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On Gamma-Gompertz-Makeham Assurances and Life Annuities

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Abstract

We focus on the gamma-Gompertz-Makeham model, and derive useful structural properties for this mortality model. We provide the basic properties like moments, remaining life expectancy, single life annuity, among many others, in closed form, and so it eliminates the need of evaluating them through numerical integration directly. The estimation of the gamma-Gompertz-Makeham model parameters is performed by using the maximum likelihood method under the traditional discrete Poisson distribution, as well as under the recently introduced discrete Bell distribution, which is an interesting alternative to the usual Poisson distribution, mainly in the presence of overdispersion. We illustrate the performance of the gamma-Gompertz-Makeham model in a human mortality database, and compute, based on the Poisson and Bell distributions, the remaining life expectancy, single life annuity and single assurance at ages 30, 55, and 80 in France, Italy, Japan, and Sweden, from 1947 to 2020 males and females separately.

Keywords: Appell function, Gompertz model, hypergeometric function, mortality models.

1. Introduction

Let Z be the gamma-distributed frailty with parameters $\lambda > 0$ and $\rho > 0$, where $\mathbb{E}(Z) = \lambda/\rho$ and $\mathbb{VAR}(Z) = \lambda/\rho^2$. It is common to assume that $\rho = \lambda$ (see, for example, Canudas-Romo, Mazzuco, and Zanotto 2018), and so $\mathbb{E}(Z) = 1$ and $\mathbb{VAR}(Z) = 1/\lambda := \sigma^2$. Under the gamma-Gompertz-Makeham mortality model, the mortality rate (or force of mortality) can be expressed as (Böhnstedt and Gampe 2019; Böhnstedt, Putter, Ouellette, Claeskens, and Gampe 2019)

$$\mu_x = \frac{\alpha \mathrm{e}^{\beta x}}{1 + \sigma^2 \frac{\alpha}{\beta}\left(\mathrm{e}^{\beta x} - 1\right)} + \gamma,$$

where $\alpha > 0$, $\beta > 0$, $\gamma \geq 0$ and $\sigma^2 > 0$. According to Böhnstedt and Gampe (2019) and Böhnstedt *et al.* (2019), the parameter σ^2 describes the heterogeneity of frailty in the gamma-Gompertz-Makeham model, as well as in the gamma-Gompertz model ($\gamma = 0$). If σ^2 is not close to zero, then there is heterogeneity in the risk of death and selection of the most robust individual will occur. On the other hand, $\sigma^2 \approx 0$ (i.e. very close to zero) may indicate that there is no heterogeneity and the force of mortality is increasing exponentially, such that $\mu_x = \alpha \mathrm{e}^{\beta x} + \gamma$. Böhnstedt and Gampe (2019) have discussed about this issue and, in addition, some statistical properties of the ML estimators are derived.

Let $\lambda = 1/\sigma^2$, and so

$$\mu_x = \gamma + \frac{\alpha e^{\beta x}}{1 + \frac{\alpha}{\lambda\beta}(e^{\beta x} - 1)} = \gamma + \frac{\beta\lambda \frac{\alpha}{\beta\lambda - \alpha} e^{\beta x}}{1 + \frac{\alpha}{\beta\lambda - \alpha} e^{\beta x}}.$$

The gamma-Gompertz-Makeham survival function for a life aged x is

$$S(x) = e^{-\gamma x}\left[\frac{1 + \frac{\alpha}{\beta\lambda - \alpha}\, e^{\beta x}}{1 + \frac{\alpha}{\beta\lambda - \alpha}}\right]^{-\lambda},$$

and

$$S(x+t) = e^{-\gamma x} e^{-\gamma t}\left[\frac{1 + \frac{\alpha}{\beta\lambda - \alpha}\, e^{\beta(x+t)}}{1 + \frac{\alpha}{\beta\lambda - \alpha}}\right]^{-\lambda}.$$

Hence,

$${}_t p_x := \frac{S(x+t)}{S(x)} = e^{-\gamma t}\left[\frac{1 + \frac{\alpha}{\beta\lambda - \alpha} e^{\beta(x+t)}}{1 + \frac{\alpha}{\beta\lambda - \alpha} e^{\beta x}}\right]^{-\lambda}.$$

To simplify the notation, let $\alpha = abk/(1+a)$, $\beta = b$, $\gamma = c$, and $\lambda = k$. Hence, it follows that

$$\mu_x = c + \frac{kbae^{bx}}{1 + ae^{bx}}, \quad S(x) = e^{-cx}\left[\frac{1 + ae^{bx}}{1 + a}\right]^{-k}, \quad {}_t p_x = e^{-ct}\left[\frac{1 + ae^{bx}e^{bt}}{1 + ae^{bx}}\right]^{-k}.$$

