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Asymptotic Properties of the LSE for Chirp-Like Signal Parameters

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Abstract

A time-continuous model of multiple chirp-like signal observed against the background of strongly or weakly dependent stationary Gaussian noise is considered in the paper. Strong consistency and asymptotic normality of the least squares estimates for such a trigonometric regression model parameters are obtained.

Keywords: multiple chirp-like signal, strongly/weakly dependent stationary Gaussian process, strong consistency, asymptotic normality, spectral measure of sinusoidal signal, Fresnel integrals.

1. Introduction and main results

The problem of detecting hidden periodicities, that is the problem of estimating unknown angular frequencies and amplitudes of the sum of harmonic oscillations, which make up a sinusoidal signal, observed on the background of a random noise, has a long history and massive bibliography. We refer here only to Artis *et al.* (2004), Quinn and Hannan (2012), Ivanov *et al.* (2015), Nandi and Kundu (2020), where a lot of links to theoretical and applied publications on this topic can be found.

We write the sinusoidal signal in the form

$$X(t) = \sum_{j=1}^{N} \left(A_j^0 \cos(\phi_j^0 t) + B_j^0 \sin(\phi_j^0 t) \right) + \varepsilon(t), \quad N \geq 1; \tag{1}$$

$t \in \mathbb{N}$, if a discrete time model is considered, or $t \in \mathbb{R}_+ = [0; +\infty)$, if the model is with continuous observation time. In the formula (1) A_j^0, B_j^0 are amplitudes, ϕ_j^0 are frequencies, stochastic process $\varepsilon(t)$, $t \in \mathbb{Z}$ or $t \in \mathbb{R}$, is a random noise masking the sum of harmonics.

Another important model in signal processing is, so called, chirp signal, that is a linearly frequency modulated signal. Signals of this type are used in radio and echolocation as a method of generating and processing a probing pulse.

Chirp signal can be written in the form:

$$X(t) = \sum_{j=1}^{N'} \left(C_j^0 \cos(\phi_j^0 t + \psi_j^0 t^2) + D_j^0 \sin(\phi_j^0 t + \psi_j^0 t^2) \right) + \varepsilon(t), \quad N' \geq 1, \tag{2}$$

time t, as in the model (1), can be discrete or continuous. Compared to model (1), there are additional parameters ψ_j^0 called chirp rates. These parameters control the rate at which the initial oscillation frequencies ϕ_j^0 increase.

Over the past 20 years for discrete time t and random noise which is a linear time series of MA(∞) type many results were obtained on consistency and asymptotic normality of the least squares estimate (LSE) and some other estimates of signal (2) parameters. Among a large number of works on this topic we point to publications Nandi and Kundu (2004), Kundu and Nandi (2008), Lahiri (2011), Lahiri, Kundu, and Mitra (2015), Kundu and Nandi (2021).

For continuous time t and Gaussian stationary strongly or weakly dependent random noise LSE strong consistency and asymptotic normality of multiple chirp signal unknown parameters were obtained in Ivanov and Hladun (2023), Ivanov and Hladun (2024).

Grover, Kundu, and Mitra (2021) considered, in particular, the LSE properties of discrete signal parameters in the model that occupies an intermediate position between signal models (1), (2) and called it "chirp-like model". The mathematical expression of chirp-like signal is of the form:

$$X(t) = \sum_{j=1}^{N} \left(A_j^0 \cos(\phi_j^0 t) + B_j^0 \sin(\phi_j^0 t) \right) + \sum_{k=1}^{N'} \left(C_k^0 \cos(\psi_k^0 t^2) + D_k^0 \sin(\psi_k^0 t^2) \right) + \varepsilon(t), \tag{3}$$

$N, N' \geq 1$, A_j^0, B_j^0, C_k^0, D_k^0 are amplitudes, ϕ_j^0 are frequencies, ψ_k^0 are chirp rates. The authors of the cited article convincingly substantiated the feasibility of using the trigonometric model (3) in signal processing. Each term of the 1st sum in (3) is an ordinary harmonic oscillation, and each term of the 2nd sum in (3) can itself be included in the so-called elementary chirp model:

$$X(t) = C^0 \cos(\psi^0 t^2) + D^0 \sin(\psi^0 t^2) + \varepsilon(t), \tag{4}$$

considered earlier in the papers Casazza and Peter (2006), Mboup and Adali (2012).

Next we consider a special case of time continuous model (3) with $N = N'$ and prove the strong consistency and asymptotic normality of the LSE of its parameters.

Assume we observe a stochastic process

$$X(t) = g(t, \theta^0) + \varepsilon(t), \quad t \in \mathbb{R}_+, \tag{5}$$

where

$$g(t, \theta^0) = \sum_{j=1}^{N} \left(A_j^0 \cos(\phi_j^0 t) + B_j^0 \sin(\phi_j^0 t) + C_j^0 \cos(\psi_j^0 t^2) + D_j^0 \sin(\psi_j^0 t^2) \right), \tag{6}$$

$$\theta^0 = \left(A_1^0, B_1^0, \phi_1^0, C_1^0, D_1^0, \psi_1^0, ..., A_N^0, B_N^0, \phi_N^0, C_N^0, D_N^0, \psi_N^0 \right)^*, \tag{7}$$

$\left(A_j^0\right)^2 + \left(B_j^0\right)^2 > 0$, $\left(C_j^0\right)^2 + \left(D_j^0\right)^2 > 0$, $j = \overline{1, N}$; the symbol a^* denotes the transposition of a matrix or vector a. $\varepsilon = \{\varepsilon(t), t \in \mathbb{R}\}$ is a stochastic process on probability space $(\Omega, \mathcal{F}, P)$ and satisfying the following condition.

A1. ε is a sample-continuous stationary Gaussian process with zero mean and covariance function $B(t) = E\varepsilon(t)\varepsilon(0)$, having one of the properties:

(i) $B(t) = L(|t|)|t|^{-\alpha}$, $\alpha \in (0, 1)$, with non-decreasing slowly varying at infinity function L;

(ii) $B(\cdot) \in L_1(\mathbb{R})$.

Let's assume that the true values of amplitudes $A_j^0, B_j^0, C_j^0, D_j^0$, $j = \overline{1, N}$, are different numbers and the true values of frequencies ϕ_j^0, $j = \overline{1, N}$, and chirp rates ψ_j^0, $j = \overline{1, N}$, are different positive numbers which form monotonically increasing sequences. For some fixed numbers $0 \le \underline{\phi} < \overline{\phi} < +\infty$, $0 \le \underline{\psi} < \overline{\psi} < +\infty$ consider the sets

$$\Psi(\underline{\psi}, \overline{\psi}) = \left\{\psi = (\psi_1, ..., \psi_N) \in \mathbb{R}^N : \underline{\psi} < \psi_1 < ... < \psi_N < \overline{\psi}\right\},$$

$$\Phi(\underline{\phi}, \overline{\phi}) = \left\{\phi = (\phi_1, ..., \phi_N) \in \mathbb{R}^N : \underline{\phi} < \phi_1 < ... < \phi_N < \overline{\phi}\right\}, \tag{8}$$

such that $\phi^0 = (\phi_1, ..., \phi_N) \in \Phi(\underline{\phi}, \overline{\phi})$, $\psi^0 = (\psi_1, ..., \psi_N) \in \Psi(\underline{\psi}, \overline{\psi})$.

Consider monotonically non-decreasing families of open sets $\Phi_T \subset \Phi\left(\underline{\phi}, \overline{\phi}\right)$, $\Psi_T \subset \Psi\left(\underline{\psi}, \overline{\psi}\right)$, $T > T_0 > 0$, containing vectors ϕ^0,ψ^0, such that $\left(\bigcup\limits_{T>T_0} \Phi_T\right)^c = \Phi^c\left(\underline{\phi}, \overline{\phi}\right)$, $\left(\bigcup\limits_{T>T_0} \Psi_T\right)^c = \Psi^c\left(\underline{\psi}, \overline{\psi}\right)$, with the following properties.

B(*i*) $\lim\limits_{T\to\infty} \inf\limits_{\substack{1\le j\le N-1 \\ \phi\in\Phi_T^c}} T\left(\phi_{j+1} - \phi_j\right) = +\infty, \quad \lim\limits_{T\to\infty} \inf\limits_{\phi\in\Phi_T^c} T\phi_1 = +\infty;$

B(*ii*) $\lim\limits_{T\to\infty} \inf\limits_{\substack{1\le j\le N-1 \\ \psi\in\Psi_T^c}} T^2\left(\psi_{j+1} - \psi_j\right) = +\infty, \quad \lim\limits_{T\to\infty} \inf\limits_{\psi\in\Psi_T^c} T^2\psi_1 = +\infty.$

Here and below in the text, we use the symbol Φ^c to denote the closure of a set Φ.

Definition 1. Any random vector

$$\theta_T = (A_{1T}, B_{1T}, \phi_{1T}, C_{1T}, D_{1T}, \psi_{1T}, ..., A_{NT}, B_{NT}, \phi_{NT}, C_{NT}, D_{NT}, \psi_{NT})^* \tag{9}$$

that minimizes the functional

$$Q_T(\theta) = \int_0^T [X(t) - g(t, \theta)]^2 \, dt \tag{10}$$

on the parametric set $\Theta_T^c \subset \mathbb{R}^{6N}$, where amplitudes $A_j, B_j, C_j, D_j, j = \overline{1, N}$, can take any values and parameters $\phi_j, \psi_j, j = \overline{1, N}$, take values in the set $\Phi_T^c \times \Psi_T^c, T > T_0 > 0$, is called LSE of the parameter θ^0.

Theorem 1. *Let the conditions **A1** and **B** be satisfied. Then LSE θ_T is a strongly consistent estimate of parameter θ^0 in the sense that $A_{jT} \to A_j^0$, $B_{jT} \to B_j^0$, $T\left(\phi_{jT} - \phi_j^0\right) \to 0$, $C_{jT} \to C_j^0$, $D_{jT} \to D_j^0$, $T^2\left(\psi_{jT} - \psi_j^0\right) \to 0$ a.s., as $T \to \infty, j = \overline{1, N}$.*

The proof of **Theorem 1** is given in section 2.

In order to formulate the theorem on asymptotic normality of LSE θ_T, we introduce an additional assumption.

A2(*i*) The process ε satisfying condition **A1(*i*)** with parameter $\alpha \in (\frac{1}{2}, 1)$ has a spectral density $f(\lambda) = \tilde{L}\left(\frac{1}{|\lambda|}\right)|\lambda|^{\alpha-1}$, where $\tilde{L}$ is a slowly varying at infinity function, and f has the 4th spectral moment.

A2(*ii*) The spectral density of the process ε satisfying the condition **A1(*ii*)** has the 4th spectral moment.

In Ivanov and Hladun (2024) an example of the Bessel covariance function that satisfies condition **A2(*i*)** is given.

Theorem 2. *Let the conditions* ***A1****,* ***A2*** *and* ***B*** *be fulfilled. Then the normed LSE* $\Big(T^{1/2}\left(A_{1T}-A_1^0\right)$, $T^{1/2}\left(B_{1T}-B_1^0\right)$, $T^{3/2}\left(\phi_{1T}-\phi_1^0\right)$, $T\left(C_{1T}-C_1^0\right)$, $T\left(D_{1T}-D_1^0\right)$, $T^3\left(\psi_{1T}-\psi_1^0\right)$,...,$T^{1/2}\left(A_{NT}-A_N^0\right)$, $T^{1/2}\left(B_{NT}-B_N^0\right)$, $T^{3/2}\left(\phi_{NT}-\phi_N^0\right)$, $T\left(C_{NT}-C_N^0\right)$, $T\left(D_{NT}-D_N^0\right)$, $T^3\left(\psi_{NT}-\psi_N^0\right)\Big)^*$ *is asymptotically, as* $T\to\infty$*, normal* $N(0,\Sigma)$*, where the covariance matrix* Σ *is given by the formulas (106)-(109).*

The proof of **Theorem 2** is given in section 3.

2. Strong consistency

In the proofs of **Theorem 1** and **2**, the properties of Fresnel integrals

$$C(x)=\int_0^x \cos(t^2)dt,\quad S(x)=\int_0^x \sin(t^2)dt,\quad x\in\mathbb{R}, \tag{11}$$

play an important role. In particular from Lemma 1 in Ivanov and Hladun (2023) it follows for any bounded functions α_T, β_T, $T\geq 0$, such that $\beta_T\to+\infty$, as $T\to\infty$,

$$1)\int_0^1 \begin{matrix}\sin\\ \cos\end{matrix}\,(\beta_T t^2)dt \leq \frac{1}{\sqrt{\beta_T}}\to 0,\ \text{as } T\to\infty; \tag{12}$$

$$2)\left|\int_0^1 \begin{matrix}\sin\\ \cos\end{matrix}\,(\alpha_T t)\ \begin{matrix}\sin\\ \cos\end{matrix}\,(\beta_T t^2)dt\right| \leq \frac{4}{\sqrt{\beta_T}}\to 0,\ \text{as } T\to\infty, \tag{13}$$

for all combinations of factors in (13).

From Theorem 1 in the same paper when condition **A1** is met, we get the following uniform laws of large numbers:

$$\sup_{\phi\in\mathbb{R}}\left|T^{-1}\int_0^T \begin{matrix}\sin\\ \cos\end{matrix}\,(\phi t)\varepsilon(t)dt\right|\to 0 \text{ a.s., as } T\to\infty; \tag{14}$$

$$\sup_{\psi\in\mathbb{R}}\left|T^{-1}\int_0^T \begin{matrix}\sin\\ \cos\end{matrix}\,(\psi t^2)\varepsilon(t)dt\right|\to 0 \text{ a.s., as } T\to\infty. \tag{15}$$

