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Asymptotic Properties of Functionals of the Squared Periodograms for Stationary Random Fields

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Abstract

The paper presents conditions for the asymptotic normality of functionals of squared periodograms based on tapered data. Stationary Gaussian random fields are considered. Two limit theorems are stated: for the first one the certain condition of integrability of the spectral density of the field is assumed, and the second result is for spectral densities with the prescribed behavior near the points of singularities.

Keywords: stationary random fields, tapered data, functionals of the periodogram, asymptotic normality.

1. Introduction and preliminaries

In the present paper we study conditions for the asymptotic normality of integral functionals of the squared periodograms based on tapered data. Note that in the spectral analysis of stationary processes and fields data are often tapered before calculating various statistics or constructing estimators of parameters of considered models. Tapering data improves asymptotic properties of statistics, it is used to reduce a leakage effect for spectra with high peaks and to treat situations with missing observations, for spatial data it helps to reduce a bias caused by so-called “edge effects” (see, e.g., Dahlhaus (1983), Dahlhaus and Künsch (1987), Guyon (1995)).

Introduce the assumption on the random fields studied in the paper:

> $X(t)$, $t \in \mathbb{Z}^d$, is a real-valued measurable stationary Gaussian random field with zero mean and a spectral density $f(\lambda)$, $\lambda \in \Lambda = (-\pi, \pi]^d$.

Let a random field $X(t)$, $t \in \mathbb{Z}^d$, be observed on a sequence L_T of increasing finite domains. We will suppose that L_T is a hypercube: $L_T = [-T, T]^d = \{t \in \mathbb{Z}^d : -T \le t^{(i)} \le T,\ i = 1, ..., d\}$. Consider the tapered values

$$\{h_T(t)\, X(t),\ t \in L_T\},$$

where $h_T(t) = h(t/T)$, $t = (t^{(1)}, ..., t^{(d)}) \in \mathbb{R}^d$, and the taper $h(t)$ factorizes as $h(t) = \prod_{i=1}^d \tilde{h}\left(t^{(i)}\right), t^{(i)} \in \mathbb{R}$, with $\tilde{h}(\cdot)$ satisfying the assumption below.

H1. $\tilde{h}(t)$, $t \in \mathbb{R}$, is a positive even function of bounded variation with bounded support: $\tilde{h}(t) = 0$ for $|t| > 1$.

Denote

$$\tilde{H}_{k,T}(\lambda) = \sum_{t=-T}^{T} [\tilde{h}_T(t)]^k e^{-i\lambda t}, \quad H_{k,T}(\lambda) = \sum_{t \in L_T} [h_T(t)]^k e^{-i(\lambda,t)} = \prod_{i=1}^{d} \tilde{H}_{k,T}(\lambda_i),$$

where $\tilde{h}_T(t) = \tilde{h}(t/T)$, $\lambda = (\lambda_1, ..., \lambda_d)$, and k is a positive integer number.

Define the finite Fourier transform of tapered data $\{h_T(t) X(t), t \in L_T\}$:

$$d_T(\lambda) = d_T^h(\lambda) = \sum_{t \in L_T} h_T(t) X(t) e^{-i(\lambda,t)}, \ \lambda \in \Lambda,$$

and the tapered periodogram of the second order (provided that $H_{2,T}(0) \neq 0$):

$$I_T(\lambda) = \frac{1}{(2\pi)^d H_{2,T}(0)} d_T(\lambda) d_T(-\lambda).$$

Consider the functional

$$J_T(\varphi) = \int_\Lambda \varphi(\lambda) I_T^2(\lambda) d\lambda, \tag{1.1}$$

where the weight function φ is is such that $\varphi f^2 \in L_1(\Lambda)$.

Limit distributions for nonlinear functionals of the periodogram have been studied extensively in the literature. We mention, for example, Taniguchi (1980), where the integrals of nonlinear functions of the periodogram (including, in particular, powers of positive orders of the periodogram) were studied for discrete time processes under the assumption of boundedness of the spectral density. In Deo and Chen (2000) the integral functionals of the squared periodogram were studied for stationary Gaussian series given by the moving average representation, the asymptotic normality result was stated under the particular assumption of summability of the coefficients of the representation and continuity of the derivative of the spectral density. In Sakhno (2012) the asymptotic normality results for such functionals were derived for Gaussian processes, covering both discrete and continuous time settings, under the general conditions of integrability of the spectral density and the weight function, and also for discrete time non-Gaussian processes under the condition of boundedness of spectral densities of all orders. Asymptotic results for the functionals of powers of the periodogram of a general order $k \geq 2$ were studied in Chiu (1988), McElroy and Holland (2009), Sakhno (2014), with applications to weighted least squares estimators in the frequency domain in Chiu (1988), and with several applications discussed in McElroy and Holland (2009), in particular, to frequency domain goodness-of-fit testing. Note that all above mentioned papers deal with the case of processes, and data tapering was used only in Sakhno (2014).

The aim of the present paper is to state the asymptotic normality of the functional (1.1) for the case of random fields and tapered data.

We present two central limit theorems for $J_T(\varphi)$: (1) under the conditions of integrability of the spectral density f and weight function φ, that is, generalizing the corresponding results for processes from Sakhno (2012) and (2) under the condition of long-range dependence of a random field X by prescribing behaviour of f and φ at the point of singularity of f.

These results generalize the next two limit theorems for the linear functionals of the periodogram

$$\widetilde{J}_T(\varphi) = \int_\Lambda \varphi(\lambda) I_T(\lambda) d\lambda, \tag{1.2}$$

which were stated in Alomari, Frias, Leonenko, Ruiz-Medina, Sakhno, and Torres (2017).