We can see that if there is no heterogeneity, i.e. $k \to \infty$, then $kba \to \alpha$ and $a \to 0$. In this case, $\mu_x \to \gamma + \alpha e^{\beta x}$, which is the mortality rate under the Gompertz-Makeham mortality model. On the other hand, if there is the heterogeneity, i.e. k is finite ($k < \infty$), we have that $\lim_{x\to\infty} \mu_x = c + kb < \infty$. In this case, the model behaves like a generalized logistic model. It is worth emphasizing that the notation used here and in the next sections (e.g., $S(x)$, $\mathbb{E}[X]$, etc.) is standard in statistics. It is also worth mentioning that actuarial science has developed its own notation, *International Actuarial Notation*, that encapsulates the probabilities and functions of greatest interest and usefulness to actuaries. We refer the reader to Dickson, Hardy, and Waters (2019) for a detailed description.

According to Souza (2022), a special topic that has been drawing the attention of researchers from different fields to date is related to deriving closed-form analytical expressions, depending on special mathematical functions, for some basic properties of the well-known mortality models like Gompertz and Gompertz-Makeham mortality models. For example, Jodrá (2009) derived a closed-form expression for the quantile function of Gompertz-Makeham model, Lenart (2014) derived explicit closed-form expressions for the moment generating function and central moments of the Gompertz model, Bowie (2021) provided closed-form expressions for annuities based on Makeham-Bread mortality laws, Castellares, Patrício, and Lemonte (2020) provided closed-form expressions to Gompertz-Makeham life expectancies, and Castellares, Patrício, and Lemonte (2022) provided analytical expressions for some statistical and actuarial properties regarding the Gompertz-Makeham model and also studied the estimation of the Gompertz-Makeham model parameters through Poisson and Bell distributions. Finally, a good historical note regarding closed-form expressions to Gompertz-Makeham life expectancies is provided by Souza (2022).

In this paper, we shall derive some statistical properties, as well as actuarial properties regarding the gamma-Gompertz-Makeham model, which are useful in practical applications. It is worth stressing that all properties we derive under the gamma-Gompertz-Makeham model are expressed in closed form. In particular, some structural properties depend on the Gaussian hypergeometric function, which is implemented in common statistical and mathematical software facilitating the use of the analytical expressions derived in this paper. It is worthwhile mentioning that the Gaussian hypergeometric function has long date and has been extremely studied and applied in several areas. Consequently, the computational implementaion of it is quite relable in statistical and mathematical software, and so the numerical result obtained from it may be more accurate computationally than the numerical result obtained from numerical integration directly, which may be prone to rounding off errors, for example.

On the other hand, as pointed out by an anounymous referee, modern computational methods have moved beyond hypergeometric functions and, hence, numerical integration may also be applied to compute the quantities studied in this paper. Here, instead of it, we express the structural properties in terms of the Gaussian hypergeometric function that, beyond it mathematical elegance, provides an easy way of computing the analytical expressions derived in this paper with minimal effort in statistical and mathematical software. Furthermore, the estimation of the gamma-Gompertz-Makeham model parameters by using the maximum likelihood method will also be addressed in this paper. In particular, we shall consider the traditional discrete Poisson distribution to do so, as well as the recently introduced discrete Bell distribution. The latter has the advantage of dealing with overdispersion, unlike the Poisson distribution. Finally, based on the human mortality database, we compute the remaining life expectancy, single life annuity and single assurance at ages 30, 55, and 80 in France, Italy, Japan, and Sweden, from 1947 to 2020 males and females separately.

2. Basic properties

Let ${}_2F_1(m,p;q;z)$ be the Gaussian hypergeometric function (see, for instance, Rainville 1960; Exton 1978; Andrews, Askey, and Roy 1999), which is defined in the form

$$ {}_2F_1(m,p;q;z) = \sum_{n=0}^{\infty} \frac{(m)_n (p)_n}{(q)_n} \frac{z^n}{n!}, \quad |z| < 1, $$

where $(m)_n$ is the (rising) Pochhammer symbol defined by $(m)_0 = 1$, and $(m)_n = m(m+1)\cdots(m+n-1)$ for $n \geq 1$. We have also that $(m)_n = \Gamma(m+n)/\Gamma(m)$, where $\Gamma(\cdot)$ denotes the complete gamma function, and ${}_2F_1(m,p;q;z) = {}_2F_1(p,m;q;z)$. We have the following theorem.