Proof of ***Theorem 1****.* Consider a system of linear equations for amplitudes estimates A_{jT}, B_{jT}, C_{jT}, D_{jT}, $j=\overline{1,N}$, that is a subsystem of the normal equations system for the LSE θ_T:

$$\frac{\partial}{\partial A_p}T^{-1}Q_T(\theta_T)=\frac{\partial}{\partial B_p}T^{-1}Q_T(\theta_T)=\frac{\partial}{\partial C_p}T^{-1}Q_T(\theta_T)=\frac{\partial}{\partial D_p}T^{-1}Q_T(\theta_T)=0,\quad p=\overline{1,N},$$

and rewrite it in more detail:

$$\begin{cases}\sum\limits_{j=1}^N\Big(A_{jT}a_{jp}^{(1)}(T)+B_{jT}b_{jp}^{(1)}(T)+C_{jT}c_{jp}^{(1)}(T)+D_{jT}d_{jp}^{(1)}(T)\Big)=e_p^{(1)}(T),\\ \sum\limits_{j=1}^N\Big(A_{jT}a_{jp}^{(2)}(T)+B_{jT}b_{jp}^{(2)}(T)+C_{jT}c_{jp}^{(2)}(T)+D_{jT}d_{jp}^{(2)}(T)\Big)=e_p^{(2)}(T),\\ \sum\limits_{j=1}^N\Big(A_{jT}a_{jp}^{(3)}(T)+B_{jT}b_{jp}^{(3)}(T)+C_{jT}c_{jp}^{(3)}(T)+D_{jT}d_{jp}^{(3)}(T)\Big)=e_p^{(3)}(T),\\ \sum\limits_{j=1}^N\Big(A_{jT}a_{jp}^{(4)}(T)+B_{jT}b_{jp}^{(4)}(T)+C_{jT}c_{jp}^{(4)}(T)+D_{jT}d_{jp}^{(4)}(T)\Big)=e_p^{(4)}(T),\quad p=\overline{1,N};\end{cases} \tag{16}$$

with

$$
\begin{aligned}
& a_{jp}^{(1)} =< \cos(\phi_{jT}t), \cos(\phi_{pT}t) >, \quad a_{jp}^{(2)} =< \cos(\phi_{jT}t), \sin(\phi_{pT}t) >, \\
& a_{jp}^{(3)} =< \cos(\phi_{jT}t), \cos(\psi_{pT}t^2) >, \quad a_{jp}^{(4)} =< \cos(\phi_{jT}t), \sin(\psi_{pT}t^2) >, \\
& b_{jp}^{(1)} =< \sin(\phi_{jT}t), \cos(\phi_{pT}t) >, \quad b_{jp}^{(2)} =< \sin(\phi_{jT}t), \sin(\phi_{pT}t) >, \\
& b_{jp}^{(3)} =< \sin(\phi_{jT}t), \cos(\psi_{pT}t^2) >, \quad b_{jp}^{(4)} =< \sin(\phi_{jT}t), \sin(\psi_{pT}t^2) >, \\
& c_{jp}^{(1)} =< \cos(\psi_{jT}t^2), \cos(\phi_{pT}t) >, \quad c_{jp}^{(2)} =< \cos(\psi_{jT}t^2), \sin(\phi_{pT}t) >, \\
& c_{jp}^{(3)} =< \cos(\psi_{jT}t^2), \cos(\psi_{pT}t^2) >, \quad c_{jp}^{(4)} =< \cos(\psi_{jT}t^2), \sin(\psi_{pT}t^2) >, \\
& d_{jp}^{(1)} =< \sin(\psi_{jT}t^2), \cos(\phi_{pT}t) >, \quad d_{jp}^{(2)} =< \sin(\psi_{jT}t^2), \sin(\phi_{pT}t) >, \\
& d_{jp}^{(3)} =< \sin(\psi_{jT}t^2), \cos(\psi_{pT}t^2) >, \quad d_{jp}^{(4)} =< \sin(\psi_{jT}t^2), \sin(\psi_{pT}t^2) >, \\
& e_{p}^{(1)} =< X(t), \cos(\phi_{pT}t) >, \quad e_{p}^{(2)} =< X(t), \sin(\phi_{pT}t) >, \\
& e_{p}^{(3)} =< X(t), \cos(\psi_{pT}t^2) >, \quad e_{p}^{(4)} =< X(t), \sin(\psi_{pT}t^2) >,
\end{aligned} \tag{17}
$$

where

$$
< u(t), v(t) >= T^{-1} \int_0^T u(t)v(t)dt.
$$

Denote by $o_T(1)$, $T > 0$, possibly, different stochastic processes such that $o_T(1) \to 0$ a.s., as $T \to \infty$. Using condition **B** and (12), (13), we get

$$
\begin{aligned}
& a_{jp}^{(1)}(T) = o_T(1), j \neq p, \quad a_{pp}^{(1)}(T) = \frac{1}{2} + o_T(1), \quad a_{jp}^{(2)}(T) = a_{jp}^{(3)}(T) = a_{jp}^{(4)}(T) = o_T(1), \\
& b_{jp}^{(2)}(T) = o_T(1), j \neq p, \quad b_{pp}^{(2)}(T) = \frac{1}{2} + o_T(1), \quad b_{jp}^{(1)}(T) = b_{jp}^{(3)}(T) = b_{jp}^{(4)}(T) = o_T(1), \\
& c_{jp}^{(3)}(T) = o_T(1), j \neq p, \quad c_{pp}^{(3)}(T) = \frac{1}{2} + o_T(1), \quad c_{jp}^{(1)}(T) = c_{jp}^{(2)}(T) = c_{jp}^{(4)}(T) = o_T(1), \\
& d_{jp}^{(4)}(T) = o_T(1), j \neq p, \quad d_{pp}^{(4)}(T) = \frac{1}{2} + o_T(1), \quad d_{jp}^{(1)}(T) = d_{jp}^{(2)}(T) = d_{jp}^{(3)}(T) = o_T(1), \\
& \qquad j, p = \overline{1, N}.
\end{aligned} \tag{18}
$$

Consider further the values $e_p^{(i)}, i = \overline{1,4}$, and set

$$
\begin{aligned}
& x_{pT}^{(1)} = \frac{\sin T(\psi_p^0 - \psi_{pT})}{T(\psi_p^0 - \psi_{pT})}; \qquad y_{pT}^{(1)} = \frac{1 - \cos T(\psi_p^0 - \psi_{pT})}{T(\psi_p^0 - \psi_{pT})}; \\
& x_{pT}^{(2)} = \int_0^1 \cos\left(T^2(\psi_p^0 - \psi_{pT})t^2\right) dt; \qquad y_{pT}^{(2)} = \int_0^1 \sin\left(T^2(\psi_p^0 - \psi_{pT})t^2\right) dt.
\end{aligned} \tag{19}
$$

Using relations (12)-(15) and condition **B** we get

$$
\begin{aligned}
& e_p^{(1)}(T) =< \varepsilon(t), \cos(\phi_{pT}t) > + \sum_{j=1}^{N} \Big(A_j^0 < \cos(\phi_j^0 t), \cos(\phi_{pT}t) > + \\
& B_j^0 < \sin(\phi_j^0 t), \cos(\phi_{pT}t) > + C_j^0 < \cos(\psi_j^0 t^2), \cos(\phi_{pT}t) > + D_j^0 < \sin(\psi_j^0 t^2), \cos(\phi_{pT}t) > \Big) \\
& \quad = A_p^0 < \cos(\phi_p^0 t), \cos(\phi_{pT}t) > + B_p^0 < \sin(\phi_p^0 t), \cos(\phi_{pT}t) > + o_T(1) \\
& \quad = \frac{A_p^0}{2} x_{pT}^{(1)} + \frac{B_p^0}{2} y_{pT}^{(1)} + o_T(1);
\end{aligned}
$$

$$
\begin{aligned}
e_p^{(2)}(T) =& < \varepsilon(t), \sin(\phi_{pT}t) > + \sum_{j=1}^{N} \Big(A_j^0 < \cos(\phi_j^0 t), \sin(\phi_{pT}t) > + \\
& B_j^0 < \sin(\phi_j^0 t), \sin(\phi_{pT}t) > + C_j^0 < \cos(\psi_j^0 t^2), \sin(\phi_{pT}t) > + D_j^0 < \sin(\psi_j^0 t^2), \sin(\phi_{pT}t) > \Big) \\
&= A_p^0 < \cos(\phi_p^0 t), \sin(\phi_{pT}t) > + B_p^0 < \sin(\phi_p^0 t), \sin(\phi_{pT}t) > + o_T(1) \\
&= \frac{B_p^0}{2} x_{pT}^{(1)} - \frac{A_p^0}{2} y_{pT}^{(1)} + o_T(1); \\
e_p^{(3)}(T) =& < \varepsilon(t), \cos(\psi_{pT}t^2) > + \sum_{j=1}^{N} \Big(A_j^0 < \cos(\phi_j^0 t), \cos(\psi_{pT}t^2) > + \\
& B_j^0 < \sin(\phi_j^0 t), \cos(\psi_{pT}t^2) > + C_j^0 < \cos(\psi_j^0 t^2), \cos(\psi_{pT}t^2) > + D_j^0 < \sin(\psi_j^0 t^2), \cos(\psi_{pT}t^2) > \Big) \\
&= C_p^0 < \cos(\psi_p^0 t^2), \cos(\psi_{pT}t^2) > + D_p^0 < \sin(\psi_p^0 t^2), \cos(\psi_{pT}t^2) > + o_T(1) \\
&= \frac{C_p^0}{2} x_{pT}^{(2)} + \frac{D_p^0}{2} y_{pT}^{(2)} + o_T(1); \\
e_p^{(4)}(T) =& < \varepsilon(t), \sin(\psi_{pT}t^2) > + \sum_{j=1}^{N} \Big(A_j^0 < \cos(\phi_j^0 t), \sin(\psi_{pT}t^2) > + \\
& B_j^0 < \sin(\phi_j^0 t), \sin(\psi_{pT}t^2) > + C_j^0 < \cos(\psi_j^0 t^2), \sin(\psi_{pT}t^2) > + D_j^0 < \sin(\psi_j^0 t^2), \sin(\psi_{pT}t^2) > \Big) \\
&= C_p^0 < \cos(\psi_p^0 t^2), \sin(\psi_{pT}t^2) > + D_p^0 < \sin(\psi_p^0 t^2), \sin(\psi_{pT}t^2) > + o_T(1) \\
&= \frac{D_p^0}{2} x_{pT}^{(2)} - \frac{C_p^0}{2} y_{pT}^{(2)} + o_T(1).
\end{aligned}
\tag{20}
$$

Applying relations (18), (20) to the system (16) we obtain the following relations for the LSE of amplitudes:

$$
\begin{aligned}
&A_{jT} = A_j^0 x_{jT}^{(1)} + B_j^0 y_{jT}^{(1)} + o_T(1); \qquad B_{jT} = B_j^0 x_{jT}^{(1)} - A_j^0 y_{jT}^{(1)} + o_T(1); \\
&C_{jT} = C_j^0 x_{jT}^{(2)} + D_j^0 y_{jT}^{(2)} + o_T(1); \qquad D_{jT} = D_j^0 x_{jT}^{(2)} - C_j^0 y_{jT}^{(2)} + o_T(1); \quad j = \overline{1, N}.
\end{aligned}
\tag{21}
$$

Since $\left|x_{jT}^{(1)}\right|, \left|y_{jT}^{(1)}\right|, \left|x_{jT}^{(2)}\right|, \left|y_{jT}^{(2)}\right| \le 1$, $j = \overline{1, N}$, then from (21) it follows that

$$
|A_{jT}|, |B_{jT}| \le |A_j^0| + |B_j^0| + o_T(1); \quad |C_{jT}|, |D_{jT}| \le |C_j^0| + |D_j^0| + o_T(1), j = \overline{1, N}. \tag{22}
$$

Write

$$
G_T(\theta_1, \theta_2) = T^{-1} \int_0^T (g(t, \theta_1) - g(t, \theta_2))^2 \, dt, \theta_1, \theta_2 \in \Theta_T^c.
$$

From LSE definition we get

$$
0 \ge T^{-1} Q_T(\theta_T) - T^{-1} Q_T(\theta^0) = G(\theta_T, \theta^0) + 2T^{-1} \int_0^T \varepsilon(t) \left[g(t, \theta^0) - g(t, \theta_T) \right] dt. \tag{23}
$$

From (14), (15) and (22) it follows that

$$
2T^{-1} \int_0^T \varepsilon(t) \left[g(t, \theta^0) - g(t, \theta_T) \right] dt \to 0 \text{ a.s., as } T \to \infty. \tag{24}
$$

Thus, taking into account (24), from (23) we obtain

$$
G(\theta_T, \theta^0) \to 0 \text{ a.s., as } T \to \infty. \tag{25}
$$

Using notation

$$g_{jT}(t) = A_{jT}\cos(\phi_{jT}t) + B_{jT}\sin(\phi_{jT}t) + C_{jT}\cos(\psi_{jT}t^2) + D_{jT}\sin(\psi_{jT}t^2)$$
$$- A_j^0\cos(\phi_j^0 t) - B_j^0\sin(\phi_j^0 t) - C_j^0\cos(\psi_j^0 t^2) - D_j^0\sin(\psi_j^0 t^2),$$

we derive

$$G(\theta_T, \theta^0) = \sum_{j=1}^{N} T^{-1}\int_0^T g_{jT}^2(t)dt + 2\sum_{j<p} T^{-1}\int_0^T g_{jT}(t)g_{pT}(t)dt. \tag{26}$$

From (12), (13), condition **B** and (22) we deduce that the 2nd sum in (26) vanishes a.s., as $T \to \infty$. On the other hand,

$$\begin{aligned}
T^{-1}\int_0^T g_{jT}^2(t)dt &= (A_{jT})^2 < \cos(\phi_{jT}t), \cos(\phi_{jT}t) > +(B_{jT})^2 < \sin(\phi_{jT}t), \sin(\phi_{jT}t) > \\
&+ (C_{jT})^2 < \cos(\psi_{jT}t^2), \cos(\psi_{jT}t^2) > +(D_{jT})^2 < \sin(\psi_{jT}t^2), \sin(\psi_{jT}t^2) > \\
&+ (A_j^0)^2 < \cos(\phi_j^0 t), \cos(\phi_j^0 t) > +(B_j^0)^2 < \sin(\phi_j^0 t), \sin(\phi_j^0 t) > \\
&+ (C_j^0)^2 < \cos(\psi_j^0 t^2), \cos(\psi_j^0 t^2) > +(D_j^0)^2 < \sin(\psi_j^0 t^2), \sin(\psi_j^0 t^2) > \\
&- 2A_{jT}A_j^0 < \cos(\phi_{jT}t), \cos(\phi_j^0 t) > -2A_{jT}B_j^0 < \cos(\phi_{jT}t), \sin(\phi_j^0 t) > \\
&- 2B_{jT}A_j^0 < \sin(\phi_{jT}t), \cos(\phi_j^0 t) > -2B_{jT}B_j^0 < \sin(\phi_{jT}t), \sin(\phi_j^0 t) > \\
&- 2C_{jT}C_j^0 < \cos(\psi_{jT}t^2), \cos(\psi_j^0 t^2) > -2C_{jT}D_j^0 < \cos(\psi_{jT}t^2), \sin(\psi_j^0 t^2) > \\
&- 2D_{jT}C_j^0 < \sin(\psi_{jT}t^2), \cos(\psi_j^0 t^2) > -2D_{jT}D_j^0 < \sin(\psi_{jT}t^2), \sin(\psi_j^0 t^2) > +o_T(1) \\
&= \frac{1}{2}\left[(A_{jT})^2 + (B_{jT})^2 + (A_j^0)^2 + (B_j^0)^2\right] - \left[A_{jT}A_j^0 + B_{jT}B_j^0\right] x_{jT}^{(1)} \\
&\quad - \left[A_{jT}B_j^0 - B_{jT}A_j^0\right] y_{jT}^{(1)} \\
&+ \frac{1}{2}\left[(C_{jT})^2 + (D_{jT})^2 + (C_j^0)^2 + (D_j^0)^2\right] - \left[C_{jT}C_j^0 + D_{jT}D_j^0\right] x_{jT}^{(2)} \\
&\quad - \left[C_{jT}D_j^0 - C_{jT}D_j^0\right] y_{jT}^{(2)} + o_T(1).
\end{aligned} \tag{27}$$

Substitution of equalities (21) into (27) gives the relation

$$\begin{aligned}
G(\theta_T, \theta^0) &= \sum_{j=1}^{N} \frac{1}{2}\left((A_j^0)^2 + (B_j^0)^2\right)\left(1 - (x_{jT}^{(1)})^2 - (y_{jT}^{(1)})^2\right) \\
&+ \sum_{j=1}^{N} \frac{1}{2}\left((C_j^0)^2 + (D_j^0)^2\right)\left(1 - (x_{jT}^{(2)})^2 - (y_{jT}^{(2)})^2\right) + o_T(1) \to 0 \text{ a.s., as } T \to \infty.
\end{aligned} \tag{28}$$

Expression (28) converges to zero a.s. if and only if for $j = \overline{1, N}$

$$(x_{jT}^{(1)})^2 + (y_{jT}^{(1)})^2 \to 1, \quad (x_{jT}^{(2)})^2 + (y_{jT}^{(2)})^2 \to 1 \text{ a.s., as } T \to \infty. \tag{29}$$