Theorem 1 (Alomari *et al.* (2017), Theorem A.1)**.** *Let $X(t)$, $t \in \mathbb{Z}^d$, be a zero-mean Gaussian random field with spectral density $f(\lambda) \in L_p$ and $\varphi(\lambda) \in L_q$, where $\frac{1}{p} + \frac{1}{q} \le \frac{1}{2}$. Then*

$$T^{d/2}(\widetilde{J}_T(\varphi) - E\widetilde{J}_T(\varphi)) \xrightarrow{D} N(0, \sigma^2) \quad as\ T \to \infty, \tag{1.3}$$

where

$$\sigma^2 = 2(2\pi)^d e(h) \int_\Lambda f^2(\lambda)\varphi^2(\lambda)d\lambda, \tag{1.4}$$

and $e(h)$ is defined as

$$e(h) = e_d(h) = \Big(\int (\tilde{h}(t))^4 dt \Big(\int (\tilde{h}(t))^2 dt \Big)^{-2} \Big)^d. \tag{1.5}$$

Theorem 2 (Alomari *et al.* (2017), Theorem A.2)**.** *Let $X(t)$, $t \in \mathbb{Z}^d$, be a zero-mean Gaussian random field with spectral density $f(\lambda)$ such that for some $0 < \alpha_i < 1$, $i = 1,..,d$, $f(\lambda) = O(\prod_{i=1}^d |\lambda_i|^{-\alpha_i})$ as $\lambda_i \to 0$, and $\varphi(\lambda) = O(\prod_{i=1}^d |\lambda_i|^{\alpha_i})$ as $\lambda_i \to 0$. The sets of discontinuities of functions $f(\lambda)$ and $\varphi(\lambda)$ have Lebesgue measure zero, and these functions are bounded for $\delta \le |\lambda| \le \pi$ for all $\delta > 0$. Then*

$$T^{d/2}(\widetilde{J}_T(\varphi) - E\widetilde{J}_T(\varphi)) \xrightarrow{D} N(0, \sigma^2) \quad as\ T \to \infty, \tag{1.6}$$

where σ^2 is the same as in Theorem 1.

Note that Theorems 1 and 2 are important, in particular, for solving the problems of parameter estimation and are used to state asymptotic normality results for minimum contrast estimators (see, e.g., Alomari *et al.* (2017), Avram, Leonenko, and Sakhno (2010a), Avram, Leonenko, and Sakhno (2010b)). As can be seen from these theorems, the limiting variance is given by the integral of the square of the spectral density, the estimate of which can be based on the integrals of the form (1.1). Therefore, investigation of integrals (1.1) is important, in particular, in the context of estimation of limiting variance of minimum contrast estimators.

In Section 2 we state the analogues of Theorems 1 and 2 for the case of the functional of the squared periodogram. Some auxiliary results needed for the proofs are given in Appendix.

2. Results and discussion

Theorem 3. *Let $X(t)$, $t \in \mathbb{Z}^d$, be a zero-mean Gaussian random field with spectral density $f(\lambda) \in L_p$ and $\varphi(\lambda) \in L_q$, where $\frac{1}{q} + 2\frac{1}{p} = \frac{1}{2}$. Then*

$$T^{d/2}(J_T(\varphi) - EJ_T(\varphi)) \xrightarrow{D} N(0, \sigma^2) \quad as\ T \to \infty, \tag{2.1}$$

where

$$\sigma^2 = 32(2\pi)^d e(h) \int_\Lambda \varphi^2(\lambda) f^4(\lambda) d\lambda, \tag{2.2}$$

$$e(h) = \Big(\int (\tilde{h}(t))^4 dt \Big(\int (\tilde{h}(t))^2 dt \Big)^{-2} \Big)^d. \tag{2.3}$$

Theorem 4. *Let $X(t)$, $t \in \mathbb{Z}^d$, be a zero-mean Gaussian random field with spectral density $f(\lambda)$ such that for some $0 < \alpha_i < 1$, $i = 1,..,d$, $f(\lambda) = O(\prod_{i=1}^d |\lambda_i|^{-\alpha_i})$ as $\lambda_i \to 0$, and $\varphi(\lambda) = O(\prod_{i=1}^d |\lambda_i|^{2\alpha_i})$ as $\lambda_i \to 0$. Suppose further that the sets of discontinuities of functions $f(\lambda)$ and $\varphi(\lambda)$ have the Lebesgue measure zero, and these functions are bounded for $\delta \le |\lambda| \le \pi$ for all $\delta > 0$. Then*

$$T^{d/2}(J_T(\varphi) - EJ_T(\varphi)) \xrightarrow{D} N(0, \sigma^2) \quad as\ T \to \infty, \tag{2.4}$$

where σ^2 is the same as in Theorem 3.

Proof of Theorem 3. The proof is obtained by following the same technique as that in the proof of Lemma 6 in Sakhno (2012), within the approach based on the evaluation of cumulants of the functional $J_T(\varphi)$. Applying the analogous calculations, we just need to keep track of normalizing factors, since now they are defined via the functions $H_{k,T}$ due to the appearance of the kernels of the Fejér's type Φ_k^T (see Appendix) in the expressions for cumulants. Therefore, we will not repeat here all the derivations, but only consider the evaluation of the variance. In Lemma 1 below we present a more general result on the asymptotics for the covariance of two functionals $J_T(\varphi_1)$ and $J_T(\varphi_2)$, which also will be used for the proof of Theorem 4.

Lemma 1.

$$\operatorname{cov}\left(T^{d/2}J_T(\varphi_1), T^{d/2}J_T(\varphi_2)\right) \to 16(2\pi)^d e(h)\int_\Lambda \varphi_1(\lambda)\left[\overline{\varphi_2}(\lambda)+\overline{\varphi_2}(-\lambda)\right]f^4(\lambda)\,d\lambda, \text{ as } T\to\infty. \quad (2.5)$$

Proof. Consider

$$\operatorname{cov}(J_T(\varphi_1), J_T(\varphi_2)) = \frac{1}{((2\pi)^d H_{2,T}(0))^4}\int_{\Lambda^2}\varphi_1(\alpha)\overline{\varphi_2}(\beta)\operatorname{cum}((d_T(\alpha)d_T(-\alpha))^2, (d_T(\beta)d_T(-\beta))^2)d\alpha d\beta. \quad (2.6)$$

Using the formula for calculation of cumulants of products of random variables (see, for example, Sakhno (2012), Appendix A, and references therein), the cumulant under the integral sign in (2.6) can be written in the form

$$\sum_{\nu=(\nu_1,\ldots,\nu_p)}\prod_{i=1}^{p}\operatorname{cum}(d_T(\mu_j), \mu_j\in\nu_i), \quad (2.7)$$

where, due to the Gaussianity assumption, the summation should be taken over all indecomposable partitions by pairs $\nu=(\nu_1,\ldots,\nu_4)$, $|\nu_i|>1$, of the table T_2 with two rows $\{\alpha,-\alpha,\alpha,-\alpha\}$ and $\{\beta,-\beta,\beta,-\beta\}$. Therefore, we have in (2.7) the sum of terms of the following form

$$\prod_{i=1}^{4} cum(d_T(\mu_i), d_T(\lambda_i)), \quad (2.8)$$

where $\mu_i,\lambda_i\in\{\alpha,-\alpha,\beta,-\beta\}$ and $\nu=\{(\mu_i,\lambda_i), i=1,\ldots,4\}$ forms an indecomposable partition of the table T_2.