Theorem 1. *If $|z| < 1$ and $Re(q) > Re(p) > 0$, then*

$$ {}_2F_1(m,p;q;z) = \frac{\Gamma(q)}{\Gamma(p)\Gamma(q-p)} \int_0^1 u^{p-1}(1-u)^{q-p-1}(1-zu)^{-m} du. $$

Proof. The proof can be found in Rainville (1960, p. 47). □

2.1. Statistical properties

In this section, we provide explicit closed-form expressions of some statistical properties of the gamma-Gompertz-Makeham model.

Characteristic function

We have the following proposition.

Proposition 1. *The characteristic function of the gamma-Gompertz-Makeham model is given by*

$$ \begin{aligned} \varphi(s) = {} & \frac{c}{bk + c - is} \, {}_2F_1\left(k, 1; k+1+\frac{c-is}{b}; \frac{1}{1+a}\right) \\ & + \frac{kab}{(1+a)(bk+c-is)} \, {}_2F_1\left(1+k, 1; k+2+\frac{c-is}{b}; \frac{1}{1+a}\right), \end{aligned} $$

where $s \in \mathbb{R}$, and $i = \sqrt{-1}$ is the imaginary unit.

Proof. We have that

$$ \begin{aligned} \varphi(s) &= \int_0^{\infty} \mathrm{e}^{isx} \mu_x S(x) dx \\ &= \int_0^{\infty} \mathrm{e}^{isx} \left(c + \frac{kba\mathrm{e}^{bx}}{1+a\mathrm{e}^{bx}}\right) \mathrm{e}^{-cx} \left[\frac{1+a\mathrm{e}^{bx}}{1+a}\right]^{-k} dx \\ &= c \int_0^{\infty} \mathrm{e}^{-(c-is)x} \left[\frac{1+a\mathrm{e}^{bx}}{1+a}\right]^{-k} dx + \frac{kba}{1+a} \int_0^{\infty} \mathrm{e}^{-(c-is-b)x} \left[\frac{1+a\mathrm{e}^{bx}}{1+a}\right]^{-k-1} dx. \end{aligned} $$

Making the change of variable $u = 1 - \mathrm{e}^{-bx}$, the result follows by using Theorem 1. □

Corollary 1. *If $c = 0$, we obtain the characteristic function of the gamma–Gompertz model, which has the form*

$$\varphi(s) = \frac{kab}{(1+a)(bk-is)}\, {}_2F_1\left(1+k, 1; k+2+\frac{-is}{b}; \frac{1}{1+a}\right),$$

where $s \in \mathbb{R}$, and $i = \sqrt{-1}$ is the imaginary unit.

Proposition 2. *The moment generating function of the gamma-Gompertz-Makeham model is given by*

$$\begin{aligned} M(s) = {} & \frac{c}{bk+c-s}\, {}_2F_1\left(k, 1; k+1+\frac{c-s}{b}; \frac{1}{1+a}\right) \\ & + \frac{kab}{(1+a)(bk+c-s)}\, {}_2F_1\left(1+k, 1; k+2+\frac{c-s}{b}; \frac{1}{1+a}\right), \quad s \in \mathbb{R}. \end{aligned}$$

Proof. We have that

$$\begin{aligned} M(s) &= \int_0^\infty \mathrm{e}^{sx} \mu_x S(x) dx \\ &= \int_0^\infty \mathrm{e}^{sx} \left(c + \frac{kba\mathrm{e}^{bx}}{1+a\mathrm{e}^{bx}}\right) \mathrm{e}^{-cx} \left[\frac{1+a\mathrm{e}^{bx}}{1+a}\right]^{-k} dx \\ &= c\int_0^\infty \mathrm{e}^{-(c-s)x} \left[\frac{1+a\mathrm{e}^{bx}}{1+a}\right]^{-k} dx + \frac{kba}{1+a}\int_0^\infty \mathrm{e}^{-(c-b)x} \left[\frac{1+a\mathrm{e}^{bx}}{1+a}\right]^{-k-1} dx. \end{aligned}$$

Making the change of variable $u = 1 - \mathrm{e}^{-bx}$, the result follows by using Theorem 1. □

Corollary 2. *The moment generating function of the gamma–Gompertz model reduces to*

$$M(s) = \frac{kab}{(1+a)(bk-s)}\, {}_2F_1\left(1+k, 1; k+1-\frac{s}{b}; \frac{1}{1+a}\right), \quad s \in \mathbb{R}.$$