Consider in more detail each expression in (29). First of all

$$(x_{jT}^{(1)})^2 + (y_{jT}^{(1)})^2 = \left(\frac{\sin T(\phi_j^0 - \phi_{jT})}{T(\phi_j^0 - \phi_{jT})}\right)^2 + \left(\frac{1 - \cos T(\phi_j^0 - \phi_{jT})}{T(\phi_j^0 - \phi_{jT})}\right)^2 = \left(\frac{\sin \frac{T}{2}(\phi_j^0 - \phi_{jT})}{\frac{T}{2}(\phi_j^0 - \phi_{jT})}\right)^2.$$

So, $(x_{jT}^{(1)})^2 + (y_{jT}^{(1)})^2 \to 1$, $j = \overline{1, N}$, if and only if

$$T(\phi_j^0 - \phi_{jT}) \to 0 \text{ a.s., as } T \to \infty, \quad j = \overline{1, N}. \tag{30}$$

Using notation $\lambda_{jT} = T^2(\psi_j^0 - \psi_{jT})$, $j = \overline{1, N}$, we have

$$(x_{jT}^{(2)})^2 + (y_{jT}^{(2)})^2 = \left(\int_0^1 \cos(\lambda_{jT}t^2)dt\right)^2 + \left(\int_0^1 \sin(\lambda_{jT}t^2)dt\right)^2. \tag{31}$$

Let $\Omega_0 \subset \Omega$, $P(\Omega_0) = 1$, is a random event for which $(x_{jT}^{(2)})^2 + (y_{jT}^{(2)})^2 \to 1$, as $T \to \infty$, $j = \overline{1, N}$. If for any elementary event $\omega \in \Omega_0$

$$\lambda_{jT} \to 0, \text{ as } T \to \infty, j = \overline{1, N}, \tag{32}$$

then $(x_{jT}^{(2)})^2 + (y_{jT}^{(2)})^2 \to 1$, as $T \to \infty$, $j = \overline{1, N}$, by Lebesgue dominated convergence theorem. Suppose that (32) is not true for some $\omega_0 \in \Omega_0$ and consider all the possible options of λ_{jT}, $j = \overline{1, N}$, behavior. Let for some $j \in \{1, ..., N\}$ $\lambda_{jT} \nrightarrow 0$, as $T \to \infty$. Then there exist $\varepsilon_0 > 0$ and sequence $\{T_n, n \geq 1\}$, $T_n \to \infty$, as $n \to \infty$, such that $|\lambda_{jT_n}| \geq \varepsilon_0$, $n \geq 1$. Let the set of values $\{\lambda_{jT_n}, n \geq 1\}$ is bounded, then there exists a subsequence $\{T_{n_k}, k \geq 1\}$, $T_{n_k} \to \infty$, as $k \to \infty$, such that $\lambda_{jT_{n_k}} \to \lambda_j \neq 0$. If the set of values $\{\lambda_{jT_n}, n \geq 1\}$ is unbounded, then for some subsequence $\{T_{n_k}, k \geq 1\}$ $\lambda_{jT_{n_k}} \to -\infty$ or $+\infty$, as $k \to \infty$.

Using notation $\lambda_{jT_{n_k}} = \lambda_{jk}$, $x_{jT_{n_k}}^{(2)} = x_{jk}^{(2)}$, $y_{jT_{n_k}}^{(2)} = y_{jk}^{(2)}$, consider the possible options of λ_{jk} convergence, as $k \to \infty$:

$$\textit{(i)}\ \lambda_{jk} \to -\infty \text{ or } +\infty; \qquad \textit{(ii)}\ \lambda_{jk} \to \lambda_j \neq 0.$$

Let's show that for the options *(i)* and *(ii)*

$$(x_{jk}^{(2)})^2 + (y_{jk}^{(2)})^2 \nrightarrow 1, \text{ as } k \to \infty. \tag{33}$$

Really,

$$\begin{aligned}(x_{jk}^{(2)})^2 + (y_{jk}^{(2)})^2 &= \left(\int_0^1 \cos(|\lambda_{jk}|t^2)dt\right)^2 + \left(\int_0^1 \sin(|\lambda_{jk}|t^2)dt\right)^2 \\ &= \frac{1}{|\lambda_{jk}|}C^2\left(\sqrt{|\lambda_{jk}|}\right) + \frac{1}{|\lambda_{jk}|}S^2\left(\sqrt{|\lambda_{jk}|}\right) \leq \frac{2}{|\lambda_{jk}|},\end{aligned} \tag{34}$$

that is for the option *(i)* the relation (33) is true.

For the option *(ii)* using Lebesgue theorem and Cauchy-Schwarz inequality we arrive at relation

$$\lim_{k\to\infty}\left((x_{jk}^{(2)})^2 + (y_{jk}^{(2)})^2\right) = \left(\int_0^1 \cos(\lambda_j t^2)dt\right)^2 + \left(\int_0^1 \sin(\lambda_j t^2)dt\right)^2 < 1. \tag{35}$$

Thus, taking into account all the points discussed after formula (31), we get

$$\lambda_{jT} = T^2(\psi_j^0 - \psi_{jT}) \to 0 \text{ a.s., as } T \to \infty, j = \overline{1, N}. \tag{36}$$

From (21), (30), (36) it follows the strong consistency of the estimates A_{jT}, B_{jT}, C_{jT}, D_{jT}, $j = \overline{1, N}$. □

3. Asymptotic normality

The proof of **Theorem 2** is preceded by 2 lemmas, and they are formulated after their proofs. Set

$$\frac{\partial}{\partial\theta_i}g(t,\theta) = g_i(t,\theta); \qquad \frac{\partial^2}{\partial\theta_i\partial\theta_j}g(t,\theta) = g_{ij}(t,\theta), \quad i,j = \overline{1, 6N}, \tag{37}$$

and write down the system of normal equations for θ_T:

$$0 = Q_T'(\theta_T) = \left(-2\int_0^T [X(t) - g(t,\theta_T)]\, g_i(t,\theta_T)dt\right)_{i=1}^{6N}. \tag{38}$$

Consider the Hesse matrix

$$Q_T''(\theta) = \left(-2 \int_0^T \left[X(t) - g(t,\theta) \right] g_{ij}(t,\theta) dt + 2 \int_0^T g_i(t,\theta) g_j(t,\theta) dt \right)_{i,j=1}^{6N} \tag{39}$$

and Taylor expansion of each row (38) with its own value of $\overline{\theta}$:

$$-\frac{1}{2} Q_T'(\theta^0) = \frac{1}{2} Q_T'(\theta_T) - \frac{1}{2} Q_T'(\theta^0) = \frac{1}{2} Q_T''(\overline{\theta})(\theta_T - \theta^0). \tag{40}$$

Let's introduce a block-diagonal matrix $\tilde{d}_T$, which contains N blocks, and each such a block, in turn, is diagonal matrix

$$d_T = diag \left(T^{1/2}, T^{1/2}, T^{3/2}, T^{1/2}, T^{1/2}, T^{5/2} \right). \tag{41}$$

Using diagonal matrix $\widetilde{d}_T$ we rewrite (40) as

$$\widetilde{d}_T(\theta_T - \theta^0) = \left(\widetilde{d}_T^{-1} \left(\frac{1}{2} Q_T''(\overline{\theta}) \right) \widetilde{d}_T^{-1} \right)^{-1} \widetilde{d}_T^{-1} \left(-\frac{1}{2} Q_T'(\theta^0) \right) \widetilde{d}_T^{-1}. \tag{42}$$

Let's first study the asymptotic behavior of the matrix

$$\begin{aligned}
\widetilde{d}_T^{-1} \left(\frac{1}{2} Q_T''(\overline{\theta}) \right) \widetilde{d}_T^{-1} &= \widetilde{d}_T^{-1} \left(\int_0^T \left[g(t,\overline{\theta}) - g(t,\theta^0) \right] g_{ij}(t,\overline{\theta}) dt \right)_{i,j=1}^{6N} \widetilde{d}_T^{-1} \\
- \widetilde{d}_T^{-1} \left(\int_0^T \varepsilon(t) g_{ij}(t,\overline{\theta}) dt \right)_{i,j=1}^{6N} &\widetilde{d}_T^{-1} + \widetilde{d}_T^{-1} \left(\int_0^T g_i(t,\overline{\theta}) g_j(t,\overline{\theta}) dt \right)_{i,j=1}^{6N} \widetilde{d}_T^{-1} \\
&= J^{(1)}(T,\overline{\theta}) - J^{(2)}(T,\overline{\theta}) + J^{(3)}(T,\overline{\theta}).
\end{aligned} \tag{43}$$

Consider matrices $J^{(1)}(T,\overline{\theta})$, $J^{(2)}(T,\overline{\theta})$ and show that these matrices tend to zero a.s., as $T \to \infty$. Matrices $J^{(1)}(T,\overline{\theta})$ and $J^{(2)}(T,\overline{\theta})$ are block-diagonal and contain N blocks of the form

$$J_k^{(1)}(T,\overline{\theta}) = d_T^{-1} \left(\int_0^T \left[g(t,\overline{\theta}) - g(t,\theta^0) \right] g_{ij}(t,\overline{\theta}) dt \right)_{i,j=6k-5}^{6k} d_T^{-1}, \tag{44}$$

$$J_k^{(2)}(T,\overline{\theta}) = d_T^{-1} \left(\int_0^T \varepsilon(t) g_{ij}(t,\overline{\theta}) dt \right)_{i,j=6k-5}^{6k} d_T^{-1}, k = \overline{1,N}. \tag{45}$$

Since the proof of convergence to zero matrix is the same for any k, we take just the case $k=1$. Consider the matrix $d_T^{-1} \left(g_{ij}(t,\overline{\theta}) \right)_{i,j=1}^{6} d_T^{-1}$ that is block-diagonal with blocks

$$\begin{bmatrix} 0 & 0 & -t \sin(\overline{\phi}_1 t) T^{-2} \\ 0 & 0 & t \cos(\overline{\phi}_1 t) T^{-2} \\ -t \sin(\overline{\phi}_1 t) T^{-2} & t \cos(\overline{\phi}_1 t) T^{-2} & -t^2 \left(\overline{A}_1 \cos(\overline{\phi}_1 t) + \overline{B}_1 \sin(\overline{\phi}_1 t) \right) T^{-3} \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & -t^2 \sin(\overline{\psi}_1 t^2) T^{-3} \\ 0 & 0 & t^2 \cos(\overline{\psi}_1 t^2) T^{-3} \\ -t^2 \sin(\overline{\psi}_1 t^2) T^{-3} & t^2 \cos(\overline{\psi}_1 t^2) T^{-3} & -t^4 \left(\overline{C}_1 \cos(\overline{\psi}_1 t^2) + \overline{D}_1 \sin(\overline{\psi}_1 t^2) \right) T^{-5} \end{bmatrix}. \tag{46}$$

From **Theorem 1** it follows for $t \in [0, T]$

$$\left|g(t,\overline{\theta}) - g(t,\theta^0)\right| \leq \sum_{k=1}^{N}\Big(|\overline{A}_k - A_k^0| + |\overline{B}_k - B_k^0| + (|A_k^0| + |B_k^0|)|\overline{\phi}_k - \phi_k^0|t$$

$$+ |\overline{C}_k - C_k^0| + |\overline{D}_k - D_k^0| + (|C_k^0| + |D_k^0|)|\overline{\psi}_k - \psi_k^0|t^2\Big) \leq \sum_{k=1}^{N}\Big(|A_{kT} - A_k^0| + |B_{kT} - B_k^0|$$

$$+ (|A_k^0| + |B_k^0|)|\phi_{kT} - \phi_k^0|T + |C_{kT} - C_k^0| + |D_{kT} - D_k^0| + (|C_k^0| + |D_k^0|)|\psi_{kT} - \psi_k^0|T^2\Big)$$

$$= \eta_T \to 0 \text{ a.s., as } T \to \infty. \tag{47}$$

Then

$$\left|J_{1,13}^{(1)}(T,\overline{\theta})\right|, \left|J_{1,23}^{(1)}(T,\overline{\theta})\right| \leq \frac{\eta_T}{2}; \quad \left|J_{1,46}^{(1)}(T,\overline{\theta})\right|, \left|J_{1,56}^{(1)}(T,\overline{\theta})\right| \leq \frac{\eta_T}{3};$$

$$\left|J_{1,33}^{(1)}(T,\overline{\theta})\right| \leq \frac{1}{3}\left(|\overline{A}_1| + |\overline{B}_1|\right)\eta_T; \quad \left|J_{1,66}^{(1)}(T,\overline{\theta})\right| \leq \frac{1}{5}\left(|\overline{C}_1| + |\overline{D}_1|\right)\eta_T, \tag{48}$$

and due to (47) and (48)

$$J^{(1)}(T,\overline{\theta}) \to 0 \text{ a.s., as } T \to \infty. \tag{49}$$

Consider then the elements of matrix $J^{(2)}(T,\overline{\theta})$:

$$\left|J_{1,13}^{(2)}(T,\overline{\theta})\right| = \left|T^{-2}\int_0^T \varepsilon(t)t\sin(\overline{\phi}_1 t)dt\right| \leq \left|T^{-2}\int_0^T \varepsilon(t)t\left(\sin(\overline{\phi}_1 t) - \sin(\phi_1^0 t)\right)dt\right|$$

$$+ \left|T^{-2}\int_0^T \varepsilon(t)t\sin(\phi_1^0 t)dt\right| = I_1(T) + I_2(T); \tag{50}$$

$$I_1(T) \leq T^{-1}\int_0^T |\varepsilon(t)|dt|\phi_{1T} - \phi_1^0|T.$$

As follows from **A1**

$$T^{-1}\int_0^T |\varepsilon(t)|dt \leq \frac{1}{2}\left(1 + T^{-1}\int_0^T \varepsilon^2(t)dt\right) \to \frac{1}{2}(1 + B(0)) \text{ a.s., as } T \to \infty,$$

and therefore $I_1(T) \to 0$ a.s., as $T \to \infty$.
On the other hand, under condition **A1(*i*)**

$$EI_2^2(T) = T^{-4}\int_0^T\int_0^T tsB(t-s)\sin(\phi_1^0 t)\sin(\phi_1^0 s)dtds \leq T^{-2}\int_0^T\int_0^T B(t-s)dtds$$

$$= \int_0^1\int_0^1 B(T(t-s))dtds = \int_{-1}^1 (1-|t|)B(Tt)dt \leq 2\int_0^1 B(Tt)dt \leq \frac{2}{1-\alpha}B(T).$$

Let $T_n = n^\beta$, and $\beta\alpha > 1$. Then $I_2(T_n) \to 0$ a.s., as $n \to \infty$. Consider the sequence of random variables

$$\sup_{T_n \leq T \leq T_{n+1}} |I_2(T) - I_2(T_n)| \leq \left(\left(\frac{T_{n+1}}{T_n}\right)^2 - 1\right) I_2(T_n) + I_3(T_n),$$

$$I_3(T_n) = T_n^{-1}\int_{T_n}^{T_{n+1}} |\varepsilon(t)|dt.$$

As far as

$$EI_3^2(T_n) \le T_n^{-2} \int\limits_{T_n}^{T_{n+1}} \int\limits_{T_n}^{T_{n+1}} E|\varepsilon(t)\varepsilon(s)|dtds \le B(0) \left(\frac{T_{n+1} - T_n}{T_n} \right)^2 = O(n^{-2}),$$

then $I_3(T_n) \to 0$ a.s., as $n \to \infty$, and $J_{1,13}^{(2)}(T, \overline{\theta}) \to 0$ a.s., as $T \to \infty$.

