have

$$\phi^Z(\xi) := \exp\{\gamma^-\Gamma(-\alpha^-)[(\beta^- + i\xi)^{\alpha^-} - (\beta^-)^{\alpha^-}] + \gamma^+\Gamma(-\alpha^+)[(\beta^+ - i\xi)^{\alpha^+} - (\beta^+)^{\alpha^+}]\}. \tag{3.5}$$

The expression for the cumulants $\kappa_n := \frac{d^n}{dz^n} \ln \phi^Z(-iz)$ is known:

$$\kappa_n = \Gamma(n-\alpha^+)\frac{\gamma^+}{(\beta^+)^{n-\alpha^+}} + (-1)^n\Gamma(n-\alpha^-)\frac{\gamma^-}{(\beta^-)^{n-\alpha^-}}, \quad n \geq 1. \tag{3.6}$$

Since $\kappa_n = \mu_n^0$, $n = 1,2,3$ and $\mu_4^0 = \kappa_4 - 3\kappa_2^2$, the mean, variance, skewness and kurtosis are given, respectively, by

$$\mu_1 = \kappa_1, \quad \mu_2^0 = \kappa_2, \quad \gamma_1 = \frac{\kappa_3}{\kappa_2^{3/2}}, \quad \gamma_2 = \frac{\kappa_4}{\kappa_2^2} - 3. \tag{3.7}$$

In Tables 4 – 7 we provide the simulation results. In the AR approximation we chose $\delta := \tau(h^{\varepsilon})$.

In order to simulate a TS r.v. we use Algorithm 0 from Kawai and Masuda (2011) together with the results already obtained in Example 3.1. Namely, in order to simulate a one-sided TS r.v. with parameters (α, β, γ) and characteristic function

$$\psi^{(\alpha,\beta,\gamma)}(\xi) = \exp\{\gamma\Gamma(-\alpha)((\beta - i\xi)^{\alpha} - \beta^{\alpha})\}, \quad \xi \in \mathbb{R},$$

Table 3: p-moments of a symmetric α-stable r.v, $p = 0.3$, (II)

α	m_p	$\hat{m}_p^{(1,DC)}$	std	$\hat{m}_p^{(2,DC)}$	std
0.4	3.134779	2.868470	0.597413	3.069524	1.426976
0.6	1.532501	1.447962	0.068008	1.526103	0.093529
0.8	1.240315	1.141892	0.029508	1.230012	0.033335
1.0	1.122326	1.010071	0.020149	1.112143	0.019984
1.2	1.059521	0.931838	0.016124	1.050564	0.014890
1.4	1.020833	0.872643	0.013867	1.013388	0.011715
1.6	0.994726	0.815668	0.012367	0.989492	0.010049
1.8	0.975973	0.739470	0.010595	0.973055	0.008825

Table 4: Characteristics (3.7) for TS r.v. with $\alpha^{\pm} = 0.2$, $\beta^{\pm} = 1.0$; (I)

parameters	theory	DC1	DC2	AR1	AR2
μ_1	0	0.065065	-0.066202	0.000760	-0.002707
μ_2^0	0.164791	0.158644	0.184591	0.026944	0.073073
γ_1	0	0.174530	0.269878	-1.372971	-0,434400
γ_2	27.584204	26.738838	20.349046	230.523477	36.000852

we simulate an α-stable r.v. $V \sim S^{+}(\alpha, \gamma)$ with characteristic function

$$\phi^{(\alpha,\gamma)}(\xi) = \exp\left\{\int_0^\infty (e^{i\xi u} - 1)\frac{\gamma du}{u^{1+\alpha}}\right\}, \quad \xi \in \mathbb{R}.$$

We say that $V \sim S^{-}(\alpha, \gamma)$, if $-V \sim S^{+}(\alpha, \gamma)$. Then we use the following algorithm.

Algorithm 3.

Step 1 *Generate $U \sim \mathrm{U}[0, 1]$ and $V \sim S^{+}(\alpha, \gamma)$;*

Step 2 *If $U \le e^{-\beta V}$, exit with V, otherwise go to Step 1.*

We modify this algorithm, but instead of simulating one-sided $S^{\pm}(\alpha, \gamma)$ r.v.'s, we simulate their approximations by, respectively, AR and DC methods.

Algorithm 4.

