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What Is the Spectral Theory of Random Fields?

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Abstract

We review the current state of the spectral theory of random functions of several variables created by Professor M. Ĭ. Yadrenko at the end of 1950s. It turns out that the spectral expansions of multi-dimensional homogeneous and isotropic random fields are governed by a pair of convex compacts and are especially simple when these compacts are simplexes. Our new result gives necessary and sufficient conditions for such a situation in terms of the group representation that defines the field.

Keywords: random field, spectral expansion.

1. Introduction

In the 1970s and 1980s, I had the honor to be first a graduate, then a doctoral student by Professor Mykhaĭlo Ĭosypovych Yadrenko, the very best teacher, the most benevolent and responsive person I have ever met in my life, and the creator of the spectral theory of random fields.

A working mathematician must know everything about something and something about everything. The theory developed by M. Ĭ. Yadrenko is a nice illustration to this thesis. Below, we will see numerous links connecting the spectral theory of random fields to various parts of science including Special Functions, Representation Theory, Invariant Theory, Convex Geometry, and Continuum Physics.

Let $(\Omega, \mathfrak{F}, \mathsf{P})$ be a probability space, and let $(S, \mathfrak{S})$ be a measurable state space. A stochastic process is a collection $\{ X(t) \colon t \in T \}$ of S-valued random variables $X(t) \colon \Omega \to S$ which are indexed by an index set T and must be measurable with respect to the σ-fields $\mathfrak{F}$ and $\mathfrak{S}$.

In many applications, the index set T is (a subset of) the real line $\mathbb{R}^1$, while the state space S is $\mathbb{R}^1$ equipped by the σ-field $\mathfrak{S} = \mathfrak{B}(\mathbb{R}^1)$ of Borel sets.

Starting from the 1920th, several applied physical papers about turbulence introduced stochastic processes, where the index set is *not* a subset of $\mathbb{R}^1$ and the state space is not necessarily equal to $\mathbb{R}^1$. We would like to mention Friedmann and Keller (1924); von Kármán (1937a,b), among others. This particular case of stochastic processes coined the name *random fields*.

The term "spectral theory of random fields" will be explained in Section 2. Here, we consider a motivating example borrowed from Malyarenko and Ostoja-Starzewski (2023).

Example 1. Let D be a domain in an affine Euclidean space $\mathbb{E}^d$ filled with a continuous medium. Let $\mathbf{X}\colon D \to U$ be a physical field (temperature, velocity of a fluid, stress tensor of a deformable body, ...) taking values in a real finite-dimensional linear space U with an inner product $(\cdot,\cdot)$. We observe that U can consist of scalars, vectors, tensors, etc. By this reason we do not use the term "vector space" and do not introduce coordinates in U at this stage.

Under some physical conditions, the movement of a gaseous or liquid medium may become turbulent. A spatially random material microstructure may appear in a deformable body. The medium becomes random, and the function $\mathbf{X}$ becomes a random field.

Choose a point $O \in D$ and call it the *origin*. The map $\mathbb{E}^d \to \mathbb{R}^d$, $A \mapsto A - O$, identifies $\mathbb{E}^d$ with $\mathbb{R}^d$. For simplicity, suppose that the random field $\{\, \mathbf{X}(\mathbf{x})\colon \mathbf{x} \in D \subset \mathbb{R}^d \,\}$ is the restriction to D of another random field with index set $\mathbb{R}^d$. We denote it by the same symbol $\mathbf{X}$.

The physical properties of the random medium do not depend on the choice of the origin. Mathematically, the random field $\mathbf{X}$ is *strictly homogeneous*, that is, for any positive integer n, for any n distinct points $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^d$, and for arbitrary $\mathbf{x} \in \mathbb{R}^d$, the U^d-valued random variables $(\mathbf{X}(\mathbf{x}_1), \dots, \mathbf{X}(\mathbf{x}_n))$ and $(\mathbf{X}(\mathbf{x}_1 + \mathbf{x}), \dots, \mathbf{X}(\mathbf{x}_n + \mathbf{x}))$ (the marginal distributions) are identical.

In what follows, we assume that the random field $\mathbf{X}$ is *second-order*, that is, $\mathsf{E}[\|\mathbf{X}(\mathbf{x})\|^2] < \infty$ for every $\mathbf{x} \in \mathbb{R}^d$. In that case, we have: the *mean value* $\mathbf{m} = \mathsf{E}[\mathbf{X}(\mathbf{x})]$ is independent of $\mathbf{x}$, and the *correlation tensor* (or correlation function when $S = \mathbb{C}^1$)

$$K(\mathbf{x} - \mathbf{y}) = \mathsf{E}[(\mathbf{X}(\mathbf{x}) - \mathbf{m}) \otimes (\mathbf{X}(\mathbf{y}) - \mathbf{m})]$$

is a $U \otimes U$-valued function of the difference $\mathbf{x} - \mathbf{y}$. We call such a field just *homogeneous*.

At the macroscopic scale, all the details of the microstructure are lost, all what remains is the *medium symmetry group*, say G, a closed subgroup of the group $\mathrm{O}(d)$ of orthogonal $d \times d$ matrices. For simplicity, we put $G = \mathrm{O}(d)$. For other possibilities, see Malyarenko and Ostoja-Starzewski (2019); Malyarenko, Ostoja-Starzewski, and Amiri-Hezaveh (2020) and the literature cited there.

The group G acts in $\mathbb{R}^d$ by matrix-vector multiplication, $\mathbf{x} \mapsto g\mathbf{x}$, and in U by an *orthogonal representation*, $\mathbf{X} \mapsto g \cdot \mathbf{X}$. That is: the identity matrix I acts trivially; $I \cdot \mathbf{X} = \mathbf{X}$; $g \cdot (h \cdot \mathbf{X}) = (gh) \cdot \mathbf{X}$; $g \cdot \mathbf{X}$ is a linear function of $\mathbf{X}$ and a continuous function of g and $\mathbf{X}$; $(g \cdot \mathbf{X}, g \cdot \mathbf{Y}) = (\mathbf{X}, \mathbf{Y})$, see Adams (1969). In particular, for the case of the temperature (resp., the velocity of a turbulent fluid, resp., the stress tensor of a deformable body) we have $U = \mathbb{R}^1$ and $g \cdot \mathbf{X} = \mathbf{X}$ (resp., $U = \mathbb{R}^d$ and $g \cdot \mathbf{X} = g\mathbf{X}$, resp., $U = \mathsf{S}^2(\mathbb{R}^d)$, the space of symmetric $d \times d$ matrices, and $g \cdot \mathbf{X} = g\mathbf{X}g^{-1}$), and so on.

Under the action of an element $g \in G$, the random field $\{\, \mathbf{X}(\mathbf{x})\colon \mathbf{x} \in \mathbb{R}^d \,\}$ becomes the field $\{\, g \cdot \mathbf{X}(g\mathbf{x})\colon \mathbf{x} \in \mathbb{R}^d \,\}$. We call $\mathbf{X}$ *strictly isotropic* if the marginal distributions of the two above fields are identical.

It is easy to see that the mean value of a strictly isotropic random field satisfies

$$\mathbf{m}(g\mathbf{x}) = g \cdot \mathbf{m}(\mathbf{x}), \tag{1}$$

while its correlation tensor satisfies

$$K(g\mathbf{x} - g\mathbf{y}) = g \cdot K(\mathbf{x} - \mathbf{y}), \tag{2}$$

where g acts on the tensor product $\mathbf{X} \otimes \mathbf{Y} \in U \otimes U$ by $g \cdot (\mathbf{X} \otimes \mathbf{Y}) = (g \cdot \mathbf{X}) \otimes (g \cdot \mathbf{Y})$. We call such a field just *isotropic*.

In particular, Section 2 explains the term *spectral expansion* of the correlation tensor of a random field and of the field itself. How to find spectral expansions of homogeneous and isotropic random fields? For the case of $U = \mathbb{R}^1$ and $g \cdot \mathbf{X} = \mathbf{X}$, this problem was solved by M. Ĭ. Yadrenko in his PhD thesis in 1961, see also the book Yadrenko (1983) and the paper

Buldygin, Kozachenko, and Leonenko (1992), where his contribution to the theory of random fields is described in details. For the case of a turbulent fluid, the problem was partially solved in Yaglom (1948). In Malyarenko (1985), the author solved the case, when G acts by a complex irreducible representation. For the fields described in Example 1, the solution has been obtained by the author in collaboration with M. Ostoja-Starzewski, see Malyarenko and Ostoja-Starzewski (2019) and the literature cited there. We give an outline of their solution in Section 3.

Concerning the spectral theory of random fields, we would like to mention the expository paper Hannan (1965a), reprinted in book form in Hannan (1965b), devoted to the algebraic approach to the above theory, as well as Jones (1963) about the spectral expansions of isotropic random fields on sphere and Ogura (1990) about chaos expansion methods for such fields.

In particular, we will see that the structure of the expansion depends on the two finite-dimensional convex compact sets which we denote by $\mathcal{C}_0$ and $\mathcal{C}_1$. The above structure is especially simple if $\mathcal{C}_1$ is a simplex. How to describe this condition in terms of the representation U?

Recall that a linear subspace U_1 of U is called *invariant* if $g \cdot \mathbf{x} \in U_1$ for all $\mathbf{x} \in U_1$. A representation U is *irreducible* if $\{\mathbf{0}\}$ and U are the two and only invariant subspaces. Let U^i runs over inequivalent irreducible representations of a compact group G, as i runs over a set $\mathcal{I}$ of indices. For every finite-dimensional representation U, there are uniquely defined nonnegative integers $\{\, m_i \colon i \in \mathcal{I}\,\}$ of which all but finitely many are zeroes such that U is the direct sum over $\mathcal{I}$ of direct sums $m_i U^i$ of m_i copies of U^i. We say that the representation U has a *simple spectrum* if $m_i \leq 1$ for all $i \in I$.