Under condition **A1(*ii*)** we obtain also that $J_{1,13}^{(2)}(T, \overline{\theta}) \to 0$ a.s., as $T \to \infty$. The proof of this fact is similar to the previous one, since

$$T^{-2} \int\limits_0^T \int\limits_0^T |B(t-s)|dtds = O(T^{-1}).$$

Similarly we get $J_{1,23}^{(2)}(T, \overline{\theta})$, $J_{1,46}^{(2)}(T, \overline{\theta})$, $J_{1,56}^{(2)}(T, \overline{\theta}) \to 0$, a.s., as $T \to \infty$. Taking into account that the proofs of $J_{1,33}^{(2)}(T, \overline{\theta})$ and $J_{1,66}^{(2)}(T, \overline{\theta})$ convergence to zero are identical, we will prove the fact for $J_{1,66}^{(2)}(T, \overline{\theta})$ only. Obviously,

$$\left| J_{1,66}^{(2)}(T, \overline{\theta}) \right| = \Bigg| T^{-5} \int\limits_0^T \varepsilon(t) \Big[\left(\overline{C}_1 - C_1^0 \right) \cos(\overline{\psi}_1 t^2) + \left(\overline{D}_1 - D_1^0 \right) \sin(\overline{\psi}_1 t^2) +$$

$$C_1^0 \left(\cos(\overline{\psi}_1 t^2) - \cos(\psi_1^0 t^2) \right) + D_1^0 \left(\sin(\overline{\psi}_1 t^2) - \sin(\psi_1^0 t^2) \right) + C_1^0 \cos(\psi_1^0 t^2) + D_1^0 \sin(\psi_1^0 t^2) \Big] dt \Bigg|$$

$$\le \left[|C_{1T} - C_1^0| + |D_{1T} - D_1^0| + (|C_1^0| + |D_1^0|)|\psi_{1T} - \psi_1^0| T^2 \right] T^{-1} \int\limits_0^T |\varepsilon(t)|dt$$

$$+ \left| C_1^0 T^{-5} \int\limits_0^T \varepsilon(t) t^4 \cos(\psi_1^0 t^2) dt \right| + \left| D_1^0 T^{-5} \int\limits_0^T \varepsilon(t) t^4 \sin(\psi_1^0 t^2) dt \right|, \qquad (51)$$

where the 1st summand tends to zero a.s. according to **Theorem 1**; convergence to zero of the 2nd and the 3rd summands is proved similarly to convergence of $I_2(T)$. Thus

$$J^{(2)}(T, \overline{\theta}) \to 0 \text{ a.s., as } T \to \infty. \qquad (52)$$

We will consider further the matrix $J^{(3)}(T, \overline{\theta})$. This matrix consists of blocks

$$J_{ij}^{(3)}(T, \overline{\theta}) = d_T^{-1} \left(\int\limits_0^T g_m(t, \overline{\theta}) g_n(t, \overline{\theta}) dt \right)_{m=6i-5, n=6j-5}^{6i,6j} d_T^{-1}, \quad i, j = \overline{1, N}. \qquad (53)$$

First examine the blocks $J_{ij}^{(3)}(T, \overline{\theta})$, $i \ne j$, and write down the elements of these blocks, dividing them into 3 natural types.

The 1st type.

$$J_{ij,11}^{(3)}(T, \overline{\theta}) = T^{-1} \int\limits_0^T \cos(\overline{\phi}_i t) \cos(\overline{\phi}_j t) dt; \quad J_{ij,12}^{(3)}(T, \overline{\theta}) = T^{-1} \int\limits_0^T \cos(\overline{\phi}_i t) \sin(\overline{\phi}_j t) dt;$$

$$J_{ij,13}^{(3)}(T, \overline{\theta}) = T^{-2} \int\limits_0^T \cos(\overline{\phi}_i t) t \left(-\overline{A}_j \sin(\overline{\phi}_j t) + \overline{B}_j \cos(\overline{\phi}_j t) \right) dt;$$

$$J_{ij,21}^{(3)}(T, \overline{\theta}) = T^{-1} \int\limits_0^T \sin(\overline{\phi}_i t) \cos(\overline{\phi}_j t) dt; \quad J_{ij,22}^{(3)}(T, \overline{\theta}) = T^{-1} \int\limits_0^T \sin(\overline{\phi}_i t) \sin(\overline{\phi}_j t) dt;$$

$$J^{(3)}_{ij,23}(T,\overline{\theta}) = T^{-2}\int_0^T \sin(\overline{\phi}_i t)t\left(-\overline{A}_j\sin(\overline{\phi}_j t) + \overline{B}_j\cos(\overline{\phi}_j t)\right)dt;$$
$$J^{(3)}_{ij,31}(T,\overline{\theta}) = T^{-2}\int_0^T t\left(-\overline{A}_i\sin(\overline{\phi}_i t) + \overline{B}_i\cos(\overline{\phi}_i t)\right)\cos(\overline{\phi}_j t)dt;$$
$$J^{(3)}_{ij,32}(T,\overline{\theta}) = T^{-2}\int_0^T t\left(-\overline{A}_i\sin(\overline{\phi}_i t) + \overline{B}_i\cos(\overline{\phi}_i t)\right)\sin(\overline{\phi}_j t)dt; \tag{54}$$
$$J^{(3)}_{ij,33}(T,\overline{\theta}) = T^{-3}\int_0^T t^2\left(-\overline{A}_i\sin(\overline{\phi}_i t) + \overline{B}_i\cos(\overline{\phi}_i t)\right)\left(-\overline{A}_j\sin(\overline{\phi}_j t) + \overline{B}_j\cos(\overline{\phi}_j t)\right)dt;$$

The 2nd type.

$$J^{(3)}_{ij,14}(T,\overline{\theta}) = T^{-1}\int_0^T \cos(\overline{\phi}_i t)\cos(\overline{\psi}_j t^2)dt; \quad J^{(3)}_{ij,15}(T,\overline{\theta}) = T^{-1}\int_0^T \cos(\overline{\phi}_i t)\sin(\overline{\psi}_j t^2)dt;$$
$$J^{(3)}_{ij,16}(T,\overline{\theta}) = T^{-3}\int_0^T \cos(\overline{\phi}_i t)t^2\left(-\overline{C}_j\sin(\overline{\psi}_j t^2) + \overline{D}_j\cos(\overline{\psi}_j t^2)\right)dt;$$
$$J^{(3)}_{ij,24}(T,\overline{\theta}) = T^{-1}\int_0^T \sin(\overline{\phi}_i t)\cos(\overline{\psi}_j t^2)dt; \quad J^{(3)}_{ij,25}(T,\overline{\theta}) = T^{-1}\int_0^T \sin(\overline{\phi}_i t)\sin(\overline{\psi}_j t^2)dt;$$
$$J^{(3)}_{ij,26}(T,\overline{\theta}) = T^{-3}\int_0^T \sin(\overline{\phi}_i t)t^2\left(-\overline{C}_j\sin(\overline{\psi}_j t^2) + \overline{D}_j\cos(\overline{\psi}_j t^2)\right)dt;$$
$$J^{(3)}_{ij,34}(T,\overline{\theta}) = T^{-2}\int_0^T t\left(-\overline{A}_i\sin(\overline{\phi}_i t) + \overline{B}_i\cos(\overline{\phi}_i t)\right)\cos(\overline{\psi}_j t^2)dt;$$
$$J^{(3)}_{ij,35}(T,\overline{\theta}) = T^{-2}\int_0^T t\left(-\overline{A}_i\sin(\overline{\phi}_i t) + \overline{B}_i\cos(\overline{\phi}_i t)\right)\sin(\overline{\psi}_j t^2)dt;$$
$$J^{(3)}_{ij,36}(T,\overline{\theta}) = T^{-4}\int_0^T t^3\left(-\overline{A}_i\sin(\overline{\phi}_i t) + \overline{B}_i\cos(\overline{\phi}_i t)\right)\left(-\overline{C}_j\sin(\overline{\psi}_j t^2) + \overline{D}_j\cos(\overline{\psi}_j t^2)\right)dt;$$
$$J^{(3)}_{ij,41}(T,\overline{\theta}) = T^{-1}\int_0^T \cos(\overline{\psi}_i t^2)\cos(\overline{\phi}_j t)dt; \quad J^{(3)}_{ij,42}(T,\overline{\theta}) = T^{-1}\int_0^T \cos(\overline{\psi}_i t^2)\sin(\overline{\phi}_j t)dt;$$
$$J^{(3)}_{ij,43}(T,\overline{\theta}) = T^{-2}\int_0^T \cos(\overline{\psi}_i t^2)t\left(-\overline{A}_j\sin(\overline{\phi}_j t) + \overline{B}_j\cos(\overline{\phi}_j t)\right)dt;$$
$$J^{(3)}_{ij,51}(T,\overline{\theta}) = T^{-1}\int_0^T \sin(\overline{\psi}_i t^2)\cos(\overline{\phi}_j t)dt; \quad J^{(3)}_{ij,52}(T,\overline{\theta}) = T^{-1}\int_0^T \sin(\overline{\psi}_i t^2)\sin(\overline{\phi}_j t)dt;$$
$$J^{(3)}_{ij,53}(T,\overline{\theta}) = T^{-2}\int_0^T \sin(\overline{\psi}_i t^2)t\left(-\overline{A}_j\sin(\overline{\phi}_j t) + \overline{B}_j\cos(\overline{\phi}_j t)\right)dt;$$
$$J^{(3)}_{ij,61}(T,\overline{\theta}) = T^{-3}\int_0^T t^2\left(-\overline{C}_i\sin(\overline{\psi}_i t^2) + \overline{D}_i\cos(\overline{\psi}_i t^2)\right)\cos(\overline{\phi}_j t)dt;$$

$$J^{(3)}_{ij,62}(T,\overline{\theta}) = T^{-3}\int_0^T t^2\left(-\overline{C}_i\sin(\overline{\psi}_i t^2) + \overline{D}_i\cos(\overline{\psi}_i t^2)\right)\sin(\overline{\phi}_j t)dt; \tag{55}$$

$$J^{(3)}_{ij,63}(T,\overline{\theta}) = T^{-4}\int_0^T t^3\left(-\overline{C}_i\sin(\overline{\psi}_i t^2) + \overline{D}_i\cos(\overline{\psi}_i t^2)\right)\left(-\overline{A}_j\sin(\overline{\phi}_j t) + \overline{B}_j\cos(\overline{\phi}_j t)\right)dt;$$

<u>The 3rd type</u>.

$$J^{(3)}_{ij,44}(T,\overline{\theta}) = T^{-1}\int_0^T \cos(\overline{\psi}_i t^2)\cos(\overline{\psi}_j t^2)dt; \quad J^{(3)}_{ij,45}(T,\overline{\theta}) = T^{-1}\int_0^T \cos(\overline{\psi}_i t^2)\sin(\overline{\psi}_j t^2)dt;$$

$$J^{(3)}_{ij,46}(T,\overline{\theta}) = T^{-3}\int_0^T \cos(\overline{\psi}_i t^2)t^2\left(-\overline{C}_j\sin(\overline{\psi}_j t^2) + \overline{D}_j\cos(\overline{\psi}_j t^2)\right)dt;$$

$$J^{(3)}_{ij,54}(T,\overline{\theta}) = T^{-1}\int_0^T \sin(\overline{\psi}_i t^2)\cos(\overline{\psi}_j t^2)dt; \quad J^{(3)}_{ij,55}(T,\overline{\theta}) = T^{-1}\int_0^T \sin(\overline{\psi}_i t^2)\sin(\overline{\psi}_j t^2)dt;$$

$$J^{(3)}_{ij,56}(T,\overline{\theta}) = T^{-3}\int_0^T \sin(\overline{\psi}_i t^2)t^2\left(-\overline{C}_j\sin(\overline{\psi}_j t^2) + \overline{D}_j\cos(\overline{\psi}_j t^2)\right)dt;$$

$$J^{(3)}_{ij,64}(T,\overline{\theta}) = T^{-3}\int_0^T t^2\left(-\overline{C}_i\sin(\overline{\psi}_i t^2) + \overline{D}_i\cos(\overline{\psi}_i t^2)\right)\cos(\overline{\psi}_j t^2)dt;$$

$$J^{(3)}_{ij,65}(T,\overline{\theta}) = T^{-3}\int_0^T t^2\left(-\overline{C}_i\sin(\overline{\psi}_i t^2) + \overline{D}_i\cos(\overline{\psi}_i t^2)\right)\sin(\overline{\psi}_j t^2)dt; \tag{56}$$

$$J^{(3)}_{ij,66}(T,\overline{\theta}) = T^{-5}\int_0^T t^4\left(-\overline{C}_i\sin(\overline{\psi}_i t^2) + \overline{D}_i\cos(\overline{\psi}_i t^2)\right)\left(-\overline{C}_j\sin(\overline{\psi}_j t^2) + \overline{D}_j\cos(\overline{\psi}_j t^2)\right)dt.$$

Let us show that due to the LSE θ_T consistency each element of all 3 types in the formulas (54)-(56) tends to zero a.s., as $T \to \infty$. Moreover, each element in (54)-(56), being multiplied by T^γ for any $\gamma \in (0,1)$, also tends to zero a.s., as $T \to \infty$. For elements of the 1st type it is obvious. For elements of the 2nd and 3rd types it follows, for example, from formulas 2.655, p. 226, of Gradshteyn and Ryzhik (2007), namely: for any bounded function a_T, $T > 0$, and bounded function b_T, $T > 0$, with the following property: there exists a neighborhood of zero V_0 such that for sufficiently large T $\beta_T \in \mathbb{R}\backslash V_0$, we have for $n \in \mathbb{N}$

$$\int_0^T t^n \begin{matrix}\sin\\ \cos\end{matrix}(\alpha_T t + \beta_T t^2)dt = O(T^{n-1}), \text{ as } T \to \infty. \tag{57}$$

In the matrix $J^{(3)}(T,\overline{\theta})$ it remains to consider a diagonal submatrix with blocks $J^{(3)}_{ij}(T,\overline{\theta})$, $i = j = k$, k=$\overline{1,N}$. Note that in the blocks $J^{(3)}_{kk}(T,\overline{\theta})$ all the elements of the 2nd type still converge to zero. Non-zero are those limits of the remaining elements of the blocks $J^{(3)}_{kk}(T,\overline{\theta})$ that contain the squares of sines and cosines under the integral signs. Thus we can formulate the following statement.