Proposition 3. *The Laplace transform of the gamma-Gompertz-Makeham model takes the form*

$$\begin{aligned} L(s) = {} & \frac{c}{bk+c+s}\, {}_2F_1\left(k, 1; k+1+\frac{c+s}{b}; \frac{1}{1+a}\right) \\ & + \frac{kab}{(1+a)(bk+c+s)}\, {}_2F_1\left(1+k, 1; k+2+\frac{c+s}{b}; \frac{1}{1+a}\right), \quad s > 0. \end{aligned}$$

Proof. We have that

$$\begin{aligned} L(s) &= \int_0^\infty \mathrm{e}^{-sx} \mu_x S(x) dx \\ &= \int_0^\infty \mathrm{e}^{-sx} \left(c + \frac{kba\mathrm{e}^{bx}}{1+a\mathrm{e}^{bx}}\right) \mathrm{e}^{-cx} \left[\frac{1+a\mathrm{e}^{bx}}{1+a}\right]^{-k} dx \\ &= c\int_0^\infty \mathrm{e}^{-(c+s)x} \left[\frac{1+a\mathrm{e}^{bx}}{1+a}\right]^{-k} dx + \frac{kba}{1+a}\int_0^\infty \mathrm{e}^{-(c+s-b)x} \left[\frac{1+a\mathrm{e}^{bx}}{1+a}\right]^{-k-1} dx. \end{aligned}$$

Making the change of variable $u = 1 - \mathrm{e}^{-bx}$, the result follows by using Theorem 1. □

Corollary 3. *The Laplace transform of the gamma–Gompertz model is*

$$L(s) = \frac{kab}{(1+a)(bk+s)}\, {}_2F_1\left(1+k, 1; k+2+\frac{s}{b}; \frac{1}{1+a}\right), \quad s > 0.$$

Moments

We have the following proposition.

Proposition 4. *Let X be gamma-Gompertz-Makeham distributed. The n-th moment of X is*

$$\mathbb{E}[X^n] = \frac{n!}{b^n} \sum_{m=n-1}^{\infty} \frac{T(m,n-1)}{m!} B(m+1, k+c/b) \times$$
$$_2F_1\left(k, m+1; k+m+1+\frac{c}{b}; \frac{1}{1+a}\right),$$

where $T(q,p)$ is Stirling numbers of the first kind, and $B(\cdot,\cdot)$ is the beta function.

Proof. We have that

$$\mathbb{E}[X^n] = n\int_0^{\infty} x^{n-1} S(x)dx = n\int_0^{\infty} x^{n-1}\mathrm{e}^{-cx}\left[\frac{1+a\mathrm{e}^{bx}}{1+a}\right]^{-k} dx.$$

Let $u = 1-\mathrm{e}^{-bx}$, and so

$$\mathbb{E}[X^n] = n\int_0^{1} \frac{1}{b^n}[-\ln(1-u)]^{n-1}(1-u)^{c/b+k-1}\left[1-\frac{u}{1+a}\right]^{-k} dx.$$

Also,

$$[-\ln(1-u)]^{n-1} = (n-1)! \sum_{m=n-1}^{\infty} \frac{1}{m!} T(m,n-1)u^m, \quad |u|<1.$$

Hence,

$$\mathbb{E}[X^n] = \frac{n!}{b^n} \sum_{m=n-1}^{\infty} \frac{1}{m!} T(m,n-1) \int_0^1 u^m (1-u)^{c/b+k-1}\left[1-\frac{u}{1+a}\right]^{-k} dx.$$

Let $z = 1/(1+a)$ in the above integral. Hence, the result follows by using Theorem 1. □

Corollary 4. *Let X be gamma–Gompertz distributed. The n-th moment of X reduces to*

$$\mathbb{E}[X^n] = \frac{n!}{b^n} \sum_{m=n-1}^{\infty} \frac{T(m,n-1)}{m!} B(m+1,k)\, {}_2F_1\left(k, m+1; k+m+1; \frac{1}{1+a}\right).$$

2.2. Actuarial properties

Here, some additional notation is in order. Let (x) denote a life aged x, where $x \geq 0$. The death of (x) can occur at any age greater than x, and we model the future lifetime of (x) by a continuous random variable which we denote by T_x. This means that $x + T_x$ represents the age-at-death random variable for (x). Also, we are generally most interested in the expected value of the present value random variable for some future payment. We refer to this as the expected present value ('EPV' for shor). It is also commonly referred to as the actuarial value or the actuarial present value. Let δ be the constant force of interest per year. We have assumed this constant force of interest per year in the results derived in the following.