Our new result is as follows.

Theorem 1. *The finite-dimensional convex compact set $\mathcal{C}_0$ (resp., $\mathcal{C}_1$) is a simplex if and only if U (resp., the restriction of U to the subgroup $\mathrm{O}(d-1)$) has a simple spectrum.*

2. What is the spectral theory of random fields?

Let V be a finite-dimensional complex linear space with inner product $(\cdot,\cdot)$, let $(\Lambda, \mathfrak{L})$ be a measurable space, and let $\boldsymbol{\zeta}$ be a measure on $(\Lambda, \mathfrak{L})$ with values in the Hilbert space $L^2(\Omega; V)$ of all V-valued random variables $\mathbf{X}$ with $\mathsf{E}[\|\mathbf{X}\|^2] < \infty$. Let j be a *real structure* on V (an analogue of the complex conjugation on $\mathbb{C}^1$). It is a conjugate-linear map $j \colon V \to V$, $j(\zeta\mathbf{v}) = \overline{\zeta} j(\mathbf{v})$ for all $\zeta \in \mathbb{C}$, satisfying in addition $j^2\mathbf{v} = \mathbf{v}$. We say that a measure Φ defined on $(\Lambda, \mathfrak{L})$ and taking values in the cone of Hermitian nonnegative-definite operators in V, is a *control measure* for $\boldsymbol{\zeta}$, if $\mathsf{E}[\boldsymbol{\zeta}(A) \otimes j(\boldsymbol{\zeta}(B))] = \Phi(A \cap B)$ for all A, $B \in \mathfrak{L}$.

The map $\mathfrak{L} \to [0,\infty)$, $A \mapsto \Phi_0(A) = \operatorname{tr}\Phi(A)$, is a finite measure on $(\Lambda, \mathfrak{L})$. For a coordinate-free definition of the trace of a linear operator, see (Adams 1969, Definition 3.29). It is possible to construct a unitary linear operator

$$L^2(\Lambda, \Phi_0) \to L^2(\Omega; V), \qquad f(\lambda) \mapsto \int_\Lambda f(\lambda)\,\mathrm{d}\boldsymbol{\zeta}(\lambda),$$

called the *stochastic integral with respect to a stochastic orthogonal measure*. For the details of the construction, see (Gikhman and Skorokhod 2004, Chapter 4, § 4).

For simplicity, assume that the stochastic orthogonal measure $\boldsymbol{\zeta}$ is centred: $\mathsf{E}[\boldsymbol{\zeta}(A)] = \mathbf{0}$ for all $A \in \mathfrak{L}$. Let $f \colon T \times \Lambda \to \mathbb{C}$ be such a map, that for every $t_0 \in T$, the function $f(t_0, \lambda)$ is square-integrable with respect to Φ_0. Consider the V-valued stochastic process $\mathbf{X}$ given by

$$\mathbf{X}(t) = \int_\Lambda f(t,\lambda)\,\mathrm{d}\boldsymbol{\zeta}(\lambda). \tag{3}$$

We say that Equation (3) is the *spectral expansion* of the stochastic process $\mathbf{X}$. A simple calculation shows that the correlation tensor of $\mathbf{X}$ is

$$K(s,t) = \int_\Lambda f(s,\lambda)\overline{f(t,\lambda)}\,\mathrm{d}\Phi(\lambda). \tag{4}$$

The converse statement holds true under a mild additional assumption.

Theorem 2 (Karhunen (1946))**.** *Assume that the set of finite linear combinations of the family* $\{ f(t_0,\lambda)\colon t_0 \in T \}$ *is dense in* $L^2(\Lambda,\Phi_0)$*. Then* (4) *implies* (3)*.*

Our answer to the question formulated in the title of this section is as follows. *The spectral theory of random fields is the art of representing either a random field in the form* (3) *or its correlation tensor in the form* (4)*.*

From now on, we assume that the index set is a topological space, and the stochastic process X is *mean-square continuous*, that is, the map $T \to L^2(\Omega;S)$, $t \mapsto X(t)$, is continuous.

For example, put $S=\mathbb{C}^1$, and let X be a centred second-order random field on an index set T. Pick n points $t_1, \dots, t_n \in T$, n arbitrary complex numbers $\zeta_1, \dots, \zeta_n$, and consider the random variable $Y=\zeta_1 X(t_1)+\cdots+\zeta_n X_n$. We have

$$\mathsf{E}[|Y|^2]=\sum_{i,j=1}^{n}\zeta_i\overline{\zeta_j}K(t_i,t_j)\geq 0,$$

which means that $K(s,t)$ is a complex-valued *positive-definite kernel* on T.

Conversely, if $K(s,t)$ is a positive-definite kernel on T, then the classical Kolmogorov Extension Theorem shows that the system of centred $\mathbb{C}^n$-valued normal random vectors

$$\{ \mathbf{X}=(X(t_1),\dots,X(t_n))^\top \}$$

with correlation matrices K satisfying $K_{ij}=B(t_i,t_j)$, determines marginal distributions of a centred Gaussian random field X with correlation function $K(s,t)$. Thus, there is a link between the spectral theory of random fields and the theory of positive-definite kernels.

The classical Herglotz Theorem of 1911, reprinted in Herglotz (1991), states that a positive-definite kernel $K(m,n)$ on the set $T=\mathbb{Z}$ of integers, that depends only on the difference $m-n$, has the form

$$K(m,n)=\int_{\mathbb{T}}\exp(\mathrm{i}(m-n)\lambda)\,\mathrm{d}\Phi(\lambda).$$

A couple of terms follow. The set $\mathbb{Z}$ is the *time domain*, or the *real space*. The torus $\mathbb{T}$ is the *frequency domain*, or the *Fourier space*. The integrand $\exp(\mathrm{i}(m-n)\lambda)$ is the *character of the group* $\mathbb{Z}$, or the *elementary positive-definite kernel*. The finite measure Φ defined on the Borel σ-field $\mathfrak{B}(\mathbb{T})$, "glues" the elementary positive-definite kernels indexed by $\lambda\in\mathbb{T}$ into the general one.

We observe that the kernel $K(m,n)$ has the form (4) with $\Lambda=\mathbb{T}$ and $f(m,\lambda)=\exp(\mathrm{i}m\lambda)$. If the set of linear combinations of the family $\{\exp(\mathrm{i}m\lambda)\colon m\in\mathbb{Z}\}$ is dense in the Hilbert space $L^2(\mathbb{T},\Phi)$, then the spectral expansion of a centred homogeneous random field on $\mathbb{Z}$ takes the form

$$X(m)=\int_{\mathbb{T}}\exp(\mathrm{i}m\lambda)\,\mathrm{d}\zeta(\lambda), \tag{5}$$

where ζ is a centred stochastic orthogonal measure on $\mathfrak{B}(\mathbb{T})$ with control measure Φ.

Remark 1. In numerous books and papers, Equation (5) is written as

$$X(m)=\int_0^{2\pi}\exp(\mathrm{i}m\lambda)\,\mathrm{d}\zeta(\lambda),$$

which is wrong.

Similarly, the spectral expansion of a centred mean-square continuous homogeneous random field on $\mathbb{R}^d$ has the form

$$X(\mathbf{x})=\int_{\hat{\mathbb{R}}^d}\exp(\mathrm{i}(\mathbf{p},\mathbf{x}))\,\mathrm{d}\zeta(\mathbf{p}).$$

This time, $\mathbb{R}^d$ is the *space domain*, $\hat{\mathbb{R}}^d$ is the *wave vector domain* or the *reciprocal space*. Instead of the Herglotz Theorem, we use another classical result due to Bochner (1932).

What about the case of a centred mean-square continuous complex-valued homogeneous and isotropic random field on the space domain? The correlation function of such a field must be a continuous positive-definite kernel $K(\mathbf{x},\mathbf{y})$ on $\mathbb{R}^d$ that depends only on the distance $\|\mathbf{x}-\mathbf{y}\|$. Another classical result by Bochner (1941) says that such a kernel is "glued" of the elementary kernels indexed by $\lambda \geq 0$ with a finite measure Φ defined on the Borel σ-field $\mathfrak{B}([0,\infty))$ as follows:

$$K(\|\mathbf{x}-\mathbf{y}\|) = 2^{(d-2)/2}\Gamma(d/2)\int_0^\infty \frac{J_{(n-2)/2}(\lambda\|\mathbf{x}-\mathbf{y}\|)}{(\lambda\|\mathbf{x}-\mathbf{y}\|)^{(d-2)/2}}\,\mathrm{d}\Phi(\lambda),$$

where Γ is the gamma function, and where $J_{(n-2)/2}$ is the Bessel function of the first kind of order $(n-2)/2$. We observe another link, this time to Special Functions. But how to write this equation in the form (3)?