The main contribution (of order $\frac{1}{T^d}$) into the covariance (2.6) is given by the terms, which correspond to the products involving the cumulants $cum(d_T(\alpha),d_T(-\alpha))$ and $cum(d_T(\beta),d_T(-\beta))$, namely, by the terms

$$cum(d_T(\alpha),d_T(-\alpha))cum(d_T(\beta),d_T(-\beta))[cum(d_T(\alpha),d_T(\beta))cum(d_T(-\alpha),d_T(-\beta)) + cum(d_T(\alpha),d_T(-\beta))cum(d_T(-\alpha),d_T(\beta))], \quad (2.9)$$

there are 16 terms of this kind in (2.8). Their contribution to the covariance is of the following form:

$$\begin{aligned}
&\frac{1}{((2\pi)^d H_{2,T}(0))^4}\int\int\varphi_1(\alpha)\overline{\varphi_2}(\beta)\\
&\times\int f(\gamma_1)H_{1,T}(\gamma_1-\alpha)H_{1,T}(-\gamma_1+\alpha)d\gamma_1\int f(\gamma_2)H_{1,T}(\gamma_2-\beta)H_{1,T}(-\gamma_2+\beta)d\gamma_2\\
&\times\left[\int f(\gamma_3)H_{1,T}(\gamma_3-\alpha)H_{1,T}(-\gamma_3-\beta)d\gamma_3\int f(\gamma_4)H_{1,T}(\gamma_4+\alpha)H_{1,T}(-\gamma_4+\beta)d\gamma_4\right.\\
&\left.+\int f(\gamma_3)H_{1,T}(\gamma_3-\alpha)H_{1,T}(-\gamma_3+\beta)d\gamma_3\int f(\gamma_4)H_{1,T}(\gamma_4+\alpha)H_{1,T}(-\gamma_4-\beta)d\gamma_4\right]d\alpha d\beta\\
&=k_T(h)\int\int\varphi_1(\alpha)\overline{\varphi_2}(\beta)\int f(\gamma_1)\Phi_2^T(\gamma_1-\alpha)d\gamma_1\int f(\gamma_2)\Phi_2^T(\gamma_2-\beta)d\gamma_2\int\int f(\gamma_3)f(\gamma_4)\\
&\times[\Phi_4^T(\gamma_3-\alpha,-\gamma_3-\beta,\gamma_4+\alpha)+\Phi_4^T(\gamma_3-\alpha,-\gamma_3+\beta,\gamma_4+\alpha)]d\gamma_3 d\gamma_4 d\alpha d\beta\\
&=I_1^X(\varphi_1,\varphi_2)+I_2^X(\varphi_1,\varphi_2)=I_1+I_2. \qquad (2.10)
\end{aligned}$$

We denote in the above formula and in what follows

$$k_T(h)=\frac{(2\pi)^{3d}H_{4,T}(0)}{((2\pi)^dH_{2,T}(0))^2}.$$

Note that all other terms in (2.8) produce the impact to the covariance (2.6) of smaller order (we refer for more delail to Sakhno (2012), see the proof of Lemma 1 therein). Since the terms I_1 and I_2 are quite similar, we analyze here just one of them. Consider:

$$\begin{aligned} I_1 &= k_T(h)\int\int\varphi_1(\alpha)\overline{\varphi_2}(\beta)\int f(\gamma_1)\Phi_2^T(\gamma_1-\alpha)d\gamma_1\int f(\gamma_2)\Phi_2^T(\gamma_2-\beta)d\gamma_2 \\ &\times\int\int f(\gamma_3)f(\gamma_4)\Phi_4^T(\gamma_3-\alpha,-\gamma_3-\beta,\gamma_4+\alpha)d\gamma_3d\gamma_4d\alpha d\beta. \end{aligned}$$

Let us represent the above integral as follows

$$\begin{aligned} & k_T(h)\int_{\Lambda^2}\int_{\Lambda^4}\varphi_1(\alpha)\overline{\varphi_2}(\beta)f(u_1+\alpha)f(u_2+\beta)f(\gamma_3)f(\gamma_4) \\ & \times\Phi_4^T(\gamma_3-\alpha,-\gamma_3-\beta,\gamma_4+\alpha)d\alpha d\beta d\gamma_3d\gamma_4\Phi_2^T(u_1)\Phi_2^T(u_2)du_1du_2 \qquad (2.11) \\ = & k_T(h)\int_{\Lambda^2}\int_{\Lambda^3}\Big[\int_\Lambda\varphi_1(\alpha)\overline{\varphi_2}(-v_1-v_2-\alpha)f(u_1+\alpha)f(u_2-v_1-v_2-\alpha) \\ & \times f(v_1+\alpha)f(v_3-\alpha)d\alpha\Big]\Phi_4^T(v_1,v_2,v_3)dv_1dv_2dv_3\Phi_2^T(u_1)\Phi_2^T(u_2)du_1du_2\,. \qquad (2.12) \end{aligned}$$

If $\varphi_i\in L_q$, $f\in L_p$ with $\frac{1}{q}+2\frac{1}{p}=\frac{1}{2}$, then the inner integral is a bounded and continuous function of arguments u_1, u_2, v_1, v_2, v_3 and we can apply formula (2.22)(see Appendix) to conclude that

$$I_1\sim\frac{(2\pi)^d}{T^d}e(h)\int_\Lambda\varphi_1(\lambda)\overline{\varphi_2}(\lambda)f^4(\lambda)\,d\lambda,\quad \text{as}\quad T\to\infty,$$

where $e(h)$ is given by (2.3). The normalizing factor $\frac{T^dH_{4,T}(0)}{H_{2,T}(0)^2}$ converges to $e(h)$ due to the asymptotic behavior $H_{k,T}(0)\sim T^d\big(\int\tilde{h}^k(t)dt\big)^d$ as $T\to\infty$. Analogously,

$$I_2\sim\frac{(2\pi)^d}{T^d}e(h)\int_\Lambda\varphi_1(\lambda)\overline{\varphi_2}(-\lambda)f^4(\lambda)\,d\lambda\quad \text{as}\quad T\to\infty,$$

and, therefore, we obtain the expression for the covariance (2.5). The lemma is proved. □