Single life annuities

We have the following proposition.

Proposition 5. *The EPV of a whole life annuity that is payable continuously at a rate of 1 per year subject to the gamma-Gompertz-Makeham mortality model is given by*

$$\bar{a}_x = \frac{1}{bk+\delta+c}\, {}_2F_1\left(k, 1; k+1+\frac{\delta+c}{b}; \frac{1}{1+a\mathrm{e}^{bx}}\right). \quad (1)$$

Proof. We have that

$$\bar{a}_x := \mathbb{E}[\bar{a}_{\overline{T_x}|}] = \int_0^\infty \mathrm{e}^{-\delta t}{}_tp_x dt,$$

where

$${}_tp_x = \mathrm{e}^{-ct}\left[\frac{1 + a\mathrm{e}^{bx}\mathrm{e}^{bt}}{1 + a\mathrm{e}^{bx}}\right]^{-k}.$$

Note that

$$\begin{aligned}\bar{a}_x &= \int_0^\infty \mathrm{e}^{-\delta t}\mathrm{e}^{-ct}\left[\frac{1 + a\mathrm{e}^{bx}\mathrm{e}^{bt}}{1 + a\mathrm{e}^{bx}}\right]^{-k} dt\\ &= [1 + a\mathrm{e}^{bx}]^k \int_0^\infty \mathrm{e}^{-(\delta+c)t}[1 + a\mathrm{e}^{bx}\mathrm{e}^{bt}]^{-k}dt.\end{aligned}$$

Using the transformation $u = 1 - \mathrm{e}^{-bt}$, we obtain

$$\begin{aligned}\bar{a}_x &= [1 + a\mathrm{e}^{bx}]^k \int_0^1 (1-u)^{\frac{\delta+c}{b}}\left[1 + a\mathrm{e}^{bx}\frac{1}{(1-u)}\right]^{-k}\frac{du}{b(1-u)}\\ &= \frac{1}{b}[1 + a\mathrm{e}^{bx}]^k \int_0^1 (1-u)^{\frac{\delta+c}{b}+k-1}[1 + a\mathrm{e}^{bx} - u]^{-k}du.\end{aligned}$$

After some algebra, it follows that

$$\bar{a}_x = \frac{1}{b}\int_0^1 (1-u)^{\frac{\delta+c}{b}+k-1}\left[1 - \frac{u}{1 + a\mathrm{e}^{bx}}\right]^{-k} du.$$

Thus, we obtain

$$\bar{a}_x = \frac{1}{b}\int_0^1 u^{1-1}(1-u)^{\frac{\delta+c}{b}+k-1}(1 - zu)^{-k}\, du,$$

where

$$z = \frac{1}{1 + a\mathrm{e}^{bx}} \in (0,1).$$

From Theorem 1, we have that

$$\int_0^1 \frac{u^{1-1}}{(1-zu)^k}(1-u)^{(\frac{\delta+c}{b}+k+1)-1-1}\, du = {}_2F_1\left(k, 1; k+1+\frac{\delta+c}{b}; z\right)\frac{b}{bk+\delta+c}.$$

Finally, it follows that

$$\bar{a}_x = \frac{1}{bk+\delta+c}{}_2F_1\left(k, 1; k+1+\frac{\delta+c}{b}; \frac{1}{1 + a\mathrm{e}^{bx}}\right),$$

and, therefore, the result holds. □

Corollary 5. *When $\delta = 0$ in equation (1), we obtain the remaining life expectancy at age x in the gamma-Gompertz-Makeham mortality model, which is given by*

$$\mathring{e}_x = \frac{1}{bk+c}\,{}_2F_1\left(k, 1; k+1+\frac{c}{b}; \frac{1}{1 + a\mathrm{e}^{bx}}\right).$$

Proposition 6. *The actuarial present value of an n-year temporary life annuity for (x) and subject to the gamma-Gompertz-Makeham mortality model is*

$$\begin{aligned}\bar{a}_{x:\overline{n}|} &= \frac{1}{bk+\delta+c}{}_2F_1\left(k, 1; k+1+\frac{\delta+c}{b}; \frac{1}{1 + a\mathrm{e}^{bx}}\right)\\ &\quad - {}_np_x\,\mathrm{e}^{-\delta n}\frac{1}{bk+\delta+c}{}_2F_1\left(k, 1; k+1+\frac{\delta+c}{b}; \frac{1}{1 + a\mathrm{e}^{b(x+n)}}\right).\end{aligned}$$