Lemma 1. *Under conditions **A(i)** and **B***

$$\tilde{d}_T^{-1}\left(\frac{1}{2}Q''_T(\overline{\theta})\right)\tilde{d}_T^{-1} \to H \text{ a.s., as } T \to \infty, \tag{58}$$

where $H = diag(H_1, ..., H_N)$ is a block-diagonal matrix with blocks $H_k = diag(H_k^{(1)}, H_k^{(2)})$, $k = \overline{1, N}$,

$$H_k^{(1)} = \frac{1}{2}\begin{bmatrix} 1 & 0 & \frac{B_k^0}{2} \\ 0 & 1 & -\frac{A_k^0}{2} \\ \frac{B_k^0}{2} & -\frac{A_k^0}{2} & \frac{(A_k^0)^2+(B_k^0)^2}{3} \end{bmatrix}, \quad H_k^{(2)} = \frac{1}{2}\begin{bmatrix} 1 & 0 & \frac{D_k^0}{3} \\ 0 & 1 & -\frac{C_k^0}{3} \\ \frac{D_k^0}{3} & -\frac{C_k^0}{3} & \frac{(C_k^0)^2+(D_k^0)^2}{5} \end{bmatrix}. \tag{59}$$

Introduce a block-diagonal matrix $\widetilde{s}_T$ with N blocks

$$s_T = diag(1, 1, 1, T^{1/2}, T^{1/2}, T^{3/2}) \tag{60}$$

and consider the random vector $\widetilde{s}_T \widetilde{d}_T^{-1}\left(-\frac{1}{2}Q_T'(\theta^0)\right) = \Xi_T$,

$$\Xi_T = \left(\Xi_{1T}^{(1)}, \Xi_{1T}^{(2)}, ..., \Xi_{kT}^{(1)}, \Xi_{kT}^{(2)}, ..., \Xi_{NT}^{(1)}, \Xi_{NT}^{(2)}\right)^*, \tag{61}$$

$$\Xi_{kT}^{(1)} = \Bigg(T^{-1/2}\int_0^T \varepsilon(t)\cos(\phi_k^0 t)dt, T^{-1/2}\int_0^T \varepsilon(t)\sin(\phi_k^0 t)dt,$$
$$T^{-3/2}\int_0^T \varepsilon(t)t\left(-A_k^0\sin(\phi_k^0 t) + B_k^0\cos(\phi_k^0 t)\right)dt\Bigg)^* = \left(\xi_{kT}^{(1)}, \xi_{kT}^{(2)}, \xi_{kT}^{(3)}\right)^*; \tag{62}$$

$$\Xi_{kT}^{(2)} = \Bigg(\int_0^T \varepsilon(t)\cos(\psi_k^0 t^2)dt, \int_0^T \varepsilon(t)\sin(\psi_k^0 t^2)dt,$$
$$T^{-1}\int_0^T \varepsilon(t)t^2\left(-C_k^0\sin(\psi_k^0 t^2) + D_k^0\cos(\psi_k^0 t^2)\right)dt\Bigg)^* = \left(\xi_{kT}^{(4)}, \xi_{kT}^{(5)}, \xi_{kT}^{(6)}\right)^*. \tag{63}$$

The normalization (60) of the vector (61) is necessary to ensure that vectors (63) do not converge to zero, as $T \to \infty$.

The problem is to find the limiting covariance matrix of the vector (61) as $T \to \infty$.
1) We'll start with the vector

$$\Xi_T^{(1)} = \left(\xi_{1T}^{(1)}, \xi_{1T}^{(2)}, \xi_{1T}^{(3)}, ..., \xi_{NT}^{(1)}, \xi_{NT}^{(2)}, \xi_{NT}^{(3)}\right)^*, \tag{64}$$

and take in the regression function (6) the sum of harmonics

$$h(t, \tau^0) = \sum_{j=1}^{N}\left(A_j^0\cos(\phi_j^0 t) + B_j^0\sin(\phi_j^0 t)\right), \tag{65}$$

$$\tau^0 = \left(A_1^0, B_1^0, \phi_1^0, ..., A_N^0, B_N^0, \phi_N^0\right)^* = \left(\tau_1^0, \tau_1^0, \tau_1^0, ..., \tau_{3N-2}^0, \tau_{3N-1}^0, \tau_{3N}^0\right)^*.$$

Set

$$h_j(t, \tau) = \frac{\partial}{\partial \tau_j}h(t, \tau), \qquad b_T^2(\tau) = diag\left(b_{jT}^2(\tau), j = \overline{1, N}\right),$$

$$b_{jT}^2(\tau) = \int_0^T h_j^2(t, \tau)dt, \qquad h_T^j(\lambda, \tau) = \int_0^T e^{i\lambda t}h_j(t, \tau)dt,$$

and introduce a family of matrix measures $\mu_T(d\lambda, \tau)$ on $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is σ-algebra of Borel subsets, with matrix densities

$$\left(\mu_T^{jl}(\lambda, \tau)\right)_{j,l=1}^{N},$$

$$\mu_T^{jl}(\lambda,\tau) = h_T^j(\lambda,\tau)\overline{h_T^l(\lambda,\tau)} \left(\int_{-\infty}^{\infty} \left| h_T^j(\lambda,\tau) \right|^2 d\lambda \int_{-\infty}^{\infty} \left| h_T^l(\lambda,\tau) \right|^2 d\lambda \right)^{-1/2}. \quad (66)$$

Note that by Plancherel identity

$$b_{jT}^2(\tau) = (2\pi)^{-1} \int_{-\infty}^{\infty} \left| h_T^j(\lambda,\tau) \right|^2 d\lambda.$$

For the function (65), the family of matrix measures (66) converges weakly to a positive definite matrix measure $\mu(d\lambda,\tau)|_{\tau=\tau_0}$, that is its elements $\mu^{jl}(d\lambda,\tau)$ are complex signed measures, the matrices $\mu(B,\tau)$, $B \in \mathcal{B}$, are non-negatively definite, and matrix $\mu(\mathbb{R},\tau)$ is positive definite.

Definition 2. (Grenander (1954), Ibragimov and Rozanov (2012), Ivanov and Leonenko (1989)). The measure $\mu(d\lambda,\tau)$ is called the spectral measure of the function $h(t,\tau)$, or, which is the same, the spectral measure of vector function $\nabla h(t,\tau)$.

For the function (65) the spectral measure $\mu(d\lambda,\tau^0)$ is a block-diagonal matrix with blocks (see, for example, Ivanov *et al.* (2013), Ivanov *et al.* (2015))

$$\begin{bmatrix} \delta_k & i\rho_k & \overline{\beta}_k \\ -i\rho_k & \delta_k & \overline{\gamma}_k \\ \beta_k & \gamma_k & \delta_k \end{bmatrix}, k = \overline{1,N}, \quad (67)$$

with $\beta_k = \frac{\sqrt{3}\left(B_k^0\delta_k + iA_k^0\rho_k\right)}{2\sqrt{(A_k^0)^2+(B_k^0)^2}}$, $\gamma_k = \frac{\sqrt{3}\left(-A_k^0\delta_k + iB_k^0\rho_k\right)}{2\sqrt{(A_k^0)^2+(B_k^0)^2}}$, measure $\delta_k = \delta_k(d\lambda)$ and signed measure $\rho_k = \rho_k(d\lambda)$ are concentrated at the points $\pm\phi_k^0$, and $\delta_k(\{\pm\phi_k^0\}) = \frac{1}{2}$, $\rho_k(\{\pm\phi_k^0\}) = \pm\frac{1}{2}$, $k = \overline{1,N}$.

Consider a random vector

$$b_T^{-1}(\tau^0) \int_0^T \varepsilon(t)\nabla h(t,\tau^0)dt, \quad (68)$$

with function $h(t,\tau^0)$ specified in (65). Using the definition of spectral measure, formulas (67) and condition **A2** we find the limit, as $T \to \infty$, of covariance matrix of vector (68) as a block-diagonal matrix

$$\sigma = 2\pi \int_{-\infty}^{\infty} f(\lambda)\mu(d\lambda,\tau^0) = (\sigma_k)_{k=1}^N$$

$$= \left(2\pi f(\phi_k^0) \begin{bmatrix} 1 & 0 & \frac{\sqrt{3}B_k^0}{2\sqrt{(A_k^0)^2+(B_k^0)^2}} \\ 0 & 1 & \frac{-\sqrt{3}A_k^0}{2\sqrt{(A_k^0)^2+(B_k^0)^2}} \\ \frac{\sqrt{3}B_k^0}{2\sqrt{(A_k^0)^2+(B_k^0)^2}} & \frac{-\sqrt{3}A_k^0}{2\sqrt{(A_k^0)^2+(B_k^0)^2}} & 1 \end{bmatrix} \right)_{k=1}^N. \quad (69)$$

When the conditions **A2(*i*)** is met the spectral density $f(\lambda)$ has a singularity at zero, that is, it is not continuous and bounded on the real line. However it can be proved (see Ivanov *et al.* (2013)) that $f(\lambda)$ is μ-admissible, namely:

$$\int_{-\infty}^{\infty} f(\lambda)\mu_T(d\lambda,\tau^0) \to \int_{-\infty}^{\infty} f(\lambda)\mu(d\lambda,\tau^0), \text{ as } T \to \infty. \quad (70)$$

Note also that for function (65)

$$b_{3k-2}(\tau^0) \sim \frac{T^{1/2}}{\sqrt{2}}, \quad b_{3k-1}(\tau^0) \sim \frac{T^{1/2}}{\sqrt{2}}, \quad b_{3k}(\tau^0) \sim T^{3/2}\sqrt{\frac{(A_k^0)^2 + (B_k^0)^2}{6}}, \quad k = \overline{1,N}. \quad (71)$$

Introduce the block-diagonal matrix Q with blocks

$$Q_k = diag(\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{\frac{(A_k^0)^2 + (B_k^0)^2}{6}}\right)), \quad k = \overline{1, N}. \tag{72}$$

Taking into account (68)-(72), the limiting covariance matrix of the vector (64)

$$G^{(11)} = \lim_{T\to\infty} E\Xi_T^{(1)} \left(\Xi_T^{(1)}\right)^*$$

consists of the elements (see formula (59))

$$G_{ii}^{(11)} = Q_i \sigma_i Q_i = 2\pi f(\phi_i^0) \frac{1}{2} \begin{bmatrix} 1 & 0 & \frac{B_i^0}{2} \\ 0 & 1 & -\frac{A_i^0}{2} \\ \frac{B_i^0}{2} & -\frac{A_i^0}{2} & \frac{(A_i^0)^2 + (B_i^0)^2}{3} \end{bmatrix} = 2\pi f(\phi_i^0) H_i^{(1)},$$

$$G_{ij}^{(11)} = 0, \quad i \neq j, \quad i, j = \overline{1, N}. \tag{73}$$

2) We will find next the limiting covariance matrix of the vector

$$\Xi_T^{(2)} = \left(\xi_{1T}^{(4)}, \xi_{1T}^{(5)}, \xi_{1T}^{(6)}, ..., \xi_{NT}^{(4)}, \xi_{NT}^{(5)}, \xi_{NT}^{(6)}\right)^*. \tag{74}$$

To do this, it is sufficient to analyze the limiting behavior of vectors $\Xi_{iT}^{(2)}$ and $\Xi_{jT}^{(2)}$, $i, j = \overline{1, N}$, covariance matrices. The problem is that the function

$$m(t) = \sum_{j=1}^{N} \left(C_j^0 \cos(\psi_j^0 t^2) + D_j^0 \sin(\psi_j^0 t^2)\right), \tag{75}$$

unlike the function (65), does not have a spectral measure, and we are forced to carry out rather tedious calculations.

It is convenient for us to write down $-C_j^0 \sin(\psi_j^0 t^2) + D_j^0 \cos(\psi_j^0 t^2) = \alpha_j^0 \cos(\psi_j^0 t^2 + \beta_j^0)$ with $\alpha_j^0 = \sqrt{(C_j^0)^2 + (D_j^0)^2}$, $\tan \beta_j^0 = \frac{C_j^0}{D_j^0}$, $j = \overline{1, N}$. We will also use the notation with variable $\lambda \geq 0$

$$u_{jT}(\lambda, \beta_j^0) = \begin{bmatrix} u_{jT}^{(11)}(\lambda, \beta_j^0) & u_{jT}^{(12)}(\lambda, \beta_j^0) \\ u_{jT}^{(21)}(\lambda, \beta_j^0) & u_{jT}^{(22)}(\lambda, \beta_j^0) \end{bmatrix}$$

$$= \begin{bmatrix} \int_0^T \cos(\lambda t) \cos(\psi_j^0 t^2 + \beta_j^0) dt & \int_0^T \cos(\lambda t) \sin(\psi_j^0 t^2 + \beta_j^0) dt \\ \int_0^T \sin(\lambda t) \cos(\psi_j^0 t^2 + \beta_j^0) dt & \int_0^T \sin(\lambda t) \sin(\psi_j^0 t^2 + \beta_j^0) dt \end{bmatrix}. \tag{76}$$

We get further

$$\begin{matrix} u_{jT}^{(11)}(\lambda, \beta_j^0) \\ u_{jT}^{(11)}(\lambda, \beta_j^0) \end{matrix} = \frac{1}{2} \int_0^T \cos\left(\psi_j^0 t^2 - \lambda t + \beta_j^0\right) dt \pm \frac{1}{2} \int_0^T \cos\left(\psi_j^0 t^2 + \lambda t + \beta_j^0\right) dt =$$

$$\frac{1}{2} \int_0^T \cos\left(\psi_j^0 \left(t - \frac{\lambda}{2\psi_j^0}\right)^2 - \left(\frac{\lambda^2}{4\psi_j^0} - \beta_j^0\right)\right) dt \pm \frac{1}{2} \int_0^T \cos\left(\psi_j^0 \left(t + \frac{\lambda}{2\psi_j^0}\right)^2 - \left(\frac{\lambda^2}{4\psi_j^0} - \beta_j^0\right)\right) dt$$

$$= \frac{1}{2\sqrt{\psi_j^0}} \int_{-\frac{\lambda}{2\sqrt{\psi_j^0}}}^{T\sqrt{\psi_j^0} - \frac{\lambda}{2\sqrt{\psi_j^0}}} \cos\left(s^2 - \left(\frac{\lambda^2}{4\psi_j^0} - \beta_j^0\right)\right) ds$$

$$\pm \frac{1}{2\sqrt{\psi_j^0}} \int\limits_{\frac{\lambda}{2\sqrt{\psi_j^0}}}^{T\sqrt{\psi_j^0}+\frac{\lambda}{2\sqrt{\psi_j^0}}} \cos\left(s^2 - \left(\frac{\lambda^2}{4\psi_j^0} - \beta_j^0\right)\right) ds$$

$$= \frac{1}{2\sqrt{\psi_j^0}} \left[\cos\left(\frac{\lambda^2}{4\psi_j^0} - \beta_j^0\right)\left(C\left(T\sqrt{\psi_j^0} - \frac{\lambda}{2\sqrt{\psi_j^0}}\right) - C\left(-\frac{\lambda}{2\sqrt{\psi_j^0}}\right)\right)\right.$$
$$+ \sin\left(\frac{\lambda^2}{4\psi_j^0} - \beta_j^0\right)\left(S\left(T\sqrt{\psi_j^0} - \frac{\lambda}{2\sqrt{\psi_j^0}}\right) - S\left(-\frac{\lambda}{2\sqrt{\psi_j^0}}\right)\right)$$
$$\pm \cos\left(\frac{\lambda^2}{4\psi_j^0} - \beta_j^0\right)\left(C\left(T\sqrt{\psi_j^0} + \frac{\lambda}{2\sqrt{\psi_j^0}}\right) - C\left(\frac{\lambda}{2\sqrt{\psi_j^0}}\right)\right)$$
$$\left.\pm \sin\left(\frac{\lambda^2}{4\psi_j^0} - \beta_j^0\right)\left(S\left(T\sqrt{\psi_j^0} + \frac{\lambda}{2\sqrt{\psi_j^0}}\right) - S\left(\frac{\lambda}{2\sqrt{\psi_j^0}}\right)\right)\right]; \quad (77)$$

$$\begin{matrix} u_{jT}^{(12)}(\lambda, \beta_j^0) \\ u_{jT}^{(21)}(\lambda, \beta_j^0) \end{matrix} = \frac{1}{2}\int\limits_0^T \sin\left(\psi_j^0 t^2 + \lambda t + \beta_j^0\right) dt \pm \frac{1}{2}\int\limits_0^T \sin\left(\psi_j^0 t^2 - \lambda t + \beta_j^0\right) dt =$$