Return to the proof of Theorem 3. The convergence of $\mathrm{Var}(T^{d/2}(J_T(\varphi)-EJ_T(\varphi)))$ to σ^2 follows from Lemma 1. Following the same arguments as in Sakhno (2012), the integrability conditions on f and φ imply also the convergence to zero of all cumulants of $T^{d/2}(J_T(\varphi)-EJ_T(\varphi))$ of order $k\ge 3$. □

Proof of Theorem 4. For the proof we use techniques and ideas from two classical papers Heyde and Gay (1993) and Fox and Taqqu (1987).

Consider firstly the case $d=1$. Following an idea from Heyde and Gay (1993) (see also Alomari *et al.* (2017)), introduce the filtered process

$$Y(t)=\nabla^{\alpha/2}X(t),$$

where $\nabla=1-B$, B is the backward shift operator ($BX(t)=X(t-1)$), $\alpha\in(0,1)$ (α is taken as in the condition of the theorem), and $\nabla^{\alpha/2}=(1-B)^{\alpha/2}:=\sum_{j=0}^\infty C_j^{\alpha/2}(-B)^j$ with the generalized binomial coeeficients of the form $C_j^{\alpha/2}=((\alpha/2)(\alpha/2-1)\dots(\alpha/2-j+1))/(j!)$, $j=0,1,\dots$. Then the process $Y(t)$ has the spectral density $f_Y(\lambda)=(2\sin|\frac{\lambda}{2}|)^\alpha f_X(\lambda)$, since

$Y(t)$ is obtained from $X(t)$ by using the filter with transfer function $D(i\lambda) = (1 - e^{i\lambda})^{\alpha/2}$ and $|D(i\lambda)|^2 = (2 \sin |\frac{1}{2}\lambda|)^{\alpha}$.

Let $\psi(\lambda) = \varphi(\lambda)/(2 \sin |\frac{1}{2}\lambda|)^{2\alpha}$ and consider the functional

$$\tilde{J}_T^Y(\psi) = \int_{-\pi}^{\pi} \psi(\lambda)(I_T^Y(\lambda))^2 d\lambda - E \int_{-\pi}^{\pi} \psi(\lambda)(I_T^Y(\lambda))^2 d\lambda,$$

where $I_T^Y(\lambda) = \frac{1}{2\pi H_{2,T}(0)} | \sum_{t \in L_T} h_T(t) e^{i\lambda t} Y(t)|^2$ is the tapered periodogram which corresponds to $\{Y(t), t \in L_T\}$.

Since the spectral density $f_Y(\lambda)$ of the process $Y(t)$ and the function $\psi(\lambda)$ satisfy conditions of Theorem 3, for the functional $\tilde{J}_T^Y(\psi)$ we have the convergence as $T \to \infty$

$$T^{1/2} \tilde{J}_T^Y(\psi) \xrightarrow{D} N(0, \sigma^2), \tag{2.13}$$

where

$$\sigma^2 = 64\pi e_1(h) \int_{-\pi}^{\pi} \psi^2(\lambda) f_Y^4(\lambda) d\lambda = 64\pi e_1(h) \int_{-\pi}^{\pi} \varphi^2(\lambda) f_X^4(\lambda) d\lambda. \tag{2.14}$$

To prove the statement of the theorem, it is sufficient to show that

$$\lim_{T \to \infty} TE|\tilde{J}_T^X(\varphi) - \tilde{J}_T^Y(\psi)|^2 = 0, \tag{2.15}$$

where

$$\tilde{J}_T^X(\varphi) = \int_{-\pi}^{\pi} \varphi(\lambda)(I_T^X(\lambda))^2 d\lambda - E \int_{-\pi}^{\pi} \varphi(\lambda)(I_T^X(\lambda))^2 d\lambda.$$

Consider

$$TE|\tilde{J}_T^X(\varphi) - \tilde{J}_T^Y(\psi)|^2 = TE|\tilde{J}_T^X(\varphi)|^2 + TE|\tilde{J}_T^Y(\psi)|^2 - 2TE|\tilde{J}_T^X(\varphi)\tilde{J}_T^Y(\psi)|. \tag{2.16}$$

For the functional which corresponds to the process $Y(t)$ we have the convergence $TE|\tilde{J}_T^Y(\psi)|^2 \to \sigma^2$ as $T \to \infty$.

We need to evaluate the asymptotic behaviour of $TE|\tilde{J}_T^X(\varphi)|^2$ and $TE|\tilde{J}_T^X(\varphi)\tilde{J}_T^Y(\psi)|$. We will show that these expressions also tend to σ^2 as $T \to \infty$.

Basing on the calculations for $\mathrm{cov}(J_T^X(\varphi_1), J_T^X(\varphi_2))$ in the proof of Lemma 1, we conclude that the asymptotic behaviour of $TE|\tilde{J}_T^X(\varphi)|^2$ is defined by that of $\tilde{I}_1^X(\varphi) = I_1^X(\varphi, \varphi)$ and $\tilde{I}_2^X(\varphi) = I_2^X(\varphi, \varphi)$, where $I_1^X(\varphi, \varphi)$ and $I_2^X(\varphi, \varphi)$ are the terms which appear in formula (2.10) (with $\varphi_1 = \varphi_2 = \varphi$)), these terms have the same behaviour as $T \to \infty$, so, we can analyze just one of them, say, $\tilde{I}_1^X(\varphi)$:

$$\begin{aligned} \tilde{I}_1^X(\varphi) &= k_T(h) \int\int \varphi(y_1)\varphi(y_2) \int f_X(\gamma_1)\Phi_2^T(\gamma_1 - y_1)d\gamma_1 \int f_X(\gamma_2)\Phi_2^T(\gamma_2 - y_2)d\gamma_2 \\ &\quad \times \int\int f_X(\gamma_3)f_X(\gamma_4)\Phi_4^T(\gamma_3 - y_1, -\gamma_3 - y_2, \gamma_4 + y_1)d\gamma_3 d\gamma_4 dy_1 dy_2 \\ &= k_T(h) \int_{\Lambda^2}\int_{\Lambda^4} \varphi(y_1)\varphi(y_2)f_X(u_1 + y_1)f_X(u_2 + y_2)f_X(\gamma_3)f_X(\gamma_4) \\ &\quad \times \Phi_4^T(\gamma_3 - y_1, -\gamma_3 - y_2, \gamma_4 + y_1)dy_1 dy_2 d\gamma_3 d\gamma_4 \Phi_2^T(u_1)\Phi_2^T(u_2)du_1 du_2. \end{aligned} \tag{2.17}$$