A brilliant solution has been found by M. Ĭ. Yadrenko. The set Λ is the non-intersecting union of countably many (two, if $d=1$) copies $\Lambda_{\ell m}$ of the set $[0,\infty)$. The index ℓ runs over nonnegative integers ($\ell \in \{0,1\}$, if $d=1$), for the integer index m we have

$$1 \leq m \leq h(\ell,d) = (2\ell+d-2)\frac{(\ell+d-3)!}{(d-2)!\ell!}$$

with convention $h(0,1)=h(1,1)=h(0,2)=1$. The same finite measure Φ is defined on every σ-field $\mathfrak{B}(\Lambda_{\ell m})$. The function $f(\mathbf{x},\lambda_{\ell m})\colon \mathbb{R}^d \times \Lambda_{\ell m}) \to \mathbb{R}^1$ has the form

$$f(\mathbf{x},\lambda_{\ell m}) = \sqrt{2^{d-1}\Gamma(d/2)\pi^{d/2}}Y_\ell^m(\theta_1,\dots,\theta_{d-2},\varphi)\frac{J_{\ell+(d-2)/2}(\lambda_{\ell m}r)}{(\lambda_{\ell m}r)^{(d-2)/2}},$$

where $(r,\theta_1,\dots,\theta_{d-2},\varphi)$ are the spherical coordinates of the point $\mathbf{x}\in\mathbb{R}^d$, and we use the notation Y_ℓ^m for the real-valued spherical harmonics, borrowed from Olver, Lozier, Boisvert, and Clark (2010), the current *de facto* standard in the area of special functions. For the exceptional case of $d=1$, we have $S^0=\{-1,1\}$, $Y_0^1(\pm 1)=\frac{1}{\sqrt{2}}$, $Y_1^1(\pm 1)=\pm\frac{1}{\sqrt{2}}$, and the spectral expansion takes the form

$$X(t) = \int_0^\infty \cos(\lambda_{01}|t|)\,\mathrm{d}\zeta_{01}(\lambda_{01}) + \int_0^\infty \sin(\lambda_{11}|t|)\,\mathrm{d}\zeta_{11}(\lambda_{11}),$$

where ζ_{01} and ζ_{11} are centred uncorrelated stochastic orthogonal measures on $\mathfrak{B}([0,\infty))$. To avoid complications, we exclude the exceptional case of $d=1$ from consideration.

How this solution has been found? In the following section, we give an outline of solution for the more sophisticated case considered in Example 1. In particular, for the case of $d=3$ and when $\mathrm{O}(3)$ acts in $\mathbb{R}^3$ by matrix-vector multiplication, Robertson (1940), using the Invariant Theory, proved that there are two functions $L_1(\mathbf{u})$ and $L_2(\mathbf{u})$ such that

$$K_{ij}(\mathbf{u}) = L_1(\mathbf{u})\delta_{ij} + L_2(\mathbf{u})\frac{u_iu_j}{\|\mathbf{u}\|^2}.$$

Yaglom (1948) proved that these functions have the form

$$\begin{aligned}
K_{ij}(\mathbf{u}) = \int_0^\infty &\left[\left(-\frac{3\sin(\lambda\|\mathbf{u}\|)}{(\lambda\|\mathbf{u}\|)^3} + \frac{\sin(\lambda\|\mathbf{u}\|)}{\lambda\|\mathbf{u}\|} + \frac{3\cos(\lambda\|\mathbf{u}\|)}{(\lambda\|\mathbf{u}\|)^2}\right)\frac{u_iu_j}{\|\mathbf{u}\|^2}\right.\\
&+\left.\left(\frac{\sin(\lambda\|\mathbf{u}\|)}{(\lambda\|\mathbf{u}\|)^3} - \frac{\cos(\lambda\|\mathbf{u}\|)}{(\lambda\|\mathbf{u}\|)^2}\right)\delta_{ij}\right]\mathrm{d}\Phi_1(\lambda)\\
&+\int_0^\infty \left[\left(\frac{3\sin(\lambda\|\mathbf{u}\|)}{(\lambda\|\mathbf{u}\|)^3} - \frac{\sin(\lambda\|\mathbf{u}\|)}{\lambda\|\mathbf{u}\|} - \frac{3\cos(\lambda\|\mathbf{u}\|)}{(\lambda\|\mathbf{u}\|)^2}\right)\frac{u_iu_j}{\|\mathbf{u}\|^2}\right.\\
&+\left.\left(\frac{\sin(\lambda\|\mathbf{u}\|)}{\lambda\|\mathbf{u}\|} - \frac{\sin(\lambda\|\mathbf{u}\|)}{(\lambda\|\mathbf{u}\|)^3} + \frac{\cos(\lambda\|\mathbf{u}\|)}{(\lambda\|\mathbf{u}\|)^2}\right)\delta_{ij}\right]\mathrm{d}\Phi_2(\lambda),
\end{aligned} \tag{6}$$

where Φ_1 and Φ_2 are two finite measures on $\mathfrak{B}([0,\infty))$ satisfying the condition $\Phi_1(\{0\}) = \Phi_2(\{0\})$. How to write this equation in the form (3)?

Remark 2. Observe that the above equation is not unique. In both integrals, we may multiply the integrand by an arbitrary positive real number and divide the measure by the same number. In that case, the condition $\Phi_1(\{0\})=\Phi_2(\{0\})$ will be replaced by another one.

Finally, we briefly discuss an alternative approach to the calculations of spectral expansions. Assume that the space U is not one-dimensional but the group $\mathrm{O}(d)$ acts trivially: $g\cdot\mathbf{x}=\mathbf{x}$, $g\in\mathrm{O}(d)$, $\mathbf{x}\in U$. Let $\{\,\mathbf{e}_i\colon 1\leq i\leq\dim U\,\}$ be an orthonormal basis in U. We would like to construct a U-valued homogeneous and isotropic random field $\mathbf{X}(\mathbf{x})$ in such a way, that the one-dimensional random fields $X_i(\mathbf{x})=(\mathbf{X}(\mathbf{x}),\mathbf{e}_i)$ belong to a prescribed class: for example, the Matérn one with zero mean and correlation function

$$\mathsf{E}[X_i(\mathbf{x})X_i(\mathbf{y})]=\frac{2^{1-\nu}\sigma^2}{\Gamma(\nu)}(a\|\mathbf{x}-\mathbf{y}\|)^\nu K_\nu(a\|\mathbf{x}-\mathbf{y}\|),$$

where $\sigma^2>0$, $a>0$, $\nu>0$, and $K_\nu(z)$ is the Bessel function of the third kind of order ν. It is not straightforward to specify the off-diagonal correlation functions $K_{ij}(\mathbf{x},\mathbf{y})=\mathsf{E}[X_i(\mathbf{x})X_j(\mathbf{y})]$, $1\leq i<j\leq\dim U$, because the matrix $K_{ij}(\mathbf{x},\mathbf{y})$ must be positive-definite. See Leonenko and Malyarenko (2017) for different approaches to this problem.

3. A description of homogeneous and isotropic random fields

It follows easily from Equation (1), that the mean value $\mathbf{m}$ of a homogeneous and isotropic random field is an arbitrary vector of the subspace $m_0U^0\subset U$, where U^0 denotes the one-dimensional trivial representation of $\mathrm{O}(d)$: $g\cdot\mathbf{x}=\mathbf{x}$ for all $g\in\mathrm{O}(d)$ and for all $\mathbf{x}\in\mathbb{R}^1$. In what follows, we consider only centred random fields with $\mathbf{m}=\mathbf{0}$.

To find all correlation tensors satisfying (2), we use the following strategy: to write down the general form of a correlation tensor of a homogeneous random field, and to find all of them that are isotropic.

The first part is easy. The tensor product $cU=\mathbb{C}^1\otimes_{\mathbb{R}}U$ is a complex linear space with the scalar-vector multiplication given by $\zeta_1(\zeta\otimes\mathbf{x})=(\zeta_1\zeta)\otimes\mathbf{x}$, ζ, $\zeta_1\in\mathbb{C}$. The space U can be embedded into cU by $\mathbf{x}\mapsto 1\otimes\mathbf{x}$. The inner product in cU is given by $(\zeta_1\otimes\mathbf{x},\zeta_2\otimes\mathbf{y})=\zeta_1\overline{\zeta_2}(\mathbf{x},\mathbf{y})$. The now cU-valued random field $\mathbf{X}$ is still homogeneous. By the result of Cramér (1940), the correlation tensor of that field has the form

$$K(\mathbf{x},\mathbf{y})=\int_{\hat{\mathbb{R}}^d}\exp(\mathrm{i}(\mathbf{p},\mathbf{x}-\mathbf{y}))\,\mathrm{d}F(\mathbf{p}),\tag{7}$$

where F is a measure on $\mathfrak{B}(\hat{\mathbb{R}}^d)$ taking values in the cone of Hermitian nonnegative-definite linear operators in cU.