Proof. We have that

$$\bar{a}_{x:\overline{n}|} := \mathbb{E}[\bar{a}_{\overline{\min\{T_x,n\}}|}] = \int_0^n \mathrm{e}^{-\delta t}\, {}_tp_x dt$$
$$= \int_0^n \mathrm{e}^{-\delta t}\mathrm{e}^{-ct}\left[\frac{1+a\mathrm{e}^{bx}\mathrm{e}^{bt}}{1+a\mathrm{e}^{bx}}\right]^{-k} dt,$$

which can be reduced to

$$\bar{a}_{x:\overline{n}|} = \bar{a}_x - {}_np_x\, \mathrm{e}^{-\delta n}\, \bar{a}_{x+n}.$$

Hence, the result follows. □

Single assurances

We have the following proposition.

Proposition 7. *For the gamma-Gompertz-Makeham model, the EPV for whole life insurance benefit payment with sum* 1 *is given by*

$$\bar{A}_x = \frac{c}{bk+2c+\delta}\, {}_2F_1\left(k,1;k+1+\frac{\delta+2c}{b};\frac{1}{1+a\mathrm{e}^{bx}}\right)$$
$$+ \frac{kba\mathrm{e}^{bx}}{(1+a\mathrm{e}^{bx})(bk+2c+\delta)}\, {}_2F_1\left(1+k,1;k+1+\frac{\delta+2c}{b};\frac{1}{1+a\mathrm{e}^{bx}}\right).$$

Proof. We have that

$$\bar{A}_x := \mathbb{E}[\mathrm{e}^{-\delta T_x}] = \int_0^\infty \mathrm{e}^{-\delta t}\, \mu_{x+t}\, {}_tp_x dt$$
$$= \int_0^\infty \mathrm{e}^{-\delta t}\left(c+\frac{kba\mathrm{e}^{b(x+t)}}{1+a\mathrm{e}^{b(x+t)}}\right)\mathrm{e}^{-ct}\left[\frac{1+a\mathrm{e}^{bx}\mathrm{e}^{bt}}{1+a\mathrm{e}^{bx}}\right]^{-k} dt.$$

Making the change of variable $u = 1-\mathrm{e}^{-bt}$, the result follows by using Theorem 1. □

Corollary 6. *For the gamma-Gompertz-Makeham model, the second moment (about zero) of the present value for whole life insurance benefit payment with sum* 1 *is given by*

$${}^2\bar{A}_x := \mathbb{E}[\mathrm{e}^{-2\delta T_x}] = \frac{c}{bk+2c+2\delta}\, {}_2F_1\left(k,1;k+1+\frac{2\delta+2c}{b};\frac{1}{1+a\mathrm{e}^{bx}}\right)$$
$$+ \frac{kba\mathrm{e}^{bx}}{(1+a\mathrm{e}^{bx})(bk+2c+2\delta)}\, {}_2F_1\left(1+k,1;k+1+\frac{2c+2\delta}{b};\frac{1}{1+a\mathrm{e}^{bx}}\right).$$

Corollary 7. *For the gamma-Gompertz model, the EPV for whole life insurance benefit payment with sum* 1 *takes the form*

$$\bar{A}_x = \frac{kba\mathrm{e}^{bx}}{(1+a\mathrm{e}^{bx})(bk+\delta)}\, {}_2F_1\left(1+k,1;k+1+\frac{\delta}{b};\frac{1}{1+a\mathrm{e}^{bx}}\right).$$

Corollary 8. *For the gamma-Gompertz model, the second moment (about zero) of the present value for whole life insurance benefit payment with sum* 1 *is given by*

$${}^2\bar{A}_x = \frac{kba\mathrm{e}^{bx}}{(1+a\mathrm{e}^{bx})(bk+2\delta)}\, {}_2F_1\left(1+k,1;k+1+\frac{2\delta}{b};\frac{1}{1+a\mathrm{e}^{bx}}\right).$$

Remark 1. *The variance of the present value of a unit benefit payable immediately on death is given by* $\mathbb{VAR}[\mathrm{e}^{-\delta T_x}] := \mathbb{E}[\mathrm{e}^{-2\delta T_x}] - \{\mathbb{E}[\mathrm{e}^{-\delta T_x}]\}^2 = {}^2\bar{A}_x - (\bar{A}_x)^2$*; see, for example,* Dickson et al. *(2019).*

Joint life annuity

The Appell function (see, for example, Appell 1880; Slater 1966; Exton 1978), which is an extension of the Gauss hypergeometric function for two variables, is defined for $|x| < 1$ and $|y| < 1$ by the double series

$$F_1(a, b_1, b_2; c; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n} m! n!} x^m y^n.$$

The Appell function converges if these arguments lie within the unit circle. Picard (1881) found out that Appell function can be expressed as a one-dimensional Euler-type integral, and so we have the following theorem.