For evaluation of $E|\tilde{J}_T^X(\varphi)\tilde{J}_T^Y(\psi)|$ we use the same arguments as for $\mathrm{cov}(J_T^X(\varphi_1), J_T^X(\varphi_2))$, the asymptotic behaviour will be defined by that of $\tilde{I}_1^{XY}(\varphi, \psi)$ and $\tilde{I}_2^{XY}(\varphi, \psi)$ which expressions are obtained by substitution into the corresponding expressions for $\tilde{I}_1^X(\varphi)$ and $\tilde{I}_2^X(\varphi)$ instead of the product

$$\varphi(\cdot)\varphi(\cdot)f_X(\cdot)f_X(\cdot)f_X(\cdot)f_X(\cdot)$$

the following product:

$$\varphi(\cdot)\psi(\cdot)f_X(\cdot)f_Y(\cdot)f_{XY}(\cdot)f_{XY}(\cdot)$$

with the corresponding arguments, where f_{XY} is the cross-spectral density of the processes X and Y. Again, it is sufficient to analyze just $\tilde{I}_1^{XY}(\varphi,\psi)$, we write down its expression:

$$\begin{aligned}\tilde{I}_1^{XY}(\varphi,\psi) &= k_T(h)\int\int\varphi(y_1)\psi(y_2)\int f_X(\gamma_1)\Phi_2^T(\gamma_1-y_1)d\gamma_1\int f_Y(\gamma_2)\Phi_2^T(\gamma_2-y_2)d\gamma_2\\ &\quad\times\int\int f_{XY}(\gamma_3)f_{XY}(\gamma_4)\Phi_4^T(\gamma_3-y_1,-\gamma_3-y_2,\gamma_4+y_1)d\gamma_3d\gamma_4dy_1dy_2\\ &= k_T(h)\int_{\Lambda^2}\int_{\Lambda^4}\varphi(y_1)\psi(y_2)f_X(u_1+y_1)f_Y(u_2+y_2)f_{XY}(\gamma_3)f_{XY}(\gamma_4)\\ &\quad\times\Phi_4^T(\gamma_3-y_1,-\gamma_3-y_2,\gamma_4+y_1)dy_1dy_2d\gamma_3d\gamma_4\Phi_2^T(u_1)\Phi_2^T(u_2)du_1du_2.\end{aligned}$$

Consider $\tilde{I}_1^X(\varphi)$. Denote

$$G(\varphi_1,\varphi_2,f_1,f_2,f_3,f_4;y_1,y_2,\gamma_1,\ldots,\gamma_4)=\varphi_1(y_1)\varphi_2(y_2)\prod_{i=1}^4 f_i(\gamma_i).$$

Then we can write

$$\begin{aligned}\tilde{I}_1^X(\varphi) &= k_T(h)\int_{\Lambda^6}G(\varphi,\varphi,f_X,f_X,f_X,f_X;y_1,y_2,\gamma_1,\ldots,\gamma_4)\Phi_4^T(\gamma_3-y_1,-\gamma_3-y_2,\gamma_4+y_1)\\ &\quad\times\Phi_2^T(\gamma_1-y_1)\Phi_2^T(\gamma_2-y_2)dy_1dy_2d\gamma_1d\gamma_2d\gamma_3d\gamma_4. \qquad (2.18)\end{aligned}$$

Introduce the measure μ_T on $\Lambda^6=[-\pi,\pi]^6$ as follows:

$$\mu_T(E)=\int_E\Phi_4^T(\gamma_3-y_1,-\gamma_3-y_2,\gamma_4+y_1)\Phi_2^T(\gamma_1-y_1)\Phi_2^T(\gamma_2-y_2)dy_1dy_2d\gamma_1d\gamma_2d\gamma_3d\gamma_4$$

for $E\subset\Lambda^6$.

First we note that the measure μ_T converges weakly to the measure μ which is concentrated on the diagonal $D=\{y_1=\ldots=y_6\}$ and satisfies $\mu\{y:a\le y_1=y_2=\ldots=y_6\le b\}=b-a$ for $\pi\le a\le b\le\pi$. Similarly to Fox and Taqqu (1987), this can be shown by considering the Fourier coefficients of μ_T and μ. Indeed, using the kernel property of Φ_k^T (see Appendix), we have the convergence $\int_{\Lambda^6}\prod_{i=1}^6 f_i(y_i)\mu_T(dy)\to\int_\Lambda\prod_{i=1}^6 f_i(z)dz$ for any bounded continuous functions f_i. Therefore, we conclude that the Fourier coefficients of μ_T converge to those of μ:

$$\int_{\Lambda^6}e^{i(n_1y_1+\cdots n_6y_6)}\mu_T(dy)\to\int_\Lambda e^{i(n_1+\cdots n_6)z}dz=\int_{\Lambda^6}e^{i(n_1y_1+\cdots n_6y_6)}\mu(dy).$$