The space cU has a natural real structure given by $j_{cU}(\zeta\otimes\mathbf{x})=\overline{\zeta}\otimes\mathbf{x}$. A cU-valued homogeneous random field $\mathbf{X}$ takes values in U if and only if the random fields $\mathbf{X}$ and $j_{cU}\mathbf{X}$ have the same finite-dimensional distributions. In particular, they have the same correlation tensor, that is

$$\mathsf{E}[\mathbf{X}(\mathbf{x})\otimes j_{cU}\mathbf{X}(\mathbf{y})]=\mathsf{E}[j_{cU}\mathbf{X}(\mathbf{x})\otimes\mathbf{X}(\mathbf{y})],\qquad\mathbf{x},\mathbf{y}\in\mathbb{R}^d.$$

The left hand side of this equation is equal to the right hand side of Equation (7), while the right hand side is given by

$$\mathsf{E}[j_{cU}\mathbf{X}(\mathbf{x})\otimes\mathbf{X}(\mathbf{y})]=\int_{\hat{\mathbb{R}}^d}\exp(-\mathrm{i}(\mathbf{p},\mathbf{x}-\mathbf{y}))\,\mathrm{d}(jF)(\mathbf{p}),\tag{8}$$

where j is the real structure on the complex linear space $\mathrm{Hom}_{\mathbb{C}}(cU,cU)$ of all $\mathbb{C}$-linear maps from cU to itself given by

$$jK=j_{cU}Kj^{-1}_{cU},\qquad K\in\mathrm{Hom}_{\mathbb{C}}(cU,cU).$$

The right hand sides of Equations (7) and (8) are equal, which happens if and only if

$$F(-IA) = (jF)(A), \qquad A \in \mathfrak{B}(\hat{\mathbb{R}}^d).$$

We proceed to the second part. Equation (2) shows that $\mathbf{X}$ is isotropic if and only if

$$F(gA) = g \cdot F(A), \qquad g \in \mathrm{O}(d), \quad A \in \mathfrak{B}(\hat{\mathbb{R}}^d).$$

In particular, the element $-I \in \mathrm{O}(d)$ always acts trivially in $U \otimes U$, so $F(-IA) = F(A) = (jF)(A)$. That is, F takes values in the cone of symmetric nonnegative-definite linear operators on U.

It is well-known that the measure F is absolutely continuous with respect to the positive finite measure $F_0 = \operatorname{tr} F$. Let $f(p)$ be the corresponding Radon–Nikodym density. Equation (7) takes the form

$$K(\mathbf{x}, \mathbf{y}) = \int_{\hat{\mathbb{R}}^d} \exp(\mathrm{i}(\mathbf{p}, \mathbf{x} - \mathbf{y})) f(\mathbf{p}) \, \mathrm{d}F_0(\mathbf{p}),$$

and this time, the measure F_0 is $\mathrm{O}(d)$-invariant, $F_0(gA) = F_0(A)$, while the measurable function f takes values in the convex compact set $\mathcal{C}$ of symmetric nonnegative-definite linear operators on U with unit trace and satisfies

$$f(g\mathbf{p}) = g \cdot f(\mathbf{p}), \qquad g \in \mathrm{O}(d), \quad \mathbf{p} \in \hat{\mathbb{R}}^d. \tag{9}$$

The measure F_0 is “glued” of $\mathrm{O}(d)$-invariant measures on the centred spheres of radius $\lambda = \|\mathbf{p}\| \in [0, \infty)$ with the help of an arbitrary finite measure Φ as follows:

$$K(\mathbf{x}, \mathbf{y}) = \int_0^\infty \int_{S^{d-1}} \exp(\mathrm{i}(\mathbf{p}, \mathbf{x} - \mathbf{y})) f(\mathbf{p}) \, \mathrm{d}\Omega \, \mathrm{d}\Phi(\lambda),$$

where Ω is the Lebesgue measure on the centred unit sphere $S^{d-1} \subset \hat{\mathbb{R}}^d$.

The space $U \otimes U$ is the direct sum of subspaces $m_l U^l$. In particular, $m_0 > 0$, because the space $m_0 U^0$ includes at least the linear subspace generated by the identity operator in U. In Equation (9), put $\mathbf{p} = \mathbf{0}$. We obtain: $f(\mathbf{0})$ is an arbitrary element of the convex compact set $\mathcal{C}_0 = \mathcal{C} \cap (m_0 U^0)$.

Similarly, put $\mathbf{p} = (0, 0, \dots, 0, \lambda)^\top$ with $\lambda > 0$. The set $\{\, g \in \mathrm{O}(d) \colon g\mathbf{p} = \mathbf{p} \,\}$ for each of these points is the group $\mathrm{O}(d-1)$ embedded into $\mathrm{O}(d)$ as $\begin{pmatrix} \mathrm{O}(d-1) & \mathbf{0}^\top \\ \mathbf{0} & 1 \end{pmatrix}$. The restriction of the representation $U \otimes U$ to $\mathrm{O}(d-1)$ is the direct sum of subspaces $n_i U^i$. Obviously, $n_0 \geq m_0$. The function $f(0, 0, \dots, 0, \lambda)$ is an arbitrary measurable map from $(0, \infty)$ to the convex compact set $\mathcal{C}_1 = \mathcal{C} \cap (n_0 U^0)$.

It's time to introduce coordinates. The space U is the direct sum of subspaces $p_i U^i$. In each subspace, we choose a basis $\{\mathbf{e}_{ijk}\}$, where the index i runs over all inequivalent irreducible representations U^i of $\mathrm{O}(d)$ with $p_i > 0$, j runs from 1 to p_i, k runs from 1 to $\dim U^i$. We assume that in the chosen basis, the matrix entries of the matrices of the representations U^i are known.

The space $U \otimes U$ has two natural bases. The first one consists of the matrices $K_{ijki'j'k'} = \mathbf{e}_{ijk} \otimes \mathbf{e}_{i'j'k'}$. Our task is to calculate the corresponding matrix entries $K_{ijki'j'k'}(\mathbf{x}, \mathbf{y})$. The second one are the matrices K^{lqs}, where the index l runs over all inequivalent irreducible representations U^l of $\mathrm{O}(d)$ with $m_l > 0$, q runs from 1 to m_l, s runs from 1 to $\dim U^l$.

The matrix entries $K^{lqs}_{ijki'j'k'}$ are similar to the famous Clebsch–Gordan coefficients used in quantum mechanics. For their calculation in the case of $d = 2$, see (Malyarenko and Ostoja-Starzewski 2019, Section 3.2), the case of $d = 3$ is analysed in Homeier and Steinborn (1996) and several subsequent publications.

On the one hand, the matrix $f(0, 0, \dots, 0, \lambda)$ takes the form

$$f(0, 0, \dots, 0, \lambda) = \sum_{t=1}^{n_0} f_t(\lambda) K^{l_t q_t s_t},$$

where $f_t(0) = 0$ for $m_0 + 1 \le t \le n_0$. Here, the tth copy of the representation U^0 of the group $\mathrm{O}(d-1)$ is the irreducible component of the restriction of the q_tth copy of the irreducible representation U^{l_t} of $\mathrm{O}(d)$ to $\mathrm{O}(d-1)$ living in the linear span of the s_tth vector of the basis of U^{l_t}.

On the other hand, by the Carathéodory Theorem, for any $\lambda \in [0,\infty)$ there are at most $\dim \mathcal{C}_1 + 1$ extreme points of $\mathcal{C}_1$, the matrices

$$K^n(\lambda) = \sum_{t=1}^{n_0} c_{nt}(\lambda) K^{l_t q_t s_t}$$

such that the matrix $f(0,0,\dots,0,\lambda)$ is the convex combination of the above matrices:

$$f(0,0,\dots,0,\lambda) = \sum_{n=1}^{\dim \mathcal{C}_1+1} \sum_{t=1}^{n_0} u_n(\lambda) c_{nt}(\lambda) K^{l_t q_t s_t},$$

where $u_n(\lambda) \ge 0$ and $u_1(\lambda) + \dots + u_{\dim \mathcal{C}_1+1}(\lambda) = 1$. Moreover, the above representation is unique if and only if $\mathcal{C}_1$ is a simplex.

The action (9) yields

$$f_{ijki'j'k'}(\mathbf{p}) = \sum_{n=1}^{\dim \mathcal{C}_1+1} \sum_{t=1}^{n_0} \sum_{v=1}^{\dim U^{l_t}} U^{l_t}_{vs_t}(g) u_n(\lambda) c_{nt}(\lambda) K^{l_t q_t v}_{ijki'j'k'}, \tag{10}$$

where g is an arbitrary element of $\mathrm{O}(d)$ that rotates the point $((0,0,\dots,0,\lambda)^\top$ to the point $\mathbf{p}$. The matrix entries $U^{l_t}_{vs_t}(g)$ depend only on the angular spherical coordinates of the point $\mathbf{p}$ and are proportional to the *real-valued spherical harmonics*

$$U^{l_t}_{vs_t}(g) = \left(\frac{2\pi^{d/2}}{\Gamma(d/2)\dim U^{l_t}}\right)^{1/2} Y^v_{l_t}(\hat{\theta}_1,\dots,\hat{\theta}_{d-2},\hat{\varphi}), \tag{11}$$

in particular, $\dim U^{l_t} = h(l_t, d)$.