Step 1 *Generate $U \sim \mathrm{U}[0, 1]$, and $V^{+} = Z_{h,\varepsilon}^{CP,+} - \ell_{h,\varepsilon}^{DC,+}$;*

Step 2 *If $U \le e^{-\beta V^{+}}$, set $Z^{+} = V^{+}$; otherwise go to Step 1;*

Step 3 *Generate $U \sim \mathrm{U}[0, 1]$, and $V^{-} = -(Z_{h,\varepsilon}^{CP,-} - \ell_{h,\varepsilon}^{DC,-})$;*

Step 4 *If $U \le e^{-\beta V^{-}}$, set $Z^{-} = V^{-}$; otherwise go to Step 3;*

Step 5 $Z^{DC,1} := Z^{+} - Z^{-}$*;*

Step 6 *Generate $W \sim \mathcal{N}(0, 1)$, $Z^{DC,2} := Z^{DC,1} + \sigma_{h,\varepsilon}^2 W$.*

Here $\sigma_{h,\varepsilon}^2$ is calculated for $n(u) = n^{+}(u) + n^{-}(u)$ with $n^{\pm}$ are given by (3.4), whereas $Z_{h,\varepsilon}^{CP,\pm}$ and $\ell_{h,\varepsilon}^{DC,\pm}$ are calculated for the one-sided $S^{\pm}(\alpha, \gamma)$ process, with only positive (respectively, negative) jumps. Note that $\gamma^{\pm} = \alpha^{\pm} M(\alpha^{\pm})$.

In Tables 5 – 7 we provide simulation results by (I). Although the d^2–distance is slightly better for the DC approximation, simulations show that in moment estimation, DC method is

Table 5: Characteristics (3.7) for TS r.v. with $\alpha^+ = 0.8$, $\alpha^- = 0.2$, $\beta^\pm = 1.0$; (I)

parameters	theory	DC1	DC2	AR1	AR2
μ_1	1.187883	-0.084557	-0.087161	0.050844	0.053271
μ_2^0	0.340571	0.310431	0.423684	0.110907	0.270157
γ_1	0.812564	1.107415	0.600519	4.194460	1.033369
γ_2	6.456588	13.896478	8.972440	36.77033	8.374111

Table 6: Characteristics (3.7) for TS r.v. with $\alpha^\pm = 0.8$, $\beta^+ = 1.0$, $\beta^- = 1.0$; (I)

parameters	theory	DC1	DC2	AR1	AR2
μ_1	0	-0.247880	-0.245937	-0.00598	-0.00925
μ_2^0	0.516351	0.482515	0.597410	0.216077	0.486467
γ_1	0	0.839875	0.648124	-0.005699	0.008762
γ_2	2.112800	18.414751	13.494138	27.957532	7.922462

better when both values of $\alpha^\pm$ are small, whereas AR method shows better results for bigger $\alpha^\pm$. On the other hand, some error can be produced by the acceptance-rejection Algorithm 3. As expected, for small $\alpha^\pm$ the DC method works better for the first approximation rather than for the second one, which systematically overestimates the moments, whereas the second approximation in the AR settings works better than the first one. In Tables 8–10 we provide the results of simulations by method (II).

4. Auxiliary results and proofs

4.1. Auxiliary results on $N^\pm$

In this subsection we provide technical results which are implied by condition (**A2**) on $N^\pm(\cdot)$ and $\psi^{L,\pm}(\cdot)$. For the sake of simplicity of notation, in this subsection we suppress the superscripts $\pm$ in the functions below.

Lemma 4.1. *We have*

$$N(2r) \asymp N(r), \quad r > 0. \tag{4.1}$$

Proof. By (**A2**) we have $N(r) \asymp \psi^U(1/r)$, so it is enough to show the doubling property (4.1) for ψ^U. Observe that the function $\psi^U(r)$ is a.e. differentiable and

$$(\psi^U(r))' = \frac{2\psi^L(r)}{r}, \quad \text{a.e.}$$

Then for $0 < r_1 < r_2$ we have

$$\int_{r_1}^{r_2} (\psi^U(r))' dr \le \int_{r_1}^{r_2} \frac{2\psi^U(r)}{r} dr,$$

implying

$$\ln\left(\frac{\psi^U(r_2)}{\psi^U(r_1)}\right) \le \ln\left(\frac{r_2}{r_1}\right)^2.$$

Taking $r_2 = 2r$, $r_1 = r$, we derive $\psi^U(2r) \le 4\psi^U(r)$. Thus,

$$\psi^U(r) \le \psi^U(2r) \le 4\psi^U(r),$$

which yields the doubling property of N. □

Table 7: Distance (3.3) for theoretical and empirical characteristic functions for TS r.v.; (I)