To evaluate the asymptotic behaviour of $\tilde{I}_1^X(\varphi)$ we follow the approach from the paper Fox and Taqqu (1987). We split the integral in (2.18) in the sum of two integrals over the domains $\Lambda^6\setminus\Lambda^6_\varepsilon$ and Λ^6_ε, where $\Lambda^6_\varepsilon=[-\varepsilon,\varepsilon]^6$. Then

$$\int_{\Lambda^6\setminus\Lambda^6_\varepsilon}G(\varphi,\varphi,f_X,f_X,f_X,f_X;y_1,y_2,\gamma_1,\ldots,\gamma_4)d\mu_T\to\int_{\Lambda^6\setminus\Lambda^6_\varepsilon}\varphi^2(z)f_X^4(z)dz,\ \text{as } T\to\infty$$

in view of the above discussed property of the measure μ_T and properties of the furnctions φ and f_X. Therefore, to prove that

$$\tilde{I}_1^X(\varphi)\sim\frac{2\pi}{T}e_1(h)\int_\Lambda\varphi^2(\lambda)f_X^4(\lambda)d\lambda,\quad\text{as}\quad T\to\infty, \qquad (2.19)$$

we need to show that

$$\begin{aligned}\lim_{\varepsilon\to0}\limsup_{T\to\infty}\frac{1}{T}\int_{\Lambda^6_\varepsilon}&G(\varphi,\varphi,f_X,f_X,f_X,f_X;y_1,y_2,\gamma_1,\ldots,\gamma_4)\Phi_4^T(\gamma_3-y_1,-\gamma_3-y_2,\gamma_4+y_1)\\ &\times\Phi_2^T(\gamma_1-y_1)\Phi_2^T(\gamma_2-y_2)dy_1dy_2d\gamma_1d\gamma_2d\gamma_3d\gamma_4=0.\end{aligned}$$

We rewrite the above integral in another form using formula (2.17) and evaluate its modulus as follows:

$$\begin{aligned}
&\int_{\Lambda_\varepsilon^2}\int_{\Lambda_\varepsilon^4} |\varphi(y_1)\varphi(y_2)f_X(u_1+y_1)f_X(u_2+y_2)f_X(\gamma_3)f_X(\gamma_4)|\frac{1}{(2\pi)^3H_{4,T}(0)}\\
&\qquad\times H_{1,T}(\gamma_3-y_1)H_{1,T}(-\gamma_3-y_2)H_{1,T}(\gamma_4+y_1)H_{1,T}(-\gamma_4+y_2)|\\
&\qquad\times|\Phi_2^T(u_1)||\Phi_2^T(u_2)|dy_1dy_2d\gamma_3d\gamma_4du_1du_2\\
&\le \int_{\Lambda_\varepsilon^4}|y_1|^{2\alpha}|y_2|^{2\alpha}|y_1|^{-\alpha}|y_2|^{-\alpha}|\gamma_3|^{-\alpha}|\gamma_4|^{-\alpha}\frac{1}{(2\pi)^3H_{4,T}(0)}\\
&\quad\times|H_{1,T}(\gamma_3-y_1)H_{1,T}(-\gamma_3-y_2)H_{1,T}(\gamma_4+y_1)H_{1,T}(-\gamma_4+y_2)|dy_1dy_2d\gamma_3d\gamma_4\\
&= \int_{\Lambda_\varepsilon^4}|y_1|^{\alpha}|y_2|^{\alpha}|\gamma_3|^{-\alpha}|\gamma_4|^{-\alpha}\frac{1}{(2\pi)^3H_{4,T}(0)}|H_{1,T}(\gamma_3-y_1)H_{1,T}(-\gamma_3-y_2)\\
&\quad\times H_{1,T}(\gamma_4+y_1)H_{1,T}(-\gamma_4+y_2)|dy_1dy_2d\gamma_3d\gamma_4 := \int_{\Lambda_\varepsilon^4}F_T(y_1,y_2,\gamma_3,\gamma_4)dy_1dy_2d\gamma_3d\gamma_4.
\end{aligned}$$

Here we used the properties of functions φ and f_X at the proximity of 0 supposed in the conditions of the theorem and properties of the kernels Φ_k^T (see Appendix).

Next we note that the following estimate holds for $H_{1,T}$: $|H_{1,T}(\lambda)| \le const\cdot l_T(\lambda)$, where $l_T(u)$ denotes 2π-periodic extension of the function $l_T^*(u)$, which is defined as: $l_T^*(u)=T$ for $|u|\le\frac{1}{T}$, and $l_T^*(u)=\frac{1}{|u|}$ for $\frac{1}{T}<|u|\le\pi$ (see, e.g., Dahlhaus (1983)).

Therefore, we can apply Proposition 6.2, part a) from Fox and Taqqu (1987) (taking there $p=2$ and $\beta=-\alpha$) to conclude that

$$\lim_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{1}{T}\int_{\Lambda_\varepsilon^4}F_T(y_1,y_2,\gamma_3,\gamma_4)dy_1dy_2d\gamma_3d\gamma_4=0.$$

and hence (2.19) holds true.

In the similar way we analyze the asymptotic behaviour of $\tilde{I}_1^{XY}(\varphi,\psi)$ as $T\to\infty$. We can write it in the form

$$\begin{aligned}
\tilde{I}_1^{XY}(\varphi,\psi) &= k_T(h)\int_{\Lambda^6}G(\varphi,\psi,f_X,f_Y,f_{XY},f_{XY};y_1,y_2,\gamma_1,\dots,\gamma_4)\\
&\quad\times\Phi_4^T(\gamma_3-y_1,-\gamma_3-y_2,\gamma_4+y_1)\Phi_2^T(\gamma_1-y_1)\Phi_2^T(\gamma_2-y_2)dy_1dy_2d\gamma_1d\gamma_2d\gamma_3d\gamma_4.
\end{aligned}$$

Then we use that $\psi(\lambda)=\frac{\varphi(\lambda)}{|D(i\lambda)|^4}$, $f_Y(\lambda)=f_X(\lambda)|D(i\lambda)|^2$, $f_{XY}(\lambda)=D(i\lambda)f_X(\lambda)$ and following the same reasoning as that for $\tilde{I}_1^X(\varphi)$ we obtain

$$\int_{\Lambda^6\setminus\Lambda_\varepsilon^6}G(\varphi,\psi,f_X,f_Y,f_{XY},f_{XY};y_1,y_2,\gamma_1,\dots,\gamma_4)d\mu_T\to\int_{\Lambda^6\setminus\Lambda_\varepsilon^6}\varphi^2(z)f_X^4(z)dz,\ \text{as } T\to\infty.$$