Combining everything together, we obtain

$$\begin{aligned} K_{ijki'j'k'}(\mathbf{x},\mathbf{y}) = \left(\frac{2\pi^{d/2}}{\Gamma(d/2)}\right)^{1/2} &\sum_{n=1}^{\dim \mathcal{C}_1+1} \sum_{t=1}^{n_0} (h(l_t,d))^{-1/2} \sum_{v=1}^{h(l_t,d)} K^{l_t q_t v}_{ijki'j'k'} \\ &\times \int_0^\infty \int_{S^{d-1}} \exp(\mathrm{i}(\mathbf{p},\mathbf{x}-\mathbf{y})) Y^v_{l_t}(\hat{\theta}_1,\dots,\hat{\theta}_{d-2},\hat{\varphi}) u_n(\lambda) c_{nt}(\lambda)\,\mathrm{d}\Omega\,\mathrm{d}\Phi(\lambda), \end{aligned} \tag{12}$$

where $(\lambda, \hat{\theta}_1,\dots,\hat{\theta}_{d-2},\hat{\varphi})$ are the spherical coordinates of the point $\mathbf{p} \in \hat{\mathbb{R}}^d$. The inner integral can be calculated using the original idea by M. Ĭ. Yadrenko. The degenerate form of the Gegenbauer addition theorem has the form

$$\exp(\mathrm{i}(\mathbf{p},\mathbf{x}-\mathbf{y})) = 2^\nu \Gamma(\nu) \sum_{\ell=0}^\infty \mathrm{i}^\ell (\ell+\nu) \frac{J_{\ell+\nu}(\|\mathbf{p}\| \cdot \|\mathbf{x}-\mathbf{y}\|)}{(\|\mathbf{p}\| \cdot \|\mathbf{x}-\mathbf{y}\|)^\nu} C^\nu_\ell(\cos\theta),$$

where $\nu = (d-2)/2$, θ is the angle between the vectors $\mathbf{p}$ and $\mathbf{x}-\mathbf{y}$, and $C^\nu_\ell(\cos\theta)$ are the Gegenbauer polynomials. The addition theorem for spherical harmonics has the form

$$\frac{C^\nu_\ell(\cos\theta)}{C^\nu_\ell(1)} = \frac{2\pi^{d/2}}{\Gamma(d/2) h(\ell,d)} \sum_{m=1}^{h(\ell,d)} Y^m_\ell(\theta_1,\dots,\theta_{d-2},\varphi) Y^m_\ell(\hat{\theta}_1,\dots,\hat{\theta}_{d-2},\hat{\varphi})$$

where $(\theta_1,\dots,\theta_{d-2},\varphi)$ and $(\hat{\theta}_1,\dots,\hat{\theta}_{d-2},\hat{\varphi})$ are the angular spherical coordinates ow two vectors and θ is the angle between them. The denominator in the left hand side is

$$C^{(d-2)/2}_\ell(1) = \frac{(\ell+d-3)!}{\ell!(d-3)!}.$$

Combining the two addition theorems together, we obtain the expansion of the plane wave in spherical harmonics as follows:

$$\begin{aligned}\exp(\mathrm{i}(\mathbf{p},\mathbf{x}-\mathbf{y})) = (2\pi)^{d/2}\sum_{\ell=0}^{\infty}\mathrm{i}^{\ell}\frac{J_{\ell+\nu}(\|\mathbf{p}\|\cdot\|\mathbf{x}-\mathbf{y}\|)}{(\|\mathbf{p}\|\cdot\|\mathbf{x}-\mathbf{y}\|)^{\nu}}\\ \times\sum_{m=1}^{h(\ell,d)}Y_{\ell}^{m}(\theta_1,\dots,\theta_{d-2},\varphi)Y_{\ell}^{m}(\hat{\theta}_1,\dots,\hat{\theta}_{d-2},\hat{\varphi}).\end{aligned} \tag{13}$$

We substitute this expansion to (12). After integration over S^{d-1}, the one and only term of the expansion, that corresponds to $\ell = l_t$ and $m = v$, gives a non-zero contribution. We obtain

$$\begin{aligned}K_{ijki'j'k'}(\mathbf{x},\mathbf{y}) &= \frac{2^{(d+1)/2}\pi^{3d/4}}{\sqrt{\Gamma(d/2)}}\sum_{n=1}^{\dim\mathcal{C}_1+1}\sum_{t=1}^{n_0}\mathrm{i}^{l_t}(h(l_t,d))^{-1/2}\\ &\times\int_0^{\infty}\frac{J_{l_t+\nu}(|\lambda\|\mathbf{x}-\mathbf{y}\|)}{(\lambda\|\mathbf{x}-\mathbf{y}\|)^{\nu}}c_{nt}(\lambda)\sum_{v=1}^{h(l_t,d)}K^{l_tq_tv}_{ijki'j'k'}Y^{v}_{l_t}(\theta_1,\dots,\theta_{d-2},\varphi)\,\mathrm{d}\Phi_n(\lambda),\end{aligned}$$

where Φ_n is a measure on $\mathfrak{B}([0,\infty))$ with the Radon–Nikodym density $u_n(\lambda)$ with respect to the measure Φ.

This equation still does not have the form (4). To overcome this difficulty, we write down the expansion (13) twice. In the first (resp., the second) expansion, we replace $\mathbf{x}-\mathbf{y}$ with $\mathbf{x}$ (resp., by $-\mathbf{y}$). The left hand side of Equation (13) becomes the product of the two right hand sides:

$$\begin{aligned}\exp(\mathrm{i}(\mathbf{p},\mathbf{x}-\mathbf{y})) &= (2\pi)^{d}\sum_{\ell',\ell''=0}^{\infty}\sum_{m'=1}^{h(\ell',d)}\sum_{m''=1}^{h(\ell'',d)}\mathrm{i}^{\ell'-\ell''}\frac{J_{\ell'+\nu}(\|\mathbf{p}\|\cdot\|\mathbf{x}\|)}{(\|\mathbf{p}\|\cdot\|\mathbf{x}\|)^{\nu}}\frac{J_{\ell''+\nu}(\|\mathbf{p}\|\cdot\|\mathbf{y}\|)}{(\|\mathbf{p}\|\cdot\|\mathbf{y}\|)^{\nu}}\\ &\times Y^{m'}_{\ell'}(\theta_1,\dots,\theta_{d-2},\varphi)Y^{m'}_{\ell'}(\hat{\theta}_1,\dots,\hat{\theta}_{d-2},\hat{\varphi})Y^{m''}_{\ell''}(\theta_1,\dots,\theta_{d-2},\varphi)Y^{m''}_{\ell''}(\hat{\theta}_1,\dots,\hat{\theta}_{d-2},\hat{\varphi}).\end{aligned}$$

Substitute this equation to (12). We observe that the integral

$$\int_{S^{d-1}}Y^{v}_{\ell_t}(\hat{\theta}_1,\dots,\hat{\theta}_{d-2},\hat{\varphi})Y^{m'}_{\ell'}(\hat{\theta}_1,\dots,\hat{\theta}_{d-2},\hat{\varphi})Y^{m''}_{\ell''}(\hat{\theta}_1,\dots,\hat{\theta}_{d-2},\hat{\varphi})\,\mathrm{d}\Omega$$

should be calculated. Equation (11) gives

$$Y^{m'}_{\ell'}(\hat{\theta}_1,\dots,\hat{\theta}_{d-2},\hat{\varphi})Y^{m''}_{\ell''}(\hat{\theta}_1,\dots,\hat{\theta}_{d-2},\hat{\varphi}) = \frac{\Gamma(d/2)\sqrt{\dim U_{\ell'}\dim U_{\ell''}}}{2\pi^{d/2}}U^{\ell'}_{mw'}(g)U^{\ell''}_{m'w''}(g),$$

where we suppose that the one-dimensional linear subspace of $\mathrm{O}(d-1)$-invariant vectors in $U_{\ell'}$ (resp., $U_{\ell''}$) is labelled by the index w' (resp., w''). The product of matrix entries in the right hand side has the form

$$\begin{aligned}U^{\ell'}_{m'w'}(g)U^{\ell''}_{m''w''}(g) &= c^{\ell'\ell''vs_t}_{m'w'm''w''l_t}U^{l_t}_{vs_t}(g)+\cdots\\ &= c^{\ell'\ell''vs_t}_{m'w'm''w''l_t}\left(\frac{2\pi^{d/2}}{\Gamma(d/2)\dim U_{l_t}}\right)^{1/2}Y^{v}_{l_t}(\hat{\theta}_1,\dots,\hat{\theta}_{d-2},\hat{\varphi})+\cdots,\end{aligned}$$

where the dots denote the terms that do not contribute to the integral. Finally, we obtain

$$\begin{aligned}&\int_{S^{d-1}}Y^{v}_{l_t}(\hat{\theta}_1,\dots,\hat{\theta}_{d-2},\hat{\varphi})Y^{m'}_{\ell'}(\hat{\theta}_1,\dots,\hat{\theta}_{d-2},\hat{\varphi})Y^{m''}_{\ell''}(\hat{\theta}_1,\dots,\hat{\theta}_{d-2},\hat{\varphi})\,\mathrm{d}\Omega\\ &\quad= \left(\frac{\Gamma(d/2)\dim U_{\ell'}\dim U_{\ell''}}{2\pi^{d/2}\dim U_{l_t}}\right)^{1/2}c^{\ell'\ell''vs_t}_{m'w'm''w''l_t}.\end{aligned}$$

Combining everything together again, we obtain

$$K_{ijki'j'k'}(\mathbf{x},\mathbf{y})=(2\pi)^d\sum_{n=1}^{\dim\mathcal{C}_1+1}\sum_{t=1}^{n_0}\sum_{\ell',\ell''=0}^{\infty}\frac{\mathrm{i}^{\ell'-\ell''}\sqrt{h(\ell',d)h(\ell'',d)}}{h(l_t,d)}\sum_{m'=1}^{h(\ell',d)}\sum_{m''=1}^{h(\ell'',d)}$$
$$\times\int_0^\infty\frac{J_{\ell'+\nu}(\lambda\|\mathbf{x}\|)}{(\lambda\|\mathbf{x}\|)^\nu}\frac{J_{\ell''+\nu}(\lambda\|\mathbf{y}\|)}{(\lambda\|\mathbf{y}\|)^\nu}c_{nt}(\lambda)\sum_{v=1}^{h(l_t,d)}c^{\ell'\ell''vs_t}_{m'w'm''w''l_t}K^{l_tq_tv}_{ijki'j'k'}\,\mathrm{d}\Phi_n(\lambda)$$
$$\times Y^{m'}_{\ell'}(\theta'_1,\dots,\theta'_{d-2},\varphi')Y^{m''}_{\ell''}(\theta''_1,\dots,\theta''_{d-2},\varphi''),$$

where $(\theta'_1,\dots,\theta'_{d-2},\varphi')$ (resp., $(\theta''_1,\dots,\theta''_{d-2},\varphi'')$) are the angular spherical coordinates of the point $\mathbf{x}$ (resp., $\mathbf{y}$).