α^+	α^+	β^+	β^-	$d^2(\hat\phi^{1,DC},\phi^Z)$	$d^2(\hat\phi^{2,DC},\phi^Z)$	$d^2(\hat\phi^{1,AR},\phi^Z)$	$d^2(\hat\phi^{2,AR},\phi^Z)$
0.10	0.10	1.0	0.5	0.920388	0.910131	0.893759	0.882829
0.10	0.50	1.0	0.5	0.891094	0.879827	0.921162	0.899328
0.50	0.95	1.0	0.5	0.966988	0.955811	0.998475	0.943920
0.95	0.95	1.0	0.5	0.837982	0.764671	0.943997	0.791415

Table 8: Characteristics (3.5) for TS r.v. with $\alpha^\pm = 0.2$, $\beta^\pm = 1.0$; (II)

parameters	theory	DC2	std
μ_1	0.000000	0.001522	0.010423
μ_2^0	0.176606	0.174702	0.031245
γ_1	0.000000	0.262261	1.370075
γ_2	25.538054	25.441146	10.569981

One can elaborate (4.1).

Lemma 4.2. *Take $R > 1$. Then there exists $\zeta = \zeta(R) > 0$ such that*

$$RN(r) \le N\left(\frac{r}{R^\zeta}\right), \quad r > 0. \tag{4.2}$$

Proof. By (4.1), there exists a constant $C > 0$ such that $CN(r) \le N(r/2)$ for all $r > 0$. Iterating this inequality k times we get

$$C^k N(r) \le N(r2^{-k}).$$

Take $\kappa := \min\{\ell > 0 : R < C^\ell\}$, i.e. $\kappa = \left[\frac{\log_2 R}{\log_2 C}\right] + 1$. Then

$$RN(r) \le N(r2^{-\kappa}) \le N(rR^{-\zeta}),$$

where $\zeta = \frac{\kappa}{\log_2 R}$. □

The above property yields the upper estimate on $\tau(t)$.

Lemma 4.3. *For $R > 1$ we have*

$$\tau(Rt) \le R^\zeta \tau(t), \quad t > 0, \tag{4.3}$$

where $\zeta = \zeta(R) > 0$ is the same as in (4.2).

Proof. Since $N(r)$ is monotone decreasing, (4.2) implies

$$(RN(r))^{-1} \le (N(rR^{-\zeta}))^{-1}, \quad r > 0,$$

which in turn implies (4.3).

□

Finally, we show that (**A1**), (**A2**) allow us to prove the following estimates for $V^{(p)}$.

Lemma 4.4. *Suppose that (**A1**), (**A2**) are satisfied. Then for any $p \ge 2$, $\varepsilon \in (0, 1/2)$*

$$V^{(p)}(h) \le 2C\tau^p(h^\varepsilon)h^{-\varepsilon}. \tag{4.4}$$

For $p = 2$ we have the lower bound

$$V^{(2)}(h) \ge c\tau^2(h^\varepsilon)h^{-\varepsilon}. \tag{4.5}$$

The constants $C, c > 0$ are independent of ε if $\varepsilon \in (0, 1/2)$.

Table 9: Characteristics (3.5) for TS r.v. with $\alpha^+ = 0.8$, $\alpha^- = 0.2$, $\beta^\pm = 1.0$; (II)

parameters	theory	DC2	std
μ_1	1.183829	1.216658	0.017094
μ_2^0	0.347145	0.318226	0.036514
γ_1	0.741511	0.687948	0.792942
γ_2	6.363491	10.518729	8.495347

Table 10: Characteristics (3.5) for TS r.v. with $\alpha^\pm = 0.8$, $\beta^\pm = 1.0$, (II)

parameters	theory	DC2	std
μ_1	0.000000	-0.003014	0.019988
μ_2^0	0.517683	0.460326	0.041929
γ_1	0.000000	-0.030982	0.478558
γ_2	2.099642	5.977117	3.789936

Proof. Observe that the function $\tau(t)$ is increasing in t. Using the definition of $\tau(t)$ we get

$$\begin{aligned} V^{(p)}(h) &= h^{-1}\int_0^h \tau^p(s^\varepsilon)\int_{0<|u|<\tau(s^\varepsilon)} |u/\tau(s^\varepsilon)|^p \nu(du)ds \le h^{-1}\tau^p(h^\varepsilon)\int_0^h \psi^L(1/\tau(s^\varepsilon))ds \\ &\le Ch^{-1}\tau^p(h^\varepsilon)\int_0^h N(\tau(s^\varepsilon))ds = Ch^{-1}\tau^p(h^\varepsilon)\int_0^h s^{-\varepsilon}ds \\ &= \frac{C}{1-\varepsilon}\tau^p(h^\varepsilon)h^{-\varepsilon}, \end{aligned} \tag{4.6}$$