$$\frac{1}{2}\int\limits_0^T \sin\left(\psi_j^0\left(t + \frac{\lambda}{2\psi_j^0}\right)^2 - \left(\frac{\lambda^2}{4\psi_j^0} - \beta_j^0\right)\right) dt \pm \frac{1}{2}\int\limits_0^T \sin\left(\psi_j^0\left(t - \frac{\lambda}{2\psi_j^0}\right)^2 - \left(\frac{\lambda^2}{4\psi_j^0} - \beta_j^0\right)\right) dt$$

$$= \frac{1}{2\sqrt{\psi_j^0}} \int\limits_{\frac{\lambda}{2\sqrt{\psi_j^0}}}^{T\sqrt{\psi_j^0}+\frac{\lambda}{2\sqrt{\psi_j^0}}} \sin\left(s^2 - \left(\frac{\lambda^2}{4\psi_j^0} - \beta_j^0\right)\right) ds$$

$$\pm \frac{1}{2\sqrt{\psi_j^0}} \int\limits_{-\frac{\lambda}{2\sqrt{\psi_j^0}}}^{T\sqrt{\psi_j^0}-\frac{\lambda}{2\sqrt{\psi_j^0}}} \sin\left(s^2 - \left(\frac{\lambda^2}{4\psi_j^0} - \beta_j^0\right)\right) ds$$

$$= \frac{1}{2\sqrt{\psi_j^0}} \left[\cos\left(\frac{\lambda^2}{4\psi_j^0} - \beta_j^0\right)\left(S\left(T\sqrt{\psi_j^0} + \frac{\lambda}{2\sqrt{\psi_j^0}}\right) - S\left(\frac{\lambda}{2\sqrt{\psi_j^0}}\right)\right)\right.$$
$$- \sin\left(\frac{\lambda^2}{4\psi_j^0} - \beta_j^0\right)\left(C\left(T\sqrt{\psi_j^0} + \frac{\lambda}{2\sqrt{\psi_j^0}}\right) - C\left(\frac{\lambda}{2\sqrt{\psi_j^0}}\right)\right)$$
$$\pm \cos\left(\frac{\lambda^2}{4\psi_j^0} - \beta_j^0\right)\left(S\left(T\sqrt{\psi_j^0} - \frac{\lambda}{2\sqrt{\psi_j^0}}\right) - S\left(-\frac{\lambda}{2\sqrt{\psi_j^0}}\right)\right)$$
$$\left.\mp \sin\left(\frac{\lambda^2}{4\psi_j^0} - \beta_j^0\right)\left(C\left(T\sqrt{\psi_j^0} - \frac{\lambda}{2\sqrt{\psi_j^0}}\right) - C\left(-\frac{\lambda}{2\sqrt{\psi_j^0}}\right)\right)\right], \quad (78)$$

where $C(x)$, $S(x)$, $x \in \mathbb{R}$, are Fresnel integrals (11).

Obviously,

$$\lim_{T\to\infty} u_{jT}(\lambda, \beta_j^0) = \begin{bmatrix} u_j^{(11)}(\lambda, \beta_j^0) & u_j^{(12)}(\lambda, \beta_j^0) \\ u_j^{(21)}(\lambda, \beta_j^0) & u_j^{(22)}(\lambda, \beta_j^0) \end{bmatrix},$$

$$\begin{aligned} &\begin{matrix} u_j^{(11)}(\lambda,\beta_j^0) \\ u_j^{(22)}(\lambda,\beta_j^0) \end{matrix} = \frac{1}{2\sqrt{\psi_j^0}}\left[\cos\left(\frac{\lambda^2}{4\psi_j^0}-\beta_j^0\right)\left(\sqrt{\frac{\pi}{8}}-C\left(-\frac{\lambda}{2\sqrt{\psi_j^0}}\right)\right)\right. \\ &+\sin\left(\frac{\lambda^2}{4\psi_j^0}-\beta_j^0\right)\left(\sqrt{\frac{\pi}{8}}-S\left(-\frac{\lambda}{2\sqrt{\psi_j^0}}\right)\right) \pm \cos\left(\frac{\lambda^2}{4\psi_j^0}-\beta_j^0\right)\left(\sqrt{\frac{\pi}{8}}-C\left(\frac{\lambda}{2\sqrt{\psi_j^0}}\right)\right) \\ &\left.\pm\sin\left(\frac{\lambda^2}{4\psi_j^0}-\beta_j^0\right)\left(\sqrt{\frac{\pi}{8}}-S\left(\frac{\lambda}{2\sqrt{\psi_j^0}}\right)\right)\right]; \end{aligned} \tag{79}$$

$$\begin{aligned} &\begin{matrix} u_j^{(12)}(\lambda,\beta_j^0) \\ u_j^{(21)}(\lambda,\beta_j^0) \end{matrix} = \frac{1}{2\sqrt{\psi_j^0}}\left[\cos\left(\frac{\lambda^2}{4\psi_j^0}-\beta_j^0\right)\left(\sqrt{\frac{\pi}{8}}-S\left(\frac{\lambda}{2\sqrt{\psi_j^0}}\right)\right)\right. \\ &-\sin\left(\frac{\lambda^2}{4\psi_j^0}-\beta_j^0\right)\left(\sqrt{\frac{\pi}{8}}-C\left(\frac{\lambda}{2\sqrt{\psi_j^0}}\right)\right) \pm \cos\left(\frac{\lambda^2}{4\psi_j^0}-\beta_j^0\right)\left(\sqrt{\frac{\pi}{8}}-S\left(-\frac{\lambda}{2\sqrt{\psi_j^0}}\right)\right) \\ &\left.\mp\sin\left(\frac{\lambda^2}{4\psi_j^0}-\beta_j^0\right)\left(\sqrt{\frac{\pi}{8}}-C\left(-\frac{\lambda}{2\sqrt{\psi_j^0}}\right)\right)\right]. \end{aligned} \tag{80}$$

From (77), (78) and Fresnel integral properties it follows that uniformly in T and λ

$$\left|u_{jT}^{(kl)}(\lambda,\beta_j^0)\right| \le \frac{4}{\sqrt{\psi_j^0}}, \quad k,l=1,2. \tag{81}$$

Besides for any $\Lambda > 0$ and $k,l=1,2$

$$\begin{aligned} &2\sqrt{\psi^0}\sup_{\lambda\in[0,\Lambda]}\left|u_{jT}^{(kl)}(\lambda,\beta_j^0)-u_j^{(kl)}(\lambda,\beta_j^0)\right| \le \sup_{\lambda\in[0,\Lambda]}\left|C\left(T\sqrt{\psi_j^0}+\frac{\lambda}{2\sqrt{\psi_j^0}}\right)-\sqrt{\frac{\pi}{8}}\right| \\ &+\sup_{\lambda\in[0,\Lambda]}\left|C\left(T\sqrt{\psi_j^0}-\frac{\lambda}{2\sqrt{\psi_j^0}}\right)-\sqrt{\frac{\pi}{8}}\right| + \sup_{\lambda\in[0,\Lambda]}\left|S\left(T\sqrt{\psi_j^0}+\frac{\lambda}{2\sqrt{\psi_j^0}}\right)-\sqrt{\frac{\pi}{8}}\right| \\ &+\sup_{\lambda\in[0,\Lambda]}\left|S\left(T\sqrt{\psi_j^0}-\frac{\lambda}{2\sqrt{\psi_j^0}}\right)-\sqrt{\frac{\pi}{8}}\right| \to 0, \text{ as } T\to\infty. \end{aligned} \tag{82}$$

Expressions (79), (80) can be written in a more compact form, bringing similar terms and using the oddness of functions $C(x)$ and $S(x)$:

$$\begin{aligned} &u_j^{(11)}(\lambda,\beta_j^0) = \sqrt{\frac{\pi}{4\psi_j^0}}\cos\left(\frac{\lambda^2}{4\psi_j^0}-\beta_j^0-\frac{\pi}{4}\right); \quad u_j^{(12)}(\lambda,\beta_j^0) = -\sqrt{\frac{\pi}{4\psi_j^0}}\sin\left(\frac{\lambda^2}{4\psi_j^0}-\beta_j^0-\frac{\pi}{4}\right); \\ &u_j^{(21)}(\lambda,\beta_j^0) = -\frac{1}{\sqrt{\psi_j^0}}\left(\cos\left(\frac{\lambda^2}{4\psi_j^0}-\beta_j^0\right)S\left(\frac{\lambda}{2\sqrt{\psi_j^0}}\right)-\sin\left(\frac{\lambda^2}{4\psi_j^0}-\beta_j^0\right)C\left(\frac{\lambda}{2\sqrt{\psi_j^0}}\right)\right); \\ &u_j^{(22)}(\lambda,\beta_j^0) = \frac{1}{\sqrt{\psi_j^0}}\left(\cos\left(\frac{\lambda^2}{4\psi_j^0}-\beta_j^0\right)C\left(\frac{\lambda}{2\sqrt{\psi_j^0}}\right)+\sin\left(\frac{\lambda^2}{4\psi_j^0}-\beta_j^0\right)S\left(\frac{\lambda}{2\sqrt{\psi_j^0}}\right)\right). \end{aligned} \tag{83}$$

We also need integrals $\int\limits_0^T t \begin{matrix}\sin\\ \cos\end{matrix}(\lambda t)\begin{matrix}\sin\\ \cos\end{matrix}(\psi_j^0t^2+\beta_j^0)dt$ and $\int\limits_0^T t^2 \begin{matrix}\sin\\ \cos\end{matrix}(\lambda t)\cos(\psi_j^0t^2+\beta_j^0)dt$. The next equalities follow from Appendix in Ivanov and Hladun (2024):

$$1)\int\limits_0^T t\cos(\lambda t)\cos(\psi_j^0t^2+\beta_j^0)dt = \frac{1}{2\psi_j^0}\left[\sin(\psi_j^0T^2+\beta_j^0)\cos(\lambda T)-\sin(\beta_j^0)+\lambda u_{jT}^{(22)}(\lambda,\beta_j^0)\right];$$

$$2) \int_0^T t\sin(\lambda t)\cos(\psi_j^0 t^2 + \beta_j^0)dt = \frac{1}{2\psi_j^0}\left[\sin(\psi_j^0 T^2 + \beta_j^0)\sin(\lambda T) - \lambda u_{jT}^{(12)}(\lambda, \beta_j^0)\right];$$

$$3) \int_0^T t\cos(\lambda t)\sin(\psi_j^0 t^2 + \beta_j^0)dt = \frac{1}{2\psi_j^0}\left[-\cos(\psi_j^0 T^2 + \beta_j^0)\cos(\lambda T) + \cos(\beta_j^0) - \lambda u_{jT}^{(21)}(\lambda, \beta_j^0)\right];$$

$$4) \int_0^T t\sin(\lambda t)\sin(\psi_j^0 t^2 + \beta_j^0)dt = \frac{1}{2\psi_j^0}\left[-\cos(\psi_j^0 T^2 + \beta_j^0)\sin(\lambda T) + \lambda u_{jT}^{(11)}(\lambda, \beta_j^0)\right];$$

$$5) \int_0^T t^2\cos(\lambda t)\cos(\psi_j^0 t^2 + \beta_j^0)dt = \frac{1}{2\psi_j^0}\Bigg[T\sin(\psi_j^0 T^2 + \beta_j^0)\cos(\lambda T) - u_{jT}^{(12)}(\lambda, \beta_j^0)$$
$$+\lambda\int_0^T t\sin(\lambda t)\sin(\psi_j^0 t^2 + \beta_j^0)dt\Bigg];$$

$$6) \int_0^T t^2\sin(\lambda t)\cos(\psi_j^0 t^2 + \beta_j^0)dt = \frac{1}{2\psi_j^0}\Bigg[T\sin(\psi_j^0 T^2 + \beta_j^0)\sin(\lambda T) - u_{jT}^{(22)}(\lambda, \beta_j^0)$$
$$-\lambda\int_0^T t\cos(\lambda t)\sin(\psi_j^0 t^2 + \beta_j^0)dt\Bigg]. \quad (84)$$

Note also that integrals 1)-4) are bounded in T by the value

$$\frac{1}{2\psi_j^0}\left(2 + \lambda\frac{4}{\sqrt{\psi_j^0}}\right) = \frac{1}{\psi_j^0} + \frac{\lambda}{(\psi_j^0)^{3/2}}. \quad (85)$$

Set $G_{ij}^{(22)}(T) = E\Xi_{iT}^{(2)}\left(\Xi_{jT}^{(2)}\right)^*$. Taking into account condition **A2** we'll use the standard formula $B(t) = 2\int_0^\infty f(\lambda)\cos(\lambda t)d\lambda$. Then by Lebesgue dominated convergence theorem

$$G_{ij,11}^{(22)}(T) = 2\int_0^\infty f(\lambda)\int_0^T\int_0^T \cos(\lambda(t-s))\cos(\psi_i^0 t^2)\cos(\psi_j^0 s^2)dtdsd\lambda$$
$$= 2\int_0^\infty f(\lambda)\left[u_{iT}^{(11)}(\lambda)u_{jT}^{(11)}(\lambda) + u_{iT}^{(21)}(\lambda)u_{jT}^{(21)}(\lambda)\right]d\lambda$$
$$\to 2\int_0^\infty f(\lambda)\left[u_i^{(11)}(\lambda)u_j^{(11)}(\lambda) + u_i^{(21)}(\lambda)u_j^{(21)}(\lambda)\right]d\lambda = G_{ij,11}^{(22)}, \text{ as } T\to\infty. \quad (86)$$

Here and bellow we use the notation $u_i^{(kl)}(\lambda, 0) = u_i^{(kl)}(\lambda)$, $u_j^{(kl)}(\lambda, 0) = u_j^{(kl)}(\lambda)$, $k, l = 1, 2$. Similarly to (86)

$$G_{ij,12}^{(22)} = 2\int_0^\infty f(\lambda)\left[u_i^{(11)}(\lambda)u_j^{(12)}(\lambda) + u_i^{(21)}(\lambda)u_j^{(22)}(\lambda)\right]d\lambda;$$
$$G_{ij,21}^{(22)} = 2\int_0^\infty f(\lambda)\left[u_j^{(11)}(\lambda)u_i^{(12)}(\lambda) + u_j^{(21)}(\lambda)u_i^{(22)}(\lambda)\right]d\lambda;$$
$$G_{ij,22}^{(22)} = 2\int_0^\infty f(\lambda)\left[u_i^{(12)}(\lambda)u_j^{(12)}(\lambda) + u_i^{(22)}(\lambda)u_j^{(22)}(\lambda)\right]d\lambda. \quad (87)$$

We have further

$$G^{(22)}_{ij,33}(T) = 2\alpha_i^0\alpha_j^0 T^{-2}\int_0^\infty f(\lambda)\int_0^T\int_0^T \cos(\lambda(t-s))t^2 s^2\cos(\psi_i^0 t^2+\beta_i^0)\cos(\psi_j^0 s^2+\beta_j^0)dtdsd\lambda$$

$$= 2\alpha_i^0\alpha_j^0 T^{-2}\int_0^\infty f(\lambda)\left[\int_0^T t^2\cos(\lambda t)\cos(\psi_i^0 t^2+\beta_i^0)dt\int_0^T t^2\cos(\lambda t)\cos(\psi_j^0 t^2+\beta_j^0)dt\right.$$

$$\left.+\int_0^T t^2\sin(\lambda t)\cos(\psi_i^0 t^2+\beta_i^0)dt\int_0^T t^2\sin(\lambda t)\cos(\psi_j^0 t^2+\beta_j^0)dt\right]d\lambda = \frac{\alpha_i^0\alpha_j^0}{2\psi_i^0\psi_j^0}T^{-2}\int_0^\infty f(\lambda)$$

$$\times\left[\left(T\sin(\psi_i^0T^2+\beta_i^0)\cos(\lambda T) - u^{(12)}_{iT}(\lambda,\beta_i^0)+\lambda\int_0^T t\sin(\lambda t)\sin(\psi_i^0t^2+\beta_i^0)dt\right)\right.$$

$$\times\left(T\sin(\psi_j^0T^2+\beta_j^0)\cos(\lambda T) - u^{(12)}_{jT}(\lambda,\beta_j^0)+\lambda\int_0^T t\sin(\lambda t)\sin(\psi_j^0t^2+\beta_j^0)dt\right)$$

$$+\left(T\sin(\psi_i^0T^2+\beta_i^0)\sin(\lambda T) - u^{(22)}_{iT}(\lambda,\beta_i^0)-\lambda\int_0^T t\cos(\lambda t)\sin(\psi_i^0t^2+\beta_i^0)dt\right)$$

$$\left.\times\left(T\sin(\psi_j^0T^2+\beta_j^0)\sin(\lambda T) - u^{(22)}_{jT}(\lambda,\beta_j^0)-\lambda\int_0^T t\cos(\lambda t)\sin(\psi_j^0t^2+\beta_j^0)dt\right)\right]d\lambda$$

$$= \frac{\alpha_i^0\alpha_j^0}{4\psi_i^0\psi_j^0}\sin(\psi_i^0T^2+\beta_i^0)\sin(\psi_j^0T^2+\beta_j^0)+O(T^{-1}),\ \text{as } T\to\infty. \tag{88}$$

Taking into account formulas (84) we see it is precisely the derivation of formula (88) that uses the existence of the 4th spectral moment stipulated by condition **A2**.