Next, the integral

$$\begin{aligned}
&\int_{\Lambda_\varepsilon^6}|G(\varphi,\psi,f_X,f_Y,f_{XY},f_{XY};y_1,y_2,\gamma_1,\dots,\gamma_4)\Phi_4^T(\gamma_3-y_1,-\gamma_3-y_2,\gamma_4+y_1)\\
&\qquad\times\Phi_2^T(\gamma_1-y_1)\Phi_2^T(\gamma_2-y_2)|dy_1dy_2d\gamma_1d\gamma_2d\gamma_3d\gamma_4
\end{aligned}$$

can be bounded by

$$\begin{aligned}
&\int_{\Lambda_\varepsilon^4}|y_1|^{\alpha}|y_2|^{\alpha}|\gamma_3|^{-\alpha/2}|\gamma_4|^{-\alpha/2}\frac{1}{(2\pi)^3H_{4,T}(0)}|H_{1,T}(\gamma_3-y_1)|\,|H_{1,T}(-\gamma_3-y_2)|\\
&\quad\times|H_{1,T}(\gamma_4+y_1)|\,|H_{1,T}(-\gamma_4+y_2)|dy_1dy_2d\gamma_3d\gamma_4 := \int_{\Lambda_\varepsilon^4}\hat{F}_T(y_1,y_2,\gamma_3,\gamma_4)dy_1dy_2d\gamma_3d\gamma_4,
\end{aligned}$$

and we can apply again Proposition 6.2, part a) from Fox and Taqqu (1987) (now with $p=2$ and $\beta=-\alpha/2$ in which case that statement still holds) to obtain that

$$\lim_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{1}{T}\int_{\Lambda_\varepsilon^4}\hat{F}_T(y_1,y_2,\gamma_3,\gamma_4)dy_1dy_2d\gamma_3d\gamma_4=0.$$

Summarizing all above derivations, we conclude that (2.15) holds true and, therefore, the statement of the theorem is proved for the case $d = 1$

The proof for the case $d = 1$ can be directly extended for $d > 1$ under the conditions imposed on the spectral density, when its singularities factorize as specified in the theorem, and due to the use of the taper which factorizes. This entails that the corresponding integrals in the course of the proof can be split as d-tuple of integrals, which appear when $d = 1$, and, therefore, all arguments can be preserved. Note that the filtered field is introduced as

$$\begin{aligned} Y(t) = Y(t_1, ..., t_d) &= \nabla_1^{\alpha_1/2} ... \nabla_d^{\alpha_d/2} X(t) \\ &= \sum_{k_1=0}^{\infty} ... \sum_{k_d=0}^{\infty} \prod_{i=1}^{d} C_{k_i}^{\alpha_i/2} X(t_1 - k_1, ..., t_d - k_d), \end{aligned}$$

and has the spectral density

$$f_Y(\lambda_1, ..., \lambda_d) = \left(\prod_{i=1}^{d} \left| 2 \sin \frac{\lambda}{2} \right|^{\alpha_i} \right) f_X(\lambda_1, ..., \lambda_d).$$

□

Remark 1. In the paper Sakhno (2012) a class of minimum contrast estimators was presented which based on an objective function given as an integral of the squared periodogram. Namely, suppose the model with spectral density $f(\lambda, \theta)$, let $w(\lambda)$ be some weight function and

$$\int_{\Lambda} f_2(\lambda; \theta) \, w(\lambda) \, d\lambda = \sigma^2(\theta),$$

so that the spectral density can be represented as

$$f_2(\lambda; \theta) = \sigma^2(\theta) \, \psi(\lambda; \theta), \quad \lambda \in \Lambda, \theta \in \Theta, \quad \text{with} \int_{\Lambda} \psi(\lambda; \theta) \, w(\lambda) \, d\lambda = 1.$$

The objective function is defined as

$$U_T(\theta) = \int_{\Lambda} I_T^2(\lambda) \psi(\lambda; \theta)^{-1} \, w(\lambda) \, d\lambda.$$

The derivation of consistency and asymptotic normality of the corresponding minimum contrast estimators is based on the asymptotic properties of the integral functionals of the squared periodogram with particular weight functions. We refer for details to Sakhno (2012). The conditions there were formulated it terms of integrability of the spectral density and weight functions. Using the results of the present paper, minimum contrast method proposed in Sakhno (2012) can be extended for the case of spectral densities with singularities.

Remark 2. Important topic in statistical data analysis is the comparison of different time series. One approach to check similarities or discrepancies between two stationary processes is to compare their spectral densities using various L_2-type statistics. For example, in the paper Preuß and Hildebrandt (2013) the comparison of processes is based on the L_2-distance of the form $D^2 := \int_{\Lambda} (f_1(\lambda) - f_2(\lambda))^2 \, d\lambda$, where f_1 and f_2 are the spectral densities of the compared processes, say, X and Y. To estimate D^2, functionals of the squared periodograms $I_T^X(\lambda)$ and $I_T^Y(\lambda)$ (corresponding to the observed data from X and Y) are used and also functionals of the product $I_T^X(\lambda) I_T^Y(\lambda)$. In Preuß and Hildebrandt (2013) (and in some other papers devoted to such problem which are referenced therein) the asymptotic normality of statistics based on squared periodograms is stated under the assumption of weak dependence of the processes X and Y formulated as conditions of summability of cumulants of the processes. The approach of the present paper can be applied to extend the results from Preuß and Hildebrandt (2013) to the case of spectral densities with singularities. We address the detailed treatment of such extension to further research.

Remark 3. The important issue for further investigation is to extend the results of Theorems 3, 4 to the convergence

$$T^{d/2}(J_T(\varphi) - \widehat{J}(\varphi)) \xrightarrow{D} N(0, \sigma^2) \text{ as } T \to \infty,$$

where $\widehat{J}(\varphi) = 2\int_\Lambda \varphi(\lambda) f^2(\lambda) d\lambda$.