where in the penultimate line we used (**A1**) and (**A2**).
Similarly,

$$\begin{aligned} V^{(2)}(h) &= h^{-1}\int_0^h \tau^2(s^\varepsilon)\psi^L(1/\tau(s^\varepsilon))ds \\ &\ge c_1 h^{-1}\int_0^h \tau^2(s^\varepsilon)s^{-\varepsilon}ds \ge c_1 h^{-1}\int_{h/2}^h \tau^2(s^\varepsilon)s^{-\varepsilon}ds = c_1\left(\frac{2-2^\varepsilon}{(1-\varepsilon)2}\right)\tau^2((h/2)^\varepsilon)h^{-\varepsilon} \\ &\ge c_2\left(\frac{2-2^\varepsilon}{(1-\varepsilon)2}\right)\tau^2((h/2)^\varepsilon)h^{-\varepsilon} \ge \frac{c_2}{2}\tau^2(h^\varepsilon)h^{-\varepsilon}. \end{aligned} \tag{4.7}$$

where in the penultimate inequality we used the doubling property of τ. In the last inequality we used (4.2) and that $2-2^\varepsilon > 1-\varepsilon$ for $\varepsilon < -\ln(\ln 2)/\ln 2$, and $-\ln(\ln 2)/\ln 2 > \frac{1}{2}$. □

4.2. Proofs of Theorem 2.1 and Lemma 2.1

The proof of Theorem 2.1 relies on Lemma 4.5 below.

Lemma 4.5. *Let $Z_{h,\varepsilon}$ be an infinitely divisible r.v. with the characteristic function*

$$\mathbb{E}e^{i\xi Z_{h,\varepsilon}} = \exp\left\{-\frac{1}{h}\int_0^h \int_{D^\pm(s)} \left(1-e^{i\xi u}+i\xi u\right)\nu(du)ds\right\}. \tag{4.8}$$

If (2.24) *holds true, then*

$$\frac{Z_{h,\varepsilon}}{\sqrt{V^{(2)}(h)}} \overset{w}{\Longrightarrow} \mathcal{N}(0,1) \quad as \quad h\to 0. \tag{4.9}$$

Proof. Using the Taylor decomposition of the expression in the exponent we get

$$-\ln \mathbb{E}e^{\frac{i\xi Z_{h,\varepsilon}}{\sqrt{V^{(2)}(h)}}} = \frac{\xi^2}{2hV^{(2)}(h)}\int_0^h \int_{D^\pm(s)} u^2\nu(du)ds + \frac{1}{h}\int_0^h \int_{D^\pm(s)} f\left(\frac{\xi}{\sqrt{V^{(2)}(h)}}, u\right)\nu(du)ds$$
$$= \frac{\xi^2}{2} + \frac{1}{h}\int_0^h \int_{D^\pm(s)} f\left(\frac{\xi}{\sqrt{V^{(2)}(h)}}, u\right)\nu(du)ds,$$

where

$$|f(\xi,u)| := \left|e^{i\xi u} - 1 - i\xi u - \frac{(i\xi u)^2}{2}\right| \le \frac{|\xi u|^3}{6}, \quad u,\xi \in \mathbb{R}.$$

Then (2.24) implies the convergence to the characteristic function of a normal distribution:

$$\left|\frac{1}{h}\int_0^h \int_{D^\pm(s)} f\left(\frac{\xi}{\sqrt{V^{(2)}(h)}}, u\right)\nu(du)ds\right| \le \frac{|\xi|^3}{6}\frac{\frac{1}{h}\int_0^h\int_{D^\pm(s)}|u|^3\nu(du)ds}{\left(\frac{1}{h}\int_0^h\int_{D^\pm(s)}|u|^2\nu(du)ds\right)^{3/2}}$$
$$\le \frac{|\xi|^3}{6}\frac{V^{(3)}(h)}{(V^{(2)}(h))^{3/2}} \longrightarrow 0, \quad h \to 0.$$

□

Proof of Theorem 2.1. The characteristic function of $R_{h,\varepsilon}(t)$ is

$$\mathbb{E}e^{i\xi R_{h,\varepsilon}(t)} = \exp\left(-\int_0^t \int_{D^\pm(s)} (1 - e^{i\xi u} + i\xi u)\nu(du)ds\right)$$
$$= \exp\left(-\frac{1}{h}\int_0^{th} \int_{D^\pm(s)} (1 - e^{i\xi u} + i\xi u)\nu(du)ds\right)$$
$$= \mathbb{E}e^{it\xi Z_{th,\varepsilon}},$$

where $Z_{h,\varepsilon}$ is a r.v. with characteristic function (4.8). Then for any fixed $t > 0$ we have

$$\frac{R_{h,\varepsilon}(t)}{\sqrt{\sigma^2_{h,\varepsilon}(t)}} \doteq \frac{\sqrt{t}Z_{th}}{\sqrt{V^{(2)}(th)}} \overset{w}{\Longrightarrow} \mathcal{N}(0,t), \quad h \to 0, \tag{4.10}$$

provided that (2.24) is satisfied. □

Observe that if we use the fixed cutting level, the result is pretty much the same. Indeed, define

$$\tilde{V}^{(k)}(h) := \int_{0<|u|\le h} |u|^k \nu(du), \tag{4.11}$$

which can be obtained from (2.21) by using constant cutting level.