On the other hand,

$$G^{(22)}_{ij,13}(T) = 2\alpha_i^0 T^{-1}\int_0^\infty f(\lambda)\int_0^T\int_0^T \cos(\lambda(t-s))t^2\cos(\psi_i^0 t^2+\beta_i^0)\cos(\psi_j^0 s^2)dtdsd\lambda$$

$$= 2\alpha_i^0 T^{-1}\int_0^\infty f(\lambda)\left[\int_0^T t^2\cos(\lambda t)\cos(\psi_i^0 t^2+\beta_i^0)dt\int_0^T \cos(\lambda t)\cos(\psi_j^0 t^2)dt\right.$$

$$\left.+\int_0^T t^2\sin(\lambda t)\cos(\psi_i^0 t^2+\beta_i^0)dt\int_0^T \sin(\lambda t)\cos(\psi_j^0 t^2)dt\right]d\lambda = \frac{\alpha_i^0}{\psi_i^0}T^{-1}\int_0^\infty f(\lambda)$$

$$\times\left[\left(T\sin(\psi_i^0T^2+\beta_i^0)\cos(\lambda T) - u^{(12)}_{iT}(\lambda,\beta_i^0)+\lambda\int_0^T t\sin(\lambda t)\sin(\psi_i^0t^2+\beta_i^0)dt\right)u^{(11)}_{jT}(\lambda)\right.$$

$$\left.+\left(T\sin(\psi_i^0T^2+\beta_i^0)\sin(\lambda T) - u^{(22)}_{iT}(\lambda,\beta_i^0)-\lambda\int_0^T t\cos(\lambda t)\sin(\psi_i^0t^2+\beta_i^0)dt\right)u^{(21)}_{jT}(\lambda)\right]d\lambda$$

$$= \frac{\alpha_i^0}{\psi_i^0}\sin(\psi_i^0T^2+\beta_i^0)\int_0^\infty f(\lambda)\left[u^{(11)}_{jT}(\lambda)\cos(\lambda T)+u^{(21)}_{jT}(\lambda)\sin(\lambda T)\right]d\lambda + O(T^{-1})\to 0, \tag{89}$$

as $T\to\infty$.

Similarly,

$$G^{(22)}_{ij,23} = G^{(22)}_{ij,31} = G^{(22)}_{ij,32} = 0. \tag{90}$$

Thus,

$$\lim_{T\to\infty}\left[G_{ij}^{(22)}(T) - diag\left(0, 0, \frac{\alpha_i^0\alpha_j^0}{4\psi_i^0\psi_j^0}\sin(\psi_i^0T^2+\beta_i^0)\sin(\psi_j^0T^2+\beta_j^0)\right)\right] = \begin{bmatrix} G_{ij,11}^{(22)} & G_{ij,12}^{(22)} & 0 \\ G_{ij,21}^{(22)} & G_{ij,22}^{(22)} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (91)$$

with functions $G_{ij,kl}^{(22)}$, $k, l = 1, 2$, given by formulas (86) and (87).

3) Finally we examine the limits of vectors $\Xi_{iT}^{(1)}$ and $\Xi_{jT}^{(2)}$, $i, j = \overline{1, N}$, covariance matrices. Write $G_{ij}^{(12)}(T) = E\Xi_{iT}^{(1)}\left(\Xi_{jT}^{(2)}\right)^*$, and show that the integrals

$$\begin{aligned}
G_{ij,11}^{(12)}(T) &= T^{-1/2}\int_0^T\int_0^T B(t-s)\cos(\phi_i^0 t)\cos(\psi_j^0 s^2)dtds;\\
G_{ij,12}^{(12)}(T) &= T^{-1/2}\int_0^T\int_0^T B(t-s)\cos(\phi_i^0 t)\sin(\psi_j^0 s^2)dtds;\\
G_{ij,13}^{(12)}(T) &= \alpha_j^0 T^{-3/2}\int_0^T\int_0^T B(t-s)\cos(\phi_i^0 t)s^2\cos(\psi_j^0 s^2+\beta_j^0)dtds;\\
G_{ij,21}^{(12)}(T) &= T^{-1/2}\int_0^T\int_0^T B(t-s)\sin(\phi_i^0 t)\cos(\psi_j^0 s^2)dtds;\\
G_{ij,22}^{(12)}(T) &= T^{-1/2}\int_0^T\int_0^T B(t-s)\sin(\phi_i^0 t)\sin(\psi_j^0 s^2)dtds;\\
G_{ij,23}^{(12)}(T) &= \alpha_j^0 T^{-3/2}\int_0^T\int_0^T B(t-s)\sin(\phi_i^0 t)s^2\cos(\psi_j^0 s^2+\beta_j^0)dtds;\\
G_{ij,31}^{(12)}(T) &= T^{-3/2}\int_0^T\int_0^T B(t-s)t\left(-A_i^0\sin(\phi_i^0 t)+B_i^0\cos(\phi_i^0 t)\right)\cos(\psi_j^0 s^2)dtds;\\
G_{ij,32}^{(12)}(T) &= T^{-3/2}\int_0^T\int_0^T B(t-s)t\left(-A_i^0\sin(\phi_i^0 t)+B_i^0\cos(\phi_i^0 t)\right)\sin(\psi_j^0 s^2)dtds; \quad (92)\\
G_{ij,33}^{(12)}(T) &= \alpha_j^0 T^{-5/2}\int_0^T\int_0^T B(t-s)t\left(-A_i^0\sin(\phi_i^0 t)+B_i^0\cos(\phi_i^0 t)\right)s^2\cos(\psi_j^0 s^2+\beta_j^0)dtds,
\end{aligned}$$

tend to zero, as $T\to\infty$.

Let $F(t, s)$ be some continuous function. Consider the change of variables in the integral

$$I = \int_0^T\int_0^T B(t-s)F(t,s)dtds = \int_0^T B(u)\left[\int_0^{T-u} F(u+v, v)dv\right]du + \int_0^T B(u)\left[\int_0^{T-u} F(v, u+v)dv\right]du = I_1 + I_2,$$

which generalizes the standard change of variables in the integral I for $F(t, s) \equiv 1$. Make also in the inner integral I_2 the change of variables $u+v \to v$. Then we get

$$I_2 = \int_0^T B(u)\left[\int_u^T F(v-u,v)dv\right] du \text{ and}$$

$$I = \int_0^T B(u)\left[\int_0^T F(v-u,v)dv - \int_0^u F(v-u,v)dv + \int_0^{T-u} F(v+u,v)dv\right] du. \quad (93)$$

Note that according to formulas (76)-(85) we obtain

$$\left|\int_0^{(T,u,T-u)} \begin{matrix}\sin\\ \cos\end{matrix}(\phi_i^0 v) \begin{matrix}\sin\\ \cos\end{matrix}(\psi_j^0 v^2 + \beta_j^0)dv\right| \le \frac{4}{\sqrt{\psi_j^0}};$$

$$\left|\int_0^{(T,u,T-u)} v \begin{matrix}\sin\\ \cos\end{matrix}(\phi_i^0 v) \begin{matrix}\sin\\ \cos\end{matrix}(\psi_j^0 v^2 + \beta_j^0)dv\right| \le \frac{1}{\psi_j^0} + \frac{2\phi_i^0}{(\psi_j^0)^{3/2}};$$

$$\left|\int_0^{(T,u,T-u)} v^2 \begin{matrix}\sin\\ \cos\end{matrix}(\phi_i^0 v) \begin{matrix}\sin\\ \cos\end{matrix}(\psi_j^0 v^2 + \beta_j^0)dv\right| \le \frac{T}{\psi_j^0} + \frac{2}{(\psi_j^0)^{3/2}} + \frac{\phi_i^0}{2(\psi_j^0)^2} + \frac{(\phi_i^0)^2}{2(\psi_j^0)^{5/2}};$$

$$\left|\int_0^{(T,u,T-u)} v^3 \begin{matrix}\sin\\ \cos\end{matrix}(\phi_i^0 v) \begin{matrix}\sin\\ \cos\end{matrix}(\psi_j^0 v^2 + \beta_j^0)dv\right|$$
$$\le \frac{2T^2}{\psi_j^0} + \frac{\phi_i^0 T}{2(\psi_j^0)^2} + \frac{1}{(\psi_j^0)^2} + \frac{3\phi_i^0}{(\psi_j^0)^{5/2}} + \frac{(\phi_i^0)^2}{4(\psi_j^0)^3} + \frac{(\phi_i^0)^3}{4(\psi_j^0)^{7/2}}. \quad (94)$$

Since the proofs that elements (92) converge to zero, as $T \to \infty$, are similar, we will focus only on the most inconvenient element $G_{ij,33}^{(12)}(T)$. Using (93) and (94), we get

$$\begin{aligned}
\left|G_{ij,33}^{(12)}(T)\right| &= \alpha_j^0 T^{-5/2}\left|\int_0^T B(u)\right.\\
&\times\left[\int_0^T (v-u)\left(-A_i^0\sin(\phi_i^0 v - \phi_i^0 u) + B_i^0\cos(\phi_i^0 v - \phi_i^0 u)\right) v^2\cos(\psi_j^0 v^2 + \beta_j^0)dv\right.\\
&- \int_0^u (v-u)\left(-A_i^0\sin(\phi_i^0 v - \phi_i^0 u) + B_i^0\cos(\phi_i^0 v - \phi_i^0 u)\right) v^2\cos(\psi_j^0 v^2 + \beta_j^0)dv\\
&\left.\left. + \int_0^{(T-u)} (v+u)\left(-A_i^0\sin(\phi_i^0 v + \phi_i^0 u) + B_i^0\cos(\phi_i^0 v + \phi_i^0 u)\right) v^2\cos(\psi_j^0 v^2 + \beta_j^0)dv\right] du\right|\\
&\le \alpha_j^0\left(|A_i^0| + |B_i^0|\right) T^{-5/2}\int_0^T |B(u)|\\
&\quad\times\left[\left|\int_0^T v^3\sin(\phi_i^0 v)\cos(\psi_j^0 v^2 + \beta_j^0)dv\right| + \left|\int_0^T v^3\cos(\phi_i^0 v)\cos(\psi_j^0 v^2 + \beta_j^0)dv\right|\right.\\
&\quad + u\left|\int_0^T v^2\sin(\phi_i^0 v)\cos(\psi_j^0 v^2 + \beta_j^0)dv\right| + u\left|\int_0^T v^2\cos(\phi_i^0 v)\cos(\psi_j^0 v^2 + \beta_j^0)dv\right|\\
&\quad + \left|\int_0^u v^3\sin(\phi_i^0 v)\cos(\psi_j^0 v^2 + \beta_j^0)dv\right| + \left|\int_0^u v^3\cos(\phi_i^0 v)\cos(\psi_j^0 v^2 + \beta_j^0)dv\right|
\end{aligned}$$

$$+ u\left|\int_0^u v^2\sin(\phi_i^0 v)\cos(\psi_j^0 v^2+\beta_j^0)dv\right| + u\left|\int_0^u v^2\cos(\phi_i^0 v)\cos(\psi_j^0 v^2+\beta_j^0)dv\right|$$

$$+\left|\int_0^{T-u} v^3\sin(\phi_i^0 v)\cos(\psi_j^0 v^2+\beta_j^0)dv\right| + \left|\int_0^{T-u} v^3\cos(\phi_i^0 v)\cos(\psi_j^0 v^2+\beta_j^0)dv\right|$$

$$+u\left|\int_0^{T-u} v^2\sin(\phi_i^0 v)\cos(\psi_j^0 v^2+\beta_j^0)dv\right| + u\left|\int_0^{T-u} v^2\cos(\phi_i^0 v)\cos(\psi_j^0 v^2+\beta_j^0)dv\right|\Bigg] du$$

$$\le 6\alpha_j^0\left(|A_i^0|+|B_i^0|\right)T^{-5/2}\int_0^T |B(u)|\left[\frac{2T^2}{\psi_j^0}+\frac{\phi_i^0 T}{2(\psi_j^0)^2}+\frac{1}{(\psi_j^0)^2}+\frac{3\phi_i^0}{(\psi_j^0)^{5/2}}+\frac{(\phi_i^0)^2}{4(\psi_j^0)^3}\right.$$

$$\left.+\frac{(\phi_i^0)^3}{4(\psi_j^0)^{7/2}}+u\left(\frac{T}{\psi_j^0}+\frac{2}{(\psi_j^0)^{3/2}}+\frac{\phi_i^0}{2(\psi_j^0)^2}+\frac{(\phi_i^0)^2}{2(\psi_j^0)^{5/2}}\right)\right] du$$

$$\le \frac{18\alpha_j^0\left(|A_i^0|+|B_i^0|\right)}{\psi_j^0}T^{-1/2}\int_0^T |B(u)|du + O(T^{-1/2}). \tag{95}$$

Under condition **A2(*i*)**

$$T^{-1/2}\int_0^T B(u)du = T^{1/2}\int_0^1 B(Tu)du \le \frac{T^{1/2}}{1-\alpha}B(T) = \frac{L(T)}{(1-\alpha)T^{\alpha-1/2}} \to 0, \text{ as } T\to\infty. \tag{96}$$

Under condition **A2(*ii*)**, obviously,

$$T^{-1/2}\int_0^T |B(u)|du \to 0, \text{ as } T\to\infty. \tag{97}$$

Thus, $G_{ij,33}^{(12)}(T) \to 0$, as $T\to\infty$. Similarly to (95), each element in (92) is bounded in modulus by $c_{ij}T^{-1/2}\int_0^T |B(u)|du + O(T^{-1/2})$, with some constant c_{ij}. So,

$$E\Xi_{iT}^{(1)}\left(\Xi_{jT}^{(2)}\right)^* \to 0, \text{ as } T\to\infty, \quad i,j=\overline{1,N}. \tag{98}$$

Lemma 2. *Under condition **A(ii)***

(i) the limiting, as $T\to\infty$, covariance matrix of the vector $\Xi_T^{(1)}$ is given by formulas (73);

(ii) the limiting, as $T\to\infty$, covariance matrix of the vector $\Xi_T^{(2)}$ can be described by relations (86), (87) and (91);

(iii) joint covariance matrix of the vectors $\Xi_T^{(1)}$ and $\Xi_T^{(2)}$ tends to zero matrix, as $T\to\infty$.