The random field $\mathbf{X}(r,\theta_1,\dots,\theta_{d-2},\varphi)$ takes the form

$$X_{ijk}(r,\theta_1,\dots,\theta_{d-2},\varphi)=(2\pi)^{d/2}\sum_{n'=1}^{\dim\mathcal{C}_1+1}\sum_{t'=1}^{n_0}\sum_{\ell'=0}^{\infty}\sum_{m'=1}^{h(\ell',d)}\int_0^\infty\frac{J_{\ell'+\nu}(\lambda r)}{(\lambda r)^\nu}\sqrt{c_{n't'}(\lambda)}\,\mathrm{d}\zeta^{m'}_{ijkn't'\ell'}(\lambda)$$
$$\times Y^{m'}_{\ell'}(\theta_1,\dots,\theta_{d-2},\varphi),$$

where $\zeta^{m'}_{ijk\ell'}$ are centered orthogonal stochastic measures on $\mathfrak{B}([0,\infty))$ satisfying the condition

$$\mathsf{E}[\zeta^{m'}_{ijkn't'\ell'}(\lambda)(A)\zeta^{m''}_{ijkn''t''\ell''}(\lambda)(B)]=\delta_{nn'}\delta_{t't''}\frac{\mathrm{i}^{\ell'-\ell''}\sqrt{h(\ell',d)h(\ell'',d)}}{h(l_t,d)}$$
$$\times\sum_{v=1}^{h(l_t,d)}c^{\ell'\ell''vs_t}_{m'w'm''w''l_t}K^{l_tq_tv}_{ijki'j'k'}\Phi_n(A\cap B)$$

for all A, $B\in\mathfrak{B}([0,\infty))$.

Example 2. Put $d=3$. The set of real irreducible representations of the group $\mathrm{O}(3)$ is $\{U^{\ell g},U^{\ell u}\colon\ell\geq 0\}$. The indices g and u are the first letter of German words *gerade* (even) and *ungerade* (odd). In even (resp., odd) representations, we have $(-g)\cdot\mathbf{x}=g\cdot\mathbf{x}$ (resp., $(-g)\cdot\mathbf{x}=-g\cdot\mathbf{x}$) for all $g\in\mathrm{O}(3)$. We put $U=U^{1u}$, which corresponds to the case, when $\mathrm{O}(3)$ acts in $U^u_1=\mathbb{R}^3$ by matrix-vector multiplication. In the basis $\{\mathbf{e}_{ijk}\}$ of the space U^{1u}, the indices i and j take only one value, and we omit them.

The space $U^{1u}\otimes U^{1u}$ is the direct sum of three irreducible components. The component U^{0g} (resp., U^{1g}, resp., U^{2g}) acts in the linear space generated by the matrix $K^{00}=\frac{1}{\sqrt{3}}I$ (resp., of skew-symmetric matrices, resp., of symmetric traceless matrices). We omit the index q in the right hand side of Equation (10), because $m_l=1$ for all l. Following quantum-mechanical conventions, we enumerate the basis vectors of the spaces $U^{\ell g}$ and $U^{\ell u}$ by integers running from $-\ell$ to ℓ. The convex compact set $\mathcal{C}_0$ is a singleton: $\mathcal{C}_0=\{\frac{1}{3}I\}$.

The restriction of the representation $U^{0g}\oplus U^{1g}\oplus U^{2g}$ to the subgroup $\mathrm{O}(2)$ contains two copies of its trivial representation U^{0+}. The first one acts in U^{0g}, the second in the linear subspace generated by the matrix $K^{20}=-\frac{1}{\sqrt{6}}\mathbf{e}_{-1}\otimes\mathbf{e}_{-1}+\frac{\sqrt{2}}{\sqrt{3}}\mathbf{e}_0\otimes\mathbf{e}_0-\frac{1}{\sqrt{6}}\mathbf{e}_1\otimes\mathbf{e}_1$. The convex compact set $\mathcal{C}_1$ is the closed interval with extreme points

$$K^1=\mathbf{e}_0\otimes\mathbf{e}_0=\frac{1}{\sqrt{3}}K^{00}+\frac{\sqrt{2}}{\sqrt{3}}K^{20},\qquad K^2=\frac{1}{2}(\mathbf{e}_{-1}\otimes\mathbf{e}_{-1}+\mathbf{e}_1\otimes\mathbf{e}_1)=\frac{1}{\sqrt{3}}K^{00}-\frac{1}{\sqrt{6}}K^{20}.$$

Equations (10) and (11) become

$$f_{kk'}(\mathbf{p})=2\sqrt{\pi}\left(\left(\frac{1}{3}\delta_{kk'}Y^0_0(\mathbf{p})+\frac{\sqrt{2}}{\sqrt{15}}\sum_{v=-2}^{2}K^{2v}_{kk'}Y^v_2(\mathbf{p})\right)u_1(\|\mathbf{p}\|)\right.$$
$$\left.+\left(\frac{1}{3}\delta_{kk'}Y^0_0(\mathbf{p})-\frac{1}{\sqrt{30}}\sum_{v=-2}^{2}K^{2v}_{kk'}Y^v_2(\mathbf{p})\right)u_2(\|\mathbf{p}\|)\right),$$

where the matrices K^{2v} have the form

$$K^{2\,-2}=\frac{1}{\sqrt{2}}(\mathbf{e}_{-1}\otimes\mathbf{e}_1+\mathbf{e}_1\otimes\mathbf{e}_{-1}),\qquad K^{2\,-1}=\frac{1}{\sqrt{2}}(\mathbf{e}_{-1}\otimes\mathbf{e}_0+\mathbf{e}_0\otimes\mathbf{e}_{-1}),$$
$$K^{21}=\frac{1}{\sqrt{2}}(\mathbf{e}_0\otimes\mathbf{e}_1+\mathbf{e}_1\otimes\mathbf{e}_0),\qquad K^{22}=\frac{1}{\sqrt{2}}(\mathbf{e}_{-1}\otimes\mathbf{e}_{-1}-\mathbf{e}_1\otimes\mathbf{e}_1),$$

and the spherical harmonics are

$$Y_0^0(\mathbf{p})=\frac{1}{2\sqrt{\pi}},\qquad Y_2^{-2}(\mathbf{p})=\frac{\sqrt{15}}{2\sqrt{\pi}}\frac{p_{-1}p_1}{\|\mathbf{p}\|^2},$$
$$Y_2^{-1}(\mathbf{p})=\frac{\sqrt{15}}{2\sqrt{\pi}}\frac{p_0p_1}{\|\mathbf{p}\|^2},\qquad Y_2^0(\mathbf{p})=\frac{\sqrt{5}}{4\sqrt{\pi}}\left(\frac{3p_0^2}{\|\mathbf{p}\|^2}-1\right),$$
$$Y_2^1(\mathbf{p})=\frac{\sqrt{15}}{2\sqrt{\pi}}\frac{p_{-1}p_0}{\|\mathbf{p}\|^2},\qquad Y_2^2(\mathbf{p})=\frac{\sqrt{15}}{4\sqrt{\pi}}\frac{p_{-1}^2-p_1^2}{\|\mathbf{p}\|^2}.$$

We substitute the plane wave expansion (13) with $d=3$ to the equation

$$K_{kk'}(\mathbf{x},\mathbf{y})=\int_0^\infty\int_{S^2}\exp(\mathrm{i}(\mathbf{p},\mathbf{x}-\mathbf{y}))f_{kk'}(\mathbf{p})\,\mathrm{d}\Omega\,\mathrm{d}\Phi(\lambda)$$

and obtain

$$K_{kk'}(\mathbf{u})=4\sqrt{2}\pi^2\sum_{n=1}^{2}\int_0^\infty\tilde{f}_{kk'}^n(\lambda,\mathbf{u})\,\mathrm{d}\Phi_n(\lambda),$$

where

$$\tilde{f}_{kk'}^1(\lambda,\mathbf{u})=\frac{J_{1/2}(\lambda\|\mathbf{u}\|)}{3\sqrt{\lambda\|\mathbf{u}\|}}\delta_{kk'}Y_0^0(\theta,\varphi)-\frac{\sqrt{2}J_{5/2}(\lambda\|\mathbf{u}\|)}{\sqrt{15\lambda\|\mathbf{u}\|}}\sum_{v=-2}^{2}K_{kk'}^{2v}Y_2^v(\theta,\varphi),$$
$$\tilde{f}_{kk'}^2(\lambda,\mathbf{u})=\frac{J_{1/2}(\lambda\|\mathbf{u}\|)}{3\sqrt{\lambda\|\mathbf{u}\|}}\delta_{kk'}Y_0^0(\theta,\varphi)+\frac{J_{5/2}(\lambda\|\mathbf{u}\|)}{\sqrt{30\lambda\|\mathbf{u}\|}}\sum_{v=-2}^{2}K_{kk'}^{2v}Y_2^v(\theta,\varphi).$$

Observe that $\frac{1}{3}I=\frac{1}{3}K^1+\frac{2}{3}K^2$. It follows that $\Phi_2(\{0\})=2\Phi_1(\{0\})$.
Using the values of the matrix entries $K_{kk'}^{2v}$ and spherical harmonics Y_2^v, we obtain

$$\sum_{v=-2}^{2}K_{kk'}^{2v}Y_2^v(\mathbf{u})=-\frac{\sqrt{5}}{2\sqrt{6\pi}}\delta_{kk'}+\frac{\sqrt{15}}{2\sqrt{2\pi}}\frac{u_ku_{k'}}{\|\mathbf{u}\|^2},$$