Lemma 4.6. *Let Z_h be an infinitely divisible r.v. with characteristic function*

$$\mathbb{E}e^{i\xi Z_h} = \exp\left\{-\int_{0<|u|\le h}\left(1 - e^{i\xi u} + i\xi u\right)\nu(du)\right\}. \tag{4.12}$$

If

$$\frac{\tilde{V}^{(3)}(h)}{(\tilde{V}^{(2)}(h))^{3/2}} \longrightarrow 0, \quad h \to 0, \tag{4.13}$$

then

$$\frac{Z_h}{\sqrt{\tilde{V}^{(2)}(h)}} \overset{w}{\Longrightarrow} \mathcal{N}(0,1) \quad \text{as} \quad h \to 0. \tag{4.14}$$

The proof is similar to that of Lemma 4.5. We omit the details.

Proof of Lemma 2.1. From Lemma 4.4 we have

$$\frac{V^{(3)}(h)}{(V^{(2)}(h))^{3/2}} \lesssim h^{\varepsilon/2} \longrightarrow 0, \quad h \to 0,$$

i.e. (2.24) is satisfied. □

Similarly, under (**A2**) we get (4.13):

$$\frac{\tilde{V}^{(3)}(h)}{(\tilde{V}^{(2)}(h))^{3/2}} \lesssim \frac{h^3\psi^L(h^{-1})}{(h^2\psi^L(h^{-1}))^{3/2}} = \frac{1}{\sqrt{\psi^L(h^{-1})}} \to 0, \quad h \to 0,$$

where $\psi^L(r) := \psi^{L,+}(r) + \psi^{L,-}(r) \to \infty$ as $r \to \infty$.

4.3. Proofs of Theorems 2.2 and 2.3

Proof of Theorem 2.2. To simplify the notation, we write in the proofs below $Z^{i,DC}(t)$ instead of $Z^{i,DC}_{h,\varepsilon}(t)$, $i = 1,2$, $R(t)$ instead of $R_{h,\varepsilon}(t)$ (cf. (2.20)) and $\sigma^2(t)$ instead of $tV^{(2)}(th)$ (cf. (2.23)).

Proof of (2.27). Observe that $R(t)$ is a martingale and $\mathbb{E}|R(t)|^2 = \sigma^2(t)$. Using the Taylor expansion and Lemma 4.4 we derive

$$\begin{aligned}\Big|\mathbb{E}f(Z(t)) - \mathbb{E}f(Z^{1,DC}(t))\Big| &\le \Big|\mathbb{E}R(t)\int_0^1 f'(Z^{1,DC}(t) + uR(t))du\Big| \\ &\le C_f\mathbb{E}|R(t)| \le C_f\sqrt{\mathbb{E}|R(t)|^2} \le C_f\sigma(t) \\ &\lesssim \tau((th)^\varepsilon)(th)^{-\varepsilon/2}\sqrt{t}.\end{aligned}$$

Note that the right-hand side is increasing in t if ε is small enough. Taking supremum in $t \in [0,T]$ we get (2.27).

Proof of (2.28). Performing integration by parts, we get

$$\begin{aligned}&|\mathbb{E}f(Z^{(1,DC)}(t) + R(t)) - \mathbb{E}f(Z^{(1,DC)}(t) + \sigma(t)B(1))| \\ &\qquad \le \int |\mathbb{E}f(x + R(t)) - \mathbb{E}f(x + \sigma(t)B(1))|\mathbb{P}(Z^{1,DC}(t) \in dx) \\ &\qquad \lesssim \sigma(t)\int |\mathbb{P}((\sigma(t))^{-1}R(t) < y) - \mathbb{P}(B(1) < y)|dy.\end{aligned}$$

We can decompose the remainder $R(t)$ as follows (cf. (1.2) and (1.3)):

$$\begin{aligned}R(t) &= \int_0^t \int_{0<|u|\le\tau((sh)^\varepsilon)} u\left(\mathcal{N}^Z(du,ds) - \nu(du)ds\right) \\ &= \sum_{j=1}^n \int_{(j-1)t/n}^{jt/n} \int_{0<|u|\le\tau((sh)^\varepsilon)} u\left(\mathcal{N}^Z(du,ds) - \nu(du)ds\right) \\ &=: \sum_{j=1}^n Y_j(t/n).\end{aligned}$$

For fixed t, $Y_j = Y_j(t/n)$ are independent, $\mathbb{E}Y_j = 0$, but not identically distributed. In order to estimate the expression on the right-hand side, we apply the Esseen theorem, see Theorem B.1 in Appendix B. For this we need to estimate $\mathbb{E}|Y_j|^k$, $k = 2,3$.