Theorem 2. *If $|x| < 1$, $|y| < 1$ and $Re(q) > Re(m) > 0$, then*

$$F_1(m, k_1, k_2, q; z_1, z_2) = \frac{\Gamma(q)}{\Gamma(m)\Gamma(q-m)} \int_0^1 t^{m-1}(1-t)^{q-m-1} \frac{(1-z_1 t)^{-k_1}}{(1-z_2 t)^{k_2}} dt.$$

Proof. This representation can be verified by means of Taylor expansion of the integrand, followed by termwise integration; see Picard (1881). □

Similar to the single life annuity issued to a life aged x for the gamma-Gompertz-Makeham mortality model, the following proposition provides a closed-form expression for computing $\bar{a}_{xy}$ in the gamma-Gompertz-Makeham mortality model.

Proposition 8. *Assume that T_x and T_y are gamma-Gompertz-Makeham distributed with parameters (a_1, b, c_1, k_1) and (a_2, b, c_2, k_2), respectively. Let T_x and T_y be independent. We have that*

$$\bar{a}_{xy} = \kappa\, F_1\left(1, k_1, k_2, \frac{\delta + c_1 + c_2}{b} + k_1 + k_2 + 1; \frac{1}{1 + a_1 e^{bx}}, \frac{1}{1 + a_2 e^{by}}\right), \tag{2}$$

where

$$\kappa = \frac{1}{\frac{\delta + c_1 + c_2}{b} + k_1 + k_2}.$$

Proof. We have that

$$\bar{a}_{xy} = \int_0^{\infty} e^{-\delta t} {}_t p_{xy} dt$$

where

$${}_t p_{xy} = {}_t p_x {}_t p_y$$

$${}_t p_{xy} = e^{-(c_1+c_2)t} \left[\frac{1 + a_1 e^{bx} e^{bt}}{1 + a_1 e^{bx}}\right]^{-k_1} \left[\frac{1 + a_2 e^{by} e^{bt}}{1 + a_2 e^{by}}\right]^{-k_2}.$$

Note that

$$\bar{a}_{xy} = [1 + a_1 e^{bx}]^{k_1} [1 + a_2 e^{by}]^{k_2} \int_0^{\infty} e^{-(\delta + c_1 + c_2)t} \frac{[1 + a_1 e^{bx} e^{bt}]^{-k_1}}{[1 + a_2 e^{by} e^{bt}]^{k_2}} dt.$$

Using the transformation $u = 1 - e^{-bt}$, and after some algebra, we obtain

$$\bar{a}_{xy} = \frac{1}{b} \int_0^1 (1-u)^{\frac{\delta + c_1 = C_2}{b} + k_1 + k_2 - 1} \left[1 - \frac{u}{1 + a_1 e^{bx}}\right]^{-k_1} \left[1 - \frac{u}{1 + a_2 e^{by}}\right]^{-k_2} du.$$

Thus, we obtain

$$\bar{a}_{xy} = \frac{1}{b} \int_0^1 u^{1-1} (1-u)^{\frac{\delta + c_1 + c_2}{b} + k_1 + k_2 - 1} (1 - z_1 u)^{-k_1} (1 - z_1 u)^{-k_1} du,$$

where

$$z_1 = \frac{1}{1 + a_1 e^{bx}}, \quad z_2 = \frac{1}{1 + a_2 e^{by}}.$$

From Theorem 2, we have that

$$\int_0^1 u^{1-1}(1-u)^{q-1-1}(1-z_1u)^{-k_1}(1-z_2u)^{-k_2}du = \frac{b}{q-1}F_1(1,k_1,k_2,q;z_1,z_2),$$

where

$$q = \frac{\delta+c_1+c_2}{b} + k_1 + k_2 + 1.$$

Therefore, it follows that

$$\bar{a}_{xy} = \kappa\, F_1\left(1,k_1,k_2,\frac{\delta+c_1+c_2}{b}+k_1+k_2+1;\frac{1}{1+a_1\mathrm{e}^{bx}},\frac{1}{1+a_2\mathrm{e}^{by}}\right),$$

where

$$\kappa = \frac{1}{\frac{\delta+c_1+c_2}{b}+k_1+k_2}.$$

□

Corollary 9. *When $\delta = 0$ in equation (2), we obtain the remaining life expectancy of $T_{xy} = \min\{T_x, T_y\}$ in the gamma-Gompertz-Makeham model, which is given by*

$$\mathring{e}_{xy} = \frac{1}{\frac{c_1+c_2}{b}+k_1+k_2}F_1\left(1,k_1,k_2,\frac{c_1+c_2}{b}+k_1+k_2+1;\frac{1}{1+a_1\mathrm{e}^{bx}},\frac{1}{1+a_2\mathrm{e}^{by}}\right).$$