The values

$$J_{1/2}(x)=\frac{\sqrt{2}}{\sqrt{\pi}}\frac{\sin x}{\sqrt{x}},\qquad J_{5/2}(x)=\frac{3\sqrt{2}}{\sqrt{\pi}}\frac{\sin x}{x^2\sqrt{x}}-\frac{\sqrt{2}}{\sqrt{\pi}}\frac{\sin x}{\sqrt{x}}-\frac{3\sqrt{2}}{\sqrt{\pi}}\frac{\cos x}{x\sqrt{x}}$$

give

$$\begin{aligned}\tilde{f}_{kk'}^1(\lambda,\mathbf{u})=\frac{1}{\pi\sqrt{2}}\Bigg[&\left(-\frac{3\sin(\lambda\|\mathbf{u}\|)}{(\lambda\|\mathbf{u}\|)^3}+\frac{\sin(\lambda\|\mathbf{u}\|)}{\lambda\|\mathbf{u}\|}+\frac{3\cos(\lambda\|\mathbf{u}\|)}{(\lambda\|\mathbf{u}\|)^2}\right)\frac{u_ku_{k'}}{\|\mathbf{u}\|^2}\\&+\left(\frac{\sin(\lambda\|\mathbf{u}\|)}{(\lambda\|\mathbf{u}\|)^3}-\frac{\cos(\lambda\|\mathbf{u}\|)}{(\lambda\|\mathbf{u}\|)^2}\right)\delta_{kk'}\Bigg],\\
\tilde{f}_{kk'}^2(\lambda,\mathbf{u})=\frac{1}{\pi\sqrt{2}}\Bigg[&\left(-\frac{3\sin(\lambda\|\mathbf{u}\|)}{2(\lambda\|\mathbf{u}\|)^3}-\frac{\sin(\lambda\|\mathbf{u}\|)}{2\lambda\|\mathbf{u}\|}-\frac{3\cos(\lambda\|\mathbf{u}\|)}{2(\lambda\|\mathbf{u}\|)^2}\right)\frac{u_ku_{k'}}{\|\mathbf{u}\|^2}\\&+\left(\frac{\sin(\lambda\|\mathbf{u}\|)}{2(\lambda\|\mathbf{u}\|)}-\frac{\sin(\lambda\|\mathbf{u}\|)}{2(\lambda\|\mathbf{u}\|)^3}+\frac{\cos(\lambda\|\mathbf{u}\|)}{2(\lambda\|\mathbf{u}\|)^2}\right)\delta_{kk'}\Bigg]\end{aligned}$$

The correlation tensor takes the form

$$\begin{aligned}K_{kk'}(\mathbf{u}) &= 4\pi\int_0^\infty\left[\left(-\frac{3\sin(\lambda\|\mathbf{u}\|)}{(\lambda\|\mathbf{u}\|)^3}+\frac{\sin(\lambda\|\mathbf{u}\|)}{\lambda\|\mathbf{u}\|}+\frac{3\cos(\lambda\|\mathbf{u}\|)}{(\lambda\|\mathbf{u}\|)^2}\right)\frac{u_k u_{k'}}{\|\mathbf{u}\|^2}\right.\\&\quad\left.+\left(\frac{\sin(\lambda\|\mathbf{u}\|)}{(\lambda\|\mathbf{u}\|)^3}-\frac{\cos(\lambda\|\mathbf{u}\|)}{(\lambda\|\mathbf{u}\|)^2}\right)\delta_{kk'}\right]\,\mathrm{d}\Phi_1(\lambda)\\&\quad+2\pi\int_0^\infty\left[\left(\frac{3\sin(\lambda\|\mathbf{u}\|)}{(\lambda\|\mathbf{u}\|)^3}-\frac{\sin(\lambda\|\mathbf{u}\|)}{\lambda\|\mathbf{u}\|}-\frac{3\cos(\lambda\|\mathbf{u}\|)}{(\lambda\|\mathbf{u}\|)^2}\right)\frac{u_k u_{k'}}{\|\mathbf{u}\|^2}\right.\\&\quad\left.+\left(\frac{\sin(\lambda\|\mathbf{u}\|)}{\lambda\|\mathbf{u}\|}-\frac{\sin(\lambda\|\mathbf{u}\|)}{(\lambda\|\mathbf{u}\|)^3}+\frac{\cos(\lambda\|\mathbf{u}\|)}{(\lambda\|\mathbf{u}\|)^2}\right)\delta_{kk'}\right]\,\mathrm{d}\Phi_2(\lambda).\end{aligned}$$

If we absorb the term 4π (resp., 2π) to the measure Φ_1 (resp., Φ_2), then we arrive at Equation (6). Our condition $\Phi_2(\{0\}) = 2\Phi_1(\{0\})$ becomes $\Phi_2(\{0\}) = \Phi_1(\{0\})$, as it should be.

To obtain the spectral expansion of the field itself, we use the expansion (13) instead and obtain

$$\begin{aligned}K_{kk'}(\mathbf{x},\mathbf{y}) &= (2\pi)^3\sum_{\ell',\ell''=0}^{\infty}\sum_{m'=-\ell'}^{\ell'}\sum_{m''=-\ell''}^{\ell''}\mathrm{i}^{\ell'-\ell''}\int_0^\infty\frac{J_{\ell'+1/2}(\lambda\|\mathbf{x}\|)}{\sqrt{\lambda\|\mathbf{x}\|}}\frac{J_{\ell''+1/2}(\lambda\|\mathbf{y}\|)}{\sqrt{\lambda\|\mathbf{y}\|}}\,\mathrm{d}\Phi_1(\lambda)\\&\quad\times\left(\frac{1}{3}\delta_{kk'}+\frac{\sqrt{2(2\ell'+1)(2\ell''+1)}}{5\sqrt{3}}\sum_{v=-2}^{2}c^{\ell'\ell''v}_{m'm''2}K^{2v}_{kk'}\right)Y^{\ell'}_{m'}(\theta',\varphi')Y^{m''}_{\ell''}(\theta'',\varphi'')\\&\quad+(2\pi)^3\sum_{\ell',\ell''=0}^{\infty}\sum_{m'=-\ell'}^{\ell'}\sum_{m''=-\ell''}^{\ell''}\mathrm{i}^{\ell'-\ell''}\int_0^\infty\frac{J_{\ell'+1/2}(\lambda\|\mathbf{x}\|)}{\sqrt{\lambda\|\mathbf{x}\|}}\frac{J_{\ell''+1/2}(\lambda\|\mathbf{y}\|)}{\sqrt{\lambda\|\mathbf{y}\|}}\,\mathrm{d}\Phi_2(\lambda)\\&\quad\times\left(\frac{1}{3}\delta_{kk'}-\frac{\sqrt{(2\ell'+1)(2\ell''+1)}}{5\sqrt{6}}\sum_{v=-2}^{2}c^{\ell'\ell''v}_{m'm''2}K^{2v}_{kk'}\right)Y^{\ell'}_{m'}(\theta',\varphi')Y^{m''}_{\ell''}(\theta'',\varphi'').\end{aligned}$$

The field takes the form

$$X_k(r,\theta,\varphi) = (2\pi)^{3/2}\sum_{n=1}^{2}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\mathrm{i}^{\ell}\int_0^\infty\frac{J_{\ell+1/2}(\lambda r)}{\sqrt{\lambda r}}\,\mathrm{d}\zeta_{k\ell mn}(\lambda)Y^{\ell}_{m}(\theta,\varphi),$$

where $\zeta_{k\ell mn}$ are centred orthogonal stochastic measures on $\mathfrak{B}([0,\infty))$ satisfying the condition

$$\mathsf{E}[\zeta_{k\ell'm'n'}(A)\zeta_{k'\ell''m''n''}(B)] = \delta_{n'n''}M^{\ell'\ell''m'm''n}_{kk'}\Phi_{n'}(A\cap B),$$

and where

$$\begin{aligned}M^{\ell'\ell''m'm''1}_{kk'} &= \frac{1}{3}\delta_{kk'}+\frac{\sqrt{2(2\ell'+1)(2\ell''+1)}}{5\sqrt{3}}\sum_{v=-2}^{2}c^{\ell'\ell''v}_{m'm''2}K^{2v}_{kk'},\\M^{\ell'\ell''m'm''2}_{kk'} &= \frac{1}{3}\delta_{kk'}-\frac{\sqrt{(2\ell'+1)(2\ell''+1)}}{5\sqrt{6}}\sum_{v=-2}^{2}c^{\ell'\ell''v}_{m'm''2}K^{2v}_{kk'}.\end{aligned}$$

4. A new result

In this Section, we prove Theorem 1.

Proof. We consider only the case of $\mathcal{C}_1$. The case of $\mathcal{C}_0$ can be proved similarly and easily.

Assume that the restriction of the representation U to $\mathrm{O}(d-1)$ has a simple spectrum $U^1\oplus\cdots\oplus U^k$. Let $\{\,\mathbf{e}_{ij}\colon 1\le j\le\dim U^i\,\}$ be a basis of the linear space U^i, $1\le i\le k$.