As a function of r, $Y_j(r)$ is a martingale with characteristic

$$\langle Y_j \rangle_r = \int_0^r \int_{0<|u|\le\tau((t(j-1)/n+s)h)^\varepsilon)} |u|^2 \nu(du)ds. \tag{4.15}$$

Hence

$$\mathbb{E}|Y_j(t/n)|^2 = \langle Y_j \rangle_{t/n} = \int_{tj/(n-1)}^{tj/n} \int_{0<|u|\le\tau((sh)^\varepsilon)} |u|^2 \nu(du)ds \tag{4.16}$$

and by (4.7)

$$B_n(t) := \sum_{j=1}^n \mathbb{E}|Y_j(t/n)|^2 = \sigma^2(t) \gtrsim t\tau^2((th)^\varepsilon)(th)^{-\varepsilon}. \tag{4.17}$$

To estimate the third moment, we apply Theorem A.2 with $F(y,s) = y\mathbb{1}_{0<|y|\le\tau((t(j-1)/n+s)h)^\varepsilon)}$:

$$\mathbb{E}|Y_j(t/n)|^3 \lesssim \int_{tj/(n-1)}^{tj/n} \int_{0<|u|\le\tau((sh)^\varepsilon)} |u|^3 \nu(du)ds + \left(\int_{tj/(n-1)}^{tj/n} \int_{0<|u|\le\tau((sh)^\varepsilon)} |u|^2 \nu(du)ds \right)^{3/2}. \tag{4.18}$$

Then

$$\begin{aligned}\sum_{j=1}^n \mathbb{E}|Y_j(t/n)|^3 &\lesssim \int_0^t \int_{0<|u|\le\tau((sh)^\varepsilon)} |u|^3 \nu(du)ds + \sum_{j=1}^n \left(\int_{tj/(n-1)}^{tj/n} \int_{0<|u|\le\tau((sh)^\varepsilon)} |u|^2 \nu(du)ds \right)^{3/2} \\ &= tV^{(3)}(th) + \sum_{j=1}^n \left(\int_{tj/(n-1)}^{tj/n} \int_{0<|u|\le\tau((sh)^\varepsilon)} |u|^2 \nu(du)ds \right)^{3/2}.\end{aligned} \tag{4.19}$$

Let us estimate the second term. Since $\tau(t)$ is increasing, using the inequality $1-(1-x)^a \le 2^{1-a}ax$ which is valid for $x \in (0,1/2]$ and $a \in (0,1)$, we get

$$\begin{aligned}\sum_{j=1}^n \left(\int_{t(j-1)/n}^{tj/n} \int_{0<|u|\le\tau((sh)^\varepsilon)} |u|^2 \nu(du)ds \right)^{3/2} &\lesssim \sum_{j=1}^n \left(\int_{t(j-1)/n}^{tj/n} \tau^2((sh)^\varepsilon)(sh)^{-\varepsilon} ds \right)^{3/2} \\ &\lesssim \left(\tau^2\left((th)^\varepsilon\right)(h)^{-\varepsilon} \left(\frac{t}{n}\right)^{1-\varepsilon} \right)^{3/2} \sum_{j=1}^n \left(\int_{j-1}^j s^{-\varepsilon} ds \right)^{3/2} \\ &\lesssim \left(\tau^2\left((th)^\varepsilon\right)(h)^{-\varepsilon} \left(\frac{t}{n}\right)^{1-\varepsilon} \right)^{3/2} \sum_{j=1}^n \left(\frac{j^{1-\varepsilon}}{1-\varepsilon} \left(1 - \left(1-\frac{1}{j}\right)^{1-\varepsilon}\right) \right)^{3/2} \\ &\lesssim \left(\tau^2\left((th)^\varepsilon\right)(h)^{-\varepsilon} \left(\frac{t}{n}\right)^{1-\varepsilon} \right)^{3/2} \left(\frac{1}{1-\varepsilon} + \sum_{j=2}^n \left(\frac{j^{1-\varepsilon}}{1-\varepsilon} \cdot \frac{2^\varepsilon(1-\varepsilon)}{j} \right)^{3/2} \right) \\ &\lesssim \left(\tau^2\left((th)^\varepsilon\right)(h)^{-\varepsilon} \left(\frac{t}{n}\right)^{1-\varepsilon} \right)^{3/2} \sum_{j=1}^n \frac{1}{j^{3\varepsilon/2}} \\ &\lesssim \left(\tau^2\left((th)^\varepsilon\right)(h)^{-\varepsilon} \left(\frac{t}{n}\right)^{1-\varepsilon} \right)^{3/2} n^{1-3\varepsilon/2} \\ &\lesssim \tau^3\left((th)^\varepsilon\right)(h)^{-3\varepsilon/2} \cdot \frac{t^{3(1-\varepsilon)/2}}{n^{1/2}}.\end{aligned} \tag{4.20}$$