For such extension one needs to investigate the bias $EJ_T(\varphi) - \widehat{J}(\varphi)$ as $T \to \infty$ and to find conditions guaranteeing the convergence

$$T^{d/2}(EJ_T(\varphi) - \widehat{J}(\varphi)) \to 0 \text{ as } T \to \infty. \tag{2.20}$$

For the case of a Gaussian field, by direct calculations, we obtain $EJ_T(\varphi) = I_1 + I_2$ with

$$\begin{aligned} I_1 &= 2\int \varphi(\lambda) \int f(\gamma)\Phi_2^T(\gamma-\lambda)d\gamma \int f(\mu)\Phi_2^T(\mu-\lambda)d\mu d\lambda \\ &= 2\int\int \left[\int \varphi(\lambda)f(\gamma-\lambda)f(\mu-\lambda)d\lambda\right]\Phi_2^T(\gamma)\Phi_2^T(\mu)d\gamma d\mu, \\ I_2 &= \frac{(2\pi)^d H_{4,T}(0)}{H_{2,T}(0)^2}\int_{\Lambda^3}\varphi(\lambda)f(\gamma)f(\mu)\Phi_4^T(\gamma-\lambda,-\gamma-\lambda,\mu+\lambda)d\gamma d\mu d\lambda, \end{aligned}$$

and, in particular, following the same arguments as in Sakhno (2012) (see the proof of Lemma 4) one can show that under the conditions of Theorem 3 it holds:

$$I_1 \to \widehat{J}(\varphi) = 2\int_\Lambda \varphi(\lambda) f^2(\lambda) d\lambda \text{ and } I_2 \to 0, \quad \text{as } T \to \infty. \tag{2.21}$$

However, we need the convergence with the normalizing factor $T^{d/2}$ as in (2.20). We can apply the analogous approach to that used in Alomari *et al.* (2017) (the proof of Lemma 4.1) and show that $I_1 - \widehat{J}(\varphi) = O(T^{-2})$, as $T \to \infty$, imposing the similar conditions on f and φ and the taper as those in Alomari *et al.* (2017). The rate of convergence of I_2 to 0 is still to be investigated separately and some additional conditions will be necessary. We address this problem in future research. We believe this can be achieved by introducing conditions which prescribe, for example, some fast power decay of tapers near the ends of the intervals where they are defined, in particular, using such kind of tapers as those in the paper Ludeña and Lavielle (1999).

Appendix

We present here some facts used in the proofs (see, e.g., Dahlhaus (1983), Guyon (1995)).

Firstly, we note that in the general case of stationary random field $X(t)$ the following formula for the cumulants of its finite Fourier transform $d_T(\lambda) = d_T^h(\lambda)$, $\lambda \in \Lambda$, can be deduced:

$$\begin{aligned} cum\left(d_T(\alpha_1), ..., d_T(\alpha_k)\right) &= \int_{\Lambda^{k-1}} f_k(\gamma_1, ..., \gamma_{k-1}) H_{1,T}(\gamma_1 - \alpha_1) ... H_{1,T}(\gamma_{k-1} - \alpha_{k-1}) \\ &\quad \times H_{1,T}\left(-\Sigma_1^{k-1}\gamma_j - \alpha_k\right) d\gamma_1 ... d\gamma_{k-1}, \end{aligned}$$

where $f_k(\gamma_1, ..., \gamma_{k-1})$ is the k-th order cumulant spectral density (provided that f_k exists). Recall that the cumulant spectral densities of orders $k = 2, 3, \ldots$ are the functions $f_k(\lambda_1, ..., \lambda_{k-1}) \in L_1(\Lambda^{k-1})$, $k = 2, 3, ...$, such that the cumulant function of the k-th order of the random field $X(t)$ is representable as

$$c_k(t_1, ..., t_{k-1}) = \int_{\Lambda^{k-1}} f_k(\lambda_1, ..., \lambda_{k-1}) e^{i\Sigma_1^{k-1}\lambda_j t_j} d\lambda_1 ... d\lambda_{k-1}.$$

If $\sum_{j=1}^{k} \lambda_j = 0$, and $H_{k,T}(0) \neq 0$, then

$$\Phi_k^T(\lambda_1, ..., \lambda_{k-1}) = \left((2\pi)^{d(k-1)} H_{k,T}(0)\right)^{-1} \prod_{j=1}^{k} H_{1,T}(\lambda_j)$$

is a multidimensional kernel of Fejér type over Λ^{k-1}, which is an approximate identity for convolution, that is, the following properties hold:

$$\sup_{T\in N} \int_{\Lambda^{k-1}} |\Phi_k^T(\lambda_1, ..., \lambda_{k-1})| d\lambda_1 ... d\lambda_{k-1} < \infty;$$
$$\lim_{T\to\infty} \int_{\Lambda^{k-1}} \Phi_k^T(\lambda_1, ..., \lambda_{k-1}) d\lambda_1 ... d\lambda_{k-1} = 1;$$
$$\lim_{T\to\infty} \int_{\Lambda^{k-1}\setminus\{|\lambda|<\varepsilon\}} \Phi_k^T(\lambda_1, ..., \lambda_{k-1}) d\lambda_1 ... d\lambda_{k-1} = 0 \text{ for all } \varepsilon > 0.$$

The above properties imply

$$\lim_{T\to\infty} \int_{\Lambda^{k-1}} \Phi_k^T(u_1, ..., u_{k-1}) G(u_1 - v_1, ..., u_{k-1} - v_{k-1}) du_1 ... du_{k-1} = G(v_1, ..., v_{k-1}), \quad (2.22)$$

for every function G which is bounded and continuous at the point $(v_1, ..., v_{k-1})$.

In the case under consideration in the present paper, where the taper factorizes, and the domain of observation is a cube $L_T = [-T, T]^d$, the above facts follow as straightforward generalizations of the corresponding results stated for dimension $d = 1$ by Dahlhaus (1983) (see also Guyon (1995)).

For $\sum_{j=1}^{k} \alpha_j = 0$, we have

$$\begin{aligned} & cum\,(d_T(\alpha_1), ..., d_T(\alpha_k)) \\ &= (2\pi)^{d(k-1)} H_{k,T}(0) \int_{\Lambda^{k-1}} \Phi_k^T(\gamma_1 - \alpha_1, ..., \gamma_{k-1} - \alpha_{k-1}) f_k(\gamma_1, ..., \gamma_{k-1}) d\gamma_1 ... d\gamma_{k-1} \\ &= (2\pi)^{d(k-1)} H_{k,T}(0) \int_{\Lambda^{k-1}} \Phi_k^T(u_1, ..., u_{k-1}) f_k(u_1 + \alpha_1, ..., u_{k-1} + \alpha_{k-1}) du_1 ... du_{k-1}. \end{aligned}$$

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