*Proof of **Theorem 2**.* Using results of Lemmas 1 and 2 we move on to LSE asymptotic normality proof of the parameters of our multiple chirp-like signal.

Consider the matrix H from Lemma 1. Elementary calculations show that

$$\det\left(H_k^{(1)}\right) = \frac{(A_k^0)^2+(B_k^0)^2}{96}, \quad \det\left(H_k^{(2)}\right) = \frac{(C_k^0)^2+(D_k^0)^2}{90},$$

and $H^{-1} = diag\left(H_1^{-1},...,H_N^{-1}\right)$ is block-diagonal matrix with blocks

$$H_k^{-1} = diag\left(\left(H_k^{(1)}\right)^{-1},\left(H_k^{(2)}\right)^{-1}\right)$$

$$\left(H_k^{(1)}\right)^{-1} = \frac{2}{(A_k^0)^2+(B_k^0)^2}\begin{bmatrix} (A_k^0)^2+4(B_k^0)^2 & -3A_k^0B_k^0 & -6B_k^0 \\ -3A_k^0B_k^0 & 4(A_k^0)^2+(B_k^0)^2 & 6A_k^0 \\ -6B_k^0 & 6A_k^0 & 12 \end{bmatrix},$$

$$\left(H_k^{(2)}\right)^{-1} = \frac{0.5}{(C_k^0)^2+(D_k^0)^2}\begin{bmatrix} 4(C_k^0)^2+9(D_k^0)^2 & -5C_k^0D_k^0 & -15D_k^0 \\ -5C_k^0D_k^0 & 9(C_k^0)^2+4(D_k^0)^2 & 15C_k^0 \\ -15D_k^0 & 15C_k^0 & 45 \end{bmatrix}, k=\overline{1,N}. \tag{99}$$

Set

$$K_T = (K_{ij})_{i,j=1}^{6N} = \left(\widetilde{d}_T^{-1}\left(\frac{1}{2}Q_T''(\overline{\theta})\right)\widetilde{d}_T^{-1}\right)^{-1} - H^{-1} \to 0 \text{ a.s., as } T\to\infty. \tag{100}$$

Consider also block-diagonal matrix $\tilde{c}_T$ with N blocks

$$c_T = diag\left(1,1,1,T^{1/2},T^{1/2},T^{1/2}\right). \tag{101}$$

Using notation (60), (61), (100), (101), rewrite (42) in the form

$$\widetilde{c}_T\widetilde{d}_T\left(\theta_T-\theta^0\right) = \widetilde{c}_T K_T \widetilde{s}_T^{-1}\Xi_T + \widetilde{c}_T H^{-1}\widetilde{s}_T^{-1}\Xi_T = V_T^{(1)} + V_T^{(2)}. \tag{102}$$

First of all we show that $V_T^{(1)} \xrightarrow{P} 0$, as $T\to\infty$. Each coordinate of vector $V_T^{(1)}$ is a linear combination of elements of the matrix $\widetilde{c}_T K_T \widetilde{s}_T^{-1}$ and coordinates of the vector Ξ_T. In accordance with formulas (73) and (91), the processes $\xi_{kT}^{(q)}$, $q=\overline{1,5}$, converge weakly, as $T\to\infty$, to Gaussian random variables with zero mean and variances $\pi f(\phi_k^0)$, $\pi f(\phi_k^0)$, $\frac{1}{3}\pi f(\phi_k^0)\left((A_k^0)^2+(B_k^0)^2\right)$, $G_{kk,11}^{(22)}$, $G_{kk,22}^{(22)}$. Denote these random variables by $\xi_k^{(q)}$, $q=\overline{1,5}$. Processes $T^{-1}\xi_{kT}^{(6)}$ converge to zero in mean square.

Matrix K_T elements tend to zero a.s., as $T\to\infty$. However, the matrix $\widetilde{c}_T K_T \widetilde{s}_T^{-1}$ contain elements

$$T^{1/2}K_{mn}, \quad m=\overline{6i-2,6i}, n=\overline{6j-5,6j-3}, i,j=\overline{1,N}, \tag{103}$$

whose convergence to zero requires additional explanation. Elements K_{mn} in (103) are cofactors of elements symmetrical to K_{mn} relatively to the main diagonal of the matrix $H_T = \widetilde{d}_T^{-1}\left(\frac{1}{2}Q_T''(\overline{\theta})\right)\widetilde{d}_T^{-1}$ divided by the $\det H_T$. Every such a cofactor is an algebraic sum of products of $6N-1$ elements of specified matrix. We state that each such product contains at least one of the elements described before formula (57) with the following property: it will converge to zero a.s., as $T\to\infty$, being multiplied by $T^{1/2}$.

Indeed, when we begin to calculate the specified cofactor, we remove one row from one diagonal 3×3 block and one column from another diagonal 3×3 block.

A product in cofactor with the maximum number of factors that do belong to 3×3 diagonal blocks will contain $6(N-1)+4=6N-2$ elements. The last factor necessarily be an element from a non-diagonal 3×3 block, which, when multiplied by $T^{1/2}$, converges to zero a.s., as $T\to\infty$.

Besides, $H_T\to H$, and then $\det(H_T)\to\det(H)$ a.s., as $T\to\infty$. Thus, elements $T^{1/2}K_{mn}\to 0$ a.s., as $T\to\infty$, $m=\overline{6i-2,6i}$, $n=\overline{6j-5,6j-3}$, $i,j=\overline{1,N}$, and $\widetilde{c}_T K_T \widetilde{s}_T^{-1}\to 0$ a.s., as $T\to\infty$.

The above considerations show that vector $V_T^{(1)}$ converges to zero at least in probability.

On the other hand, the vector $V_T^{(2)} = \left(V_{1T}^{(2)},...,V_{kT}^{(2)},...,V_{NT}^{(2)}\right)^*$ with

$$V_{kT}^{(2)} = \left(\left(H_k^{(1)}\right)^{-1}\left(\xi_{kT}^{(1)},\xi_{kT}^{(2)},\xi_{kT}^{(3)}\right)^*, \left(H_k^{(2)}\right)^{-1}\left(\xi_{kT}^{(4)},\xi_{kT}^{(5)},T^{-1}\xi_{kT}^{(6)}\right)^*\right)^*. \tag{104}$$

Thus, $\widetilde{c}_T\widetilde{d}_T\left(\theta_T-\theta^0\right)$ weakly converges to the Gaussian random vector

$$\begin{aligned} V_T^{(2)} &= \left(V_1^{(2)},...,V_k^{(2)},...,V_N^{(2)}\right)^*, \\ V_k^{(2)} &= \left(\left(H_k^{(1)}\right)^{-1}\left(\xi_k^{(1)},\xi_k^{(2)},\xi_k^{(3)}\right)^*, \left(H_k^{(2)}\right)^{-1}\left(\xi_k^{(4)},\xi_k^{(5)},0\right)^*\right)^*, \end{aligned} \tag{105}$$

with covariance matrices that are made of blocks 6×6

$$\Sigma = (\Sigma_{ij})_{i,j=1}^{N},$$

$$\Sigma_{ii} = \begin{bmatrix} 2\pi f(\phi_i^0) R_i^{(1)} & 0 \\ 0 & R_i^{(2)} E\left(\xi_i^{(4)}, \xi_i^{(5)}\right)^* \left(\xi_i^{(4)}, \xi_i^{(5)}\right) \left(R_i^{(2)}\right)^* \end{bmatrix};$$

$$\Sigma_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & R_i^{(2)} E\left(\xi_i^{(4)}, \xi_i^{(5)}\right)^* \left(\xi_j^{(4)}, \xi_j^{(5)}\right) \left(R_j^{(2)}\right)^* \end{bmatrix}, i \neq j, \tag{106}$$

zeros denote 3rd order zeros matrices;

$$R_i^{(1)} = \frac{2}{(A_i^0)^2 + (B_i^0)^2} \begin{bmatrix} (A_i^0)^2 + 4(B_i^0)^2 & -3A_i^0 B_i^0 & -6B_i^0 \\ -3A_i^0 B_i^0 & 4(A_i^0)^2 + (B_i^0)^2 & 6A_i^0 \\ -6B_i^0 & 6A_i^0 & 12 \end{bmatrix};$$

$$R_i^{(2)} = \frac{0.5}{(C_k^0)^2 + (D_k^0)^2} \begin{bmatrix} 4(C_i^0)^2 + 9(D_i^0)^2 & -5C_i^0 D_i^0 \\ -5C_i^0 D_i^0 & 9(C_i^0)^2 + 4(D_i^0)^2 \\ -15D_i^0 & 15C_i^0 \end{bmatrix}; \tag{107}$$

$$E\xi_i^{(4)}\xi_j^{(4)} = 2\int_0^\infty f(\lambda)\left[u_i^{(11)}(\lambda)u_j^{(11)}(\lambda) + u_i^{(21)}(\lambda)u_j^{(21)}(\lambda)\right] d\lambda;$$

$$E\xi_i^{(4)}\xi_j^{(5)} = 2\int_0^\infty f(\lambda)\left[u_i^{(11)}(\lambda)u_j^{(12)}(\lambda) + u_i^{(21)}(\lambda)u_j^{(22)}(\lambda)\right] d\lambda;$$

$$E\xi_i^{(4)}\xi_j^{(5)} = 2\int_0^\infty f(\lambda)\left[u_j^{(11)}(\lambda)u_i^{(12)}(\lambda) + u_j^{(21)}(\lambda)u_i^{(21)}(\lambda)\right] d\lambda;$$

$$E\xi_i^{(5)}\xi_j^{(5)} = 2\int_0^\infty f(\lambda)\left[u_i^{(12)}(\lambda)u_j^{(12)}(\lambda) + u_i^{(22)}(\lambda)u_j^{(22)}(\lambda)\right] d\lambda; \tag{108}$$

$$u_i^{(11)}(\lambda) = \sqrt{\frac{\pi}{4\psi_i^0}} \cos\left(\frac{\lambda^2}{4\psi_i^0} - \frac{\pi}{4}\right); \quad u_i^{(12)}(\lambda) = -\sqrt{\frac{\pi}{4\psi_i^0}} \sin\left(\frac{\lambda^2}{4\psi_i^0} - \frac{\pi}{4}\right);$$

$$u_i^{(21)}(\lambda) = -\frac{1}{\sqrt{\psi_i^0}}\left(\cos\left(\frac{\lambda^2}{4\psi_i^0}\right) S\left(\frac{\lambda}{2\sqrt{\psi_i^0}}\right) - \sin\left(\frac{\lambda^2}{4\psi_i^0}\right) C\left(\frac{\lambda}{2\sqrt{\psi_i^0}}\right)\right);$$

$$u_i^{(22)}(\lambda) = \frac{1}{\sqrt{\psi_i^0}}\left(\cos\left(\frac{\lambda^2}{4\psi_i^0}\right) C\left(\frac{\lambda}{2\sqrt{\psi_i^0}}\right) + \sin\left(\frac{\lambda^2}{4\psi_i^0}\right) S\left(\frac{\lambda}{2\sqrt{\psi_i^0}}\right)\right). \tag{109}$$

□

Corollary 1. *Under conditions **A(i)**, **A(ii)** and **B** for any* $\delta_1 \in \left(0, \frac{1}{2}\right)$, $\delta_2 \in (0,1)$ *random variables* $T^{1/2-\delta_1}(A_{iT} - A_i^0)$, $T^{1/2-\delta_1}(B_{iT} - B_i^0)$, $T^{3/2-\delta_1}(\phi_{iT} - \phi_i^0)$, $T^{1-\delta_2}(C_{iT} - C_i^0)$, $T^{1-\delta_2}(D_{iT} - D_i^0)$, $T^{3-\delta_2}(\psi_{iT} - \psi_i^0) \xrightarrow{P} 0$, *as* $T \to \infty$, $i = \overline{1, N}$.

Corollary 2. *If* $rank(\mathcal{R}) = 5N$ *(see bellow), then* $rank(\Sigma) = 5N$. *This means that limiting Gaussian distribution of normed LSE* θ_T *is singular.*

Proof. The matrix Σ from (106) can be written as the product of three block matrices as follows:

$$\Sigma = \begin{bmatrix} R_1 & 0 & 0 & \dots & 0 \\ 0 & R_2 & 0 & \dots & 0 \\ 0 & 0 & R_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & R_N \end{bmatrix} \times \begin{bmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} & \mathcal{R}_{13} & \dots & \mathcal{R}_{1N} \\ \mathcal{R}_{21} & \mathcal{R}_{22} & \mathcal{R}_{23} & \dots & \mathcal{R}_{2N} \\ \mathcal{R}_{31} & \mathcal{R}_{32} & \mathcal{R}_{33} & \dots & \mathcal{R}_{3N} \\ \dots & \dots & \dots & \dots & \dots \\ \mathcal{R}_{N1} & \mathcal{R}_{N2} & \mathcal{R}_{N3} & \dots & \mathcal{R}_{NN} \end{bmatrix}$$

$$\times \begin{bmatrix} R_1^* & 0 & 0 & \dots & 0 \\ 0 & R_2^* & 0 & \dots & 0 \\ 0 & 0 & R_3^* & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & R_N^* \end{bmatrix} = R\mathcal{R}R^*, \qquad (110)$$

where $R_i = diag\left(R_i^{(1)}, R_i^{(2)}\right)$, $i = \overline{1, N}$, are block-diagonal matrices with blocks from (107) and matrix R is of order $6N \times 5N$, R^* is of order $5N \times 6N$. Matrix $\mathcal{R}$ is covariance matrix of the random vector $\left(\xi_1^{(1)}, \xi_1^{(2)}, \xi_1^{(3)}, \xi_1^{(4)}, \xi_1^{(5)}, ..., \xi_N^{(1)}, \xi_N^{(2)}, \xi_N^{(3)}, \xi_N^{(4)}, \xi_N^{(5)}\right)^*$ and we assume that it is not singular. Let's find the *rank* of the matrix R.

Consider a square submatrix M of R consisting of all its rows that contain the first five rows of each matrix R_i, $i = \overline{1, N}$. Then

$$\det M = \prod_{i=1}^{N} \det R_i^{(1)} \det \left(\frac{0.5}{(C_i^0)^2 + (D_i^0)^2} \begin{bmatrix} 4(C_i^0)^2 + 9(D_i^0)^2 & -5C_i^0 D_i^0 \\ -5C_i^0 D_i^0 & 9(C_i^0)^2 + 4(D_i^0)^2 \end{bmatrix} \right)$$
$$= 864^N \prod_{i=1}^{N} \frac{1}{(A_i^0)^2 + (B_i^0)^2} > 0. \qquad (111)$$

Thus, $rank(R) = rank(R^*) = 5N$, $rank(R\mathcal{R}) = rank(R) = 5N$, and $rank(\Sigma) = rank(R\mathcal{R}R^*) = rank(R\mathcal{R}) = 5N$. □

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