By (4.6) we have $tV^{(3)}(th) \le \frac{Ct}{1-\varepsilon}\tau^3\left((th)^\varepsilon\right)(th)^{-\varepsilon}$. Thus, if we chose $n \ge [t(th)^{-\varepsilon}]$, then

$$\begin{aligned}L_n(t) := B_n^{-3/2}(t) \sum_{j=1}^n \mathbb{E}|Y_j(t/n)|^3 &\lesssim \left[\frac{(th)^{\varepsilon/2}}{t^{1/2}} + \frac{1}{n^{1/2}} \right] \\ &\lesssim (th)^{\varepsilon/2} \left(\frac{1}{\sqrt{t}} \vee 1 \right),\end{aligned} \tag{4.21}$$

and $|\ln L_n(t)| \lesssim (|\ln(th)^{\varepsilon}| + |\ln t|)$. Then by (B.4)

$$
\begin{aligned}
\sigma(t)\int |\mathbb{P}((\sigma(t))^{-1}R(t)<y) - \mathbb{P}(B(1)<y)|dy \\
&\lesssim \left(tV^{(2)}(th)\right)^{1/2} L_n(t)|\ln L_n(t)| \int_{\mathbb{R}} \frac{dy}{1+y^2} \\
&\lesssim (1\vee\sqrt{t})\tau\left((th)^{\varepsilon}\right)\left(|\ln(th)^{\varepsilon}| + |\ln t|\right).
\end{aligned}
$$

Since $\tau(t)$ and $\tau(t)|\ln t|$ are increasing, this proves (2.28).

□

Proof of Theorem 2.3. Proof of (2.34). Observe that condition (2.33) guarantees that the probability $Q_t(A) := \mathbb{P}(Z^{1,DC}(t)\in A)$ can be estimated from above as follows:

$$
Q_t(B(z,r)) \lesssim r^{\kappa}, \quad r\in(0,1), \tag{4.22}
$$

Indeed,

$$
Q_t(A) = e^{-\lambda_{h,\varepsilon}(t)}\left(\delta_0(A) + \sum_{k=1}^{\infty}\frac{\mu_{h,\varepsilon}^{(*k)}(t,A)}{k!}\right), \tag{4.23}
$$

where $\mu_{h,\varepsilon}^{(*k)}(t,\cdot)$ is the k-folds space convolution of $\mu_{h,\varepsilon}(t,\cdot)$. Take now $A = B(z,r)$, where z is as in the assumption of the theorem and $r\in(0,1)$. By (2.33) we have for $r\in(0,1)$

$$
\mu_{h,\varepsilon}^{(*k)}(t,B(z,r)) \le C^k r^{\kappa}.
$$

Denote $\phi(h) := h^{\frac{\varepsilon(2-\alpha)}{2\alpha}-\frac{\delta}{2}}$, where α is from (2.29), $\delta < \frac{\varepsilon(2-\alpha)}{\alpha}$ (then $\phi(h)\to 0$ as $h\to 0$). Take now $r=\phi(h)$. By the Chebyshev inequality we get

$$
\begin{aligned}
\left|\mathbb{P}(Z(t)\le z) - \mathbb{P}(Z^{1,DC}(t)\le z)\right| &= \left|\int_0^z \left(\mathbb{P}(R(t)\le w-y) - 1\right)Q_t(dy)\right| \\
&= \int_0^z \mathbb{P}(R(t)>w-y)Q_t(dy) \\
&\le \int_{|y-w|\le\phi(h)} Q_t(dy)(dy) + \int_{|y-w|>\phi(h)} \frac{\sigma^2(t)}{|y-w|^2}Q_t(dy) \\
&\lesssim \left(\phi^{\kappa}(h) + \frac{\sigma^2(t)}{\phi^2(h)}\right) \\
&\lesssim \left(\phi^{\kappa}(h) + T^{1+\frac{\varepsilon(2-\alpha)}{\alpha}}h^{\delta}\right) \\
&\lesssim \left(1\vee T^{1+\frac{\varepsilon(2-\alpha)}{\alpha}}\right)h^{\frac{\kappa\varepsilon(2-\alpha)}{(2+\kappa)\alpha}},
\end{aligned}
$$

where in the last line we choose δ such that the terms $\phi^{\kappa}(h)$ and h^{δ} are comparable in h.