The tensor product $U^i \otimes U^l$ contains the trivial irreducible representation of $\mathrm{O}(d-1)$ if and only if $\ell = i$. The above representation lives in the linear space generated by the matrix

$$K^i = \frac{1}{\dim U^i}(\mathbf{e}_{i1} \otimes \mathbf{e}_{i1} + \cdots + \mathbf{e}_{i\dim U^i} \otimes \mathbf{e}_{i\dim U^i}).$$

These matrices are the k extreme points of the $(k-1)$-dimensional simplex $\mathcal{C}_1$.

To prove the converse statement, assume that the restriction of the representation U to $\mathrm{O}(d-1)$ has a spectrum $m_1U^1 \oplus \cdots \oplus m_kU^k$. Let $\{\, \mathbf{e}_{ilj} \colon 1 \leq j \leq \dim U^i \,\}$ be a basis of the lth copy of the linear space U^i, $1 \leq i \leq k$, $1 \leq l \leq m_i$. The ith connected component of the set of extreme points of $\mathcal{C}_1$ consists of the symmetric nonnegative-definite matrices of the form

$$K = \sum_{l',l''=1}^{m_i} a_{l'l''}(\mathbf{e}_{il'1} \otimes \mathbf{e}_{il''1} + \cdots + \mathbf{e}_{il'\dim U^i} \otimes \mathbf{e}_{il''\dim U^i})$$

with unit trace. If there is i with $m_1 \geq 2$, then the above component is not a singleton and $\mathcal{C}_1$ is not a simplex. □

5. Concluding remarks

The spectral theory of random fields is under further development. We mention the theory of random cross-sections of homogeneous vector bundles, see Malyarenko (2011, 2024), spectral theory of fractional random fields, see Broadbridge, Nanayakkara, and Olenko (2022), Leonenko, Olenko, and Vaz (2024), random fields with singular spectrum, see Leonenko (1999), random fields on a sphere, see Marinucci and Peccati (2011). There are many other directions and open problems that were not mentioned here.

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References

Adams JF (1969). *Lectures on Lie groups.* W. A. Benjamin, New York–Amsterdam.

Bochner S (1932). *Vorlesungen über Fouriersche Integrale*, volume 12 of *Mathematik und ihre Anwendungen in Monographien und Lehrbüchern.* Akademische Verlagsgesellschaft M.B.H., Leipzig.

Bochner S (1941). "Hilbert Distances and Positive Definite Functions." *Annals of Mathematics. Second Series*, **42**, 647–656. ISSN 0003-486X. doi:10.2307/1969252.

Broadbridge P, Nanayakkara R, Olenko A (2022). "On Multifractionality of Spherical Random Fields with Cosmological Applications." *The ANZIAM Journal. The Australian & New Zealand Industrial and Applied Mathematics Journal*, **64**(2), 90–118. ISSN 1446-1811. doi:10.1017/S1446181122000104.

Buldygin VV, Kozachenko YV, Leonenko NN (1992). "On the Works of M. Ĭ. Yadrenko in the Theory of Random Fields." *Ukrainian Mathematical Journal*, **44**(11), 1460–1465. ISSN 0041-6053. doi:10.1007/BF01071506.

Cramér H (1940). "On the Theory of Stationary Random Processes." *Annals of Mathematics. Second Series*, **41**, 215–230. ISSN 0003-486X. doi:10.2307/1968827.

Friedmann AA, Keller LP (1924). "Differentialgleichungen für Turbulente Bewegung einer Kompressiblen Flüssigkeit." In *Proceedings of the 1st International Congress for Applied Mechanics*, pp. 395–405. Delft.

Gikhman II, Skorokhod AV (2004). *The Theory of Stochastic Processes. I.* Classics in Mathematics. Springer-Verlag, Berlin. ISBN 3-540-20284-6. Translated from the Russian by S. Kotz, Reprint of the 1974 edition.

Hannan EJ (1965a). "Group Representations and Applied Probability." *Journal of Applied Probability*, **2**, 1–68. ISSN 0021-9002. doi:10.2307/3211873.

Hannan EJ (1965b). *Group Representations and Applied Probability.* Methuen's Supplementary Review Series in Applied Probability, Vol. 3. Methuen & Co., Ltd., London.

Herglotz G (1991). *Über Potenzreihen mit Positivem, Reellen Teil im Einheitskreis*, volume 16 of *Teubner-Archiv zur Mathematik [Teubner Archive on Mathematics]*, pp. 6–19. B. G. Teubner Verlagsgesellschaft mbH, Stuttgart. ISBN 9783322909268. doi:10.1007/978-3-322-90926-8_1. Gewidmet dem großen Mathematiker Issai Schur (1875–1941). [Dedicated to the great mathematician Issai Schur (1875–1941)], Edited and with a foreword and afterword by Bernd Fritzsche and Bernd Kirstein, With contributions by W. Ledermann and W. K. Hayman, With English, French and Russian summaries.

Homeier HHH, Steinborn EO (1996). "Some Properties of the Coupling Coefficients of Real Spherical Harmonics and Their Relation to Gaunt Coefficients." *Journal of Molecular Structure: THEOCHEM*, **368**, 31–37. ISSN 0166-1280. doi:https://doi.org/10.1016/S0166-1280(96)90531-X. Proceedings of the Second Electronic Computational Chemistry Conference, URL https://www.sciencedirect.com/science/article/pii/S016612809690531X.

Jones RH (1963). "Stochastic Processes on a Sphere." *Annals of Mathematical Statistics*, **34**, 213–218. ISSN 0003-4851. doi:10.1214/aoms/1177704257.

Karhunen K (1946). "Zur Spektraltheorie Stochastischer Prozesse." *Annales Academiae Scientiarum Fennicae. Series A. I. Mathematica-Physica*, **1946**(34), 7. ISSN 0365-2300.

Leonenko M (1999). *Limit Theorems for Random Fields with Singular Spectrum*, volume 465 of *Mathematics and Its Applications*. Kluwer Academic Publishers, Dordrecht. ISBN 0-7923-5635-7. doi:10.1007/978-94-011-4607-4.

Leonenko M, Malyarenko A (2017). "Matérn Class Tensor-valued Random Fields and Beyond." *Journal of Statistical Physics*, **168**(6), 1276–1301. ISSN 0022-4715. doi:10.1007/s10955-017-1847-2.

Leonenko M, Olenko A, Vaz J (2024). "On Fractional Spherically Restricted Hyperbolic Diffusion Random Field." *Communications in Nonlinear Science and Numerical Simulation*, **131**, Paper No. 107866, 15. ISSN 1007-5704. doi:10.1016/j.cnsns.2024.107866.

Malyarenko A (1985). "Spectral Decomposition of Multidimensional Homogeneous and Isotropic Random Fields." *Akademiya Nauk Ukrainskoĭ SSR. Doklady. Seriya A. Fiziko-Matematicheskie i Tekhnicheskie Nauki*, (7), 20–22. ISSN 0201-8446.

Malyarenko A (2011). "Invariant Random Fields in Vector Bundles and Application to Cosmology." *Annales de l'Institut Henri Poincaré Probabilités et Statistiques*, **47**(4), 1068–1095. ISSN 0246-0203. doi:10.1214/10-AIHP409.

Malyarenko A (2024). *Probabilistic Models of Cosmic Backgrounds.* CRC Press, Boca Raton, FL. ISBN 9781032381985. URL https://www.routledge.com/Probabilistic-Models-of-Cosmic-Backgrounds/Malyarenko/p/book/9781032381985.

Malyarenko A, Ostoja-Starzewski M (2019). *Tensor-valued Random Fields for Continuum Physics.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge. ISBN 978-1-108-42985-6.

Malyarenko A, Ostoja-Starzewski M (2023). "Tensor- and Spinor-valued Random Fields with Applications to Continuum Physics and Cosmology." *Probability Surveys*, **20**, 1–86. doi:10.1214/22-ps12.

Malyarenko A, Ostoja-Starzewski M, Amiri-Hezaveh A (2020). *Random Fields of Piezoelectricity and Piezomagnetism—correlation Structures.* SpringerBriefs in Applied Sciences and Technology. Springer, Cham. ISBN 978-3-030-60064-8; 978-3-030-60063-1. doi:10.1007/978-3-030-60064-8. SpringerBriefs in Mathematical Methods.

Marinucci D, Peccati G (2011). *Random Fields on the Sphere*, volume 389 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge. ISBN 978-0-521-17561-6. doi:10.1017/CBO9780511751677. Representation, limit theorems and cosmological applications.

Ogura H (1990). "Representations of the Random Fields on a Sphere." *Memoirs of the Faculty of Engineering. Kyoto University*, **52**(2), 81–105. ISSN 0023-6063.

Olver FWJ, Lozier DW, Boisvert RF, Clark CW (2010). *NIST Handbook of Mathematical Functions.* U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge. ISBN 978-0-521-14063-8. With 1 CD-ROM (Windows, Macintosh and UNIX).

Robertson HP (1940). "The Invariant Theory of Isotropic Turbulence." *Mathematical Proceedings of the Cambridge Philosophical Society*, **36**, 209–223.

von Kármán T (1937a). "The Fundamentals of the Statistical Theory of Turbulence." *Journal of the Aeronautical Sciences*, **4**, 131–138.

von Kármán T (1937b). "On the Statistical Theory of Turbulence." *Proceedings of the National Academy of Sciences of the United States of America*, **23**, 98–105.

Yadrenko MĬ (1983). *Spectral Theory of Random Fields.* Translation Series in Mathematics and Engineering. Optimization Software, Inc., Publications Division, New York. ISBN 0-911575-00-6. Translated from the Russian.

Yaglom AM (1948). "Homogeneous and Isotropic Turbulence in a Viscous Compressible Fluid." *Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz.*, **12**, 501–522.

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