Proof of (2.35). We have

$$
\begin{aligned}
&\sup_w\left|\mathbb{P}(Z(t)\le w) - \mathbb{P}(Z^{2,DC}(t)\le w)\right| \\
&\le \int_{\mathbb{R}} |\mathbb{P}(R(t)\le w-v) - \mathbb{P}(\sigma(t)B(1)\le w-v)|\,Q_t(dv) \\
&\le \sup_w\left|\mathbb{P}^x((\sigma(t))^{-1}R(t)\le w) - \mathbb{P}(B(1)\le w)\right| \\
&\lesssim \left(t^{-1/2}\vee 1\right)(th)^{\varepsilon/2},
\end{aligned}
$$

where in the last line we used (4.21) and (B.3). □

A.

In this Appendix we quote the Burkholder-Davis-Gundy inequality, used in our proofs. The following result can be found in (Kühn and Schilling 2023, Th.4.20), see also (Kunita 2004, Thm.2.11) or (Protter and Talay 1997, Lem.4.1).

Theorem A.1. *Let $X(t)$ be a one-dimensional stochastic process of the form*

$$X(t) = \int_0^t \int_{y\neq 0} F(y,s)\tilde{\mathbb{N}}(dy,ds). \tag{A.1}$$

where $F(y,s)$ is a predictable stochastic process, $\tilde{\mathbb{N}}(dy,ds)$ is a compensated Poisson random measure with compensator $\nu(dy)ds$, the measure $\nu(\cdot)$ satisfies $\int_{y\neq 0}\min(1,|y|^2)\nu(dy)$, cf. (1.2). If $\int_0^T\int_{y\neq 0}|F(y,s)|^2\nu(du)ds < \infty$, then for all $p \geq 2$

$$\begin{aligned}\mathbb{E}\left(\sup_{t\leq T}|X(t)|^p\right) \lesssim \mathbb{E}&\left[\left(\int_0^T\int_{y\neq 0}|F(y,s)|^p\nu(dy)ds\right)\right] \\ &+ \mathbb{E}\left[\left(\int_0^T\int_{y\neq 0}|F(y,s)|^2\nu(dy)ds\right)^{p/2}\right].\end{aligned} \tag{A.2}$$

For $p = 2$ we have the two-sided inequality because of the Ito isometry, see (Kunita 2004, (1.7)), also (Schilling 2016, Th.13.3).

Remark A.1. *Let $F(y,s)$ be a predictable stochatsic process satisfying*

$$\mathbb{E}\int_0^T\int_{y\neq 0}|F(y,s)|^2\nu(dy)ds < \infty. \tag{A.3}$$

Then

$$\mathbb{E}\left[\int_0^T\int_{y\neq 0}F(y,s)\tilde{\mathbb{N}}(dy,ds)\right]^2 = \mathbb{E}\int_0^T\int_{y\neq 0}|F(y,s)|^2\nu(dy)ds < \infty. \tag{A.4}$$

B.

The Esseen theorem for independent r.v.'s is quoted from (Petrov 1975, Th.3, p.111) and (Petrov 1975, Th.11,p.133), see also Esseen (1945).

Theorem B.1. *Let Y_j, $j = 1,\dots,n$ be independent r.v.'s such that $\mathbb{E}Y_j = 0$, $\mathbb{E}|Y_j|^3 < \infty$, $i = 1,\dots,n$. Denote*

$$\sigma_j^2 = \mathbb{E}|Y_j|^2, \quad B_n := \sum_{j=1}^n \sigma_k^2, \quad L_n := B_n^{-3/2}\sum_{j=1}^n \mathbb{E}|Y_j|^3, \tag{B.1}$$

Then for

$$F_n(x) := \mathbb{P}\left(\frac{1}{\sqrt{B_n}}\sum_{j=1}^n Y_j < x\right), \quad \Phi(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-y^2/2}dy \tag{B.2}$$

we have

$$\sup_x |F_n(x) - \Phi(x)| \lesssim L_n. \tag{B.3}$$

If $L_n \to 0$ as $n \to \infty$, then

$$|F_n(x) - \Phi(x)| \lesssim \frac{L_n|\ln L_n|}{1+x^2}, \quad x \in \mathbb{R}. \tag{B.4}$$

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