Also, $\mathring{e}_{\overline{xy}} := \mathbb{E}\left[T_{\overline{xy}}\right] = \mathring{e}_x + \mathring{e}_y - \mathring{e}_{xy}$.

Corollary 10. *Assume that T_x and T_y are gamma-Gompertz-Makeham distributed with parameters (a_1, b, c_1, k_1) and (a_2, b, c_2, k_2), respectively. Let T_x and T_y be independent. We have that $\bar{a}_{\overline{xy}} = \bar{a}_x + \bar{a}_y - \bar{a}_{xy}$; see, for example, Dickson* et al. *(2019).*

Corollary 11. *Assume that T_x and T_y are gamma-Gompertz-Makeham distributed with the same parameters (a, b, c, k). Let T_x and T_y be independent. We have that*

$$\bar{a}_{xy} = \frac{1}{\frac{\delta+2c}{b}+2k}F_1\left(1,k,k,\frac{\delta+2c}{b}+2k+1;\frac{1}{1+a\mathrm{e}^{bx}},\frac{1}{1+a\mathrm{e}^{by}}\right).$$

3. Maximum likelihood estimation

Let D_x be the number of deaths in a given age interval $[x, x+1)$ for $x = 0, \ldots, m$. Also, let E_x denote the number of person-years with age x exposed to the risk of dying (see, for example, Brillinger 1986; Macdonald, Currie, and Richards 2018). Also, define $\boldsymbol{D} = (D_0, D_1, \ldots, D_m)'$ and $\boldsymbol{E} = (E_0, E_1, \ldots, E_m)'$. In addition, let $\boldsymbol{\theta} = (\alpha, \beta, \gamma, \sigma^2)'$ be the parameter vector that characterizes the force of mortality at age x, which is given by

$$\mu_x = \frac{\alpha\mathrm{e}^{\beta x}}{1+\sigma^2\frac{\alpha}{\beta}(\mathrm{e}^{\beta x}-1)} + \gamma.$$

Finally, we assume that the number of deaths and the number of person-years exposed to the risk of dying can be observed.

3.1. Poisson distribution

The standard approach to estimate mortality models was presented by Brillinger (1986), where a Poisson distribution is assumed for the number of deaths D_x with $\mathbb{E}(D_x) = \mathbb{VAR}(D_x) = \mu_x E_x$. Thus, the probability mass function of the random variable D_x is

$$\Pr_{\mathcal{P}}[D_x = z] = \frac{\mathrm{e}^{-\mu_x E_x}(\mu_x E_x)^z}{z!}, \quad z = 0, 1, 2, \ldots.$$

By considering the standard assumption on the death count, the likelihood function for the parameter vector $\boldsymbol{\theta} = (\alpha, \beta, \gamma, \sigma^2)'$ is given by

$$L_{\mathcal{P}}(\boldsymbol{\theta}) \equiv L_{\mathcal{P}}(\boldsymbol{\theta}|\boldsymbol{D}, \boldsymbol{E}) = \prod_x \frac{\mathrm{e}^{-\mu_x E_x}(\mu_x E_x)^{D_x}}{D_x!},$$

and the log-likelihood function, unless constant terms, takes the form

$$\ell_{\mathcal{P}}(\boldsymbol{\theta}) \equiv \ell_{\mathcal{P}}(\boldsymbol{\theta}|\boldsymbol{D}, \boldsymbol{E}) = \ln(L_{\mathcal{P}}(\boldsymbol{\theta})) = \sum_x [D_x \ln(\mu_x E_x) - \mu_x E_x].$$

The maximum likelihood (ML) estimator $\widehat{\boldsymbol{\theta}} = (\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\sigma}^2)'$ of $\boldsymbol{\theta} = (\alpha, \beta, \gamma, \sigma^2)'$ is obtained by maximizing the log-likelihood function with respect to the model parameters. The score functions can be expressed as

$$\frac{\partial \ell_{\mathcal{P}}(\boldsymbol{\theta}|\boldsymbol{D}, \boldsymbol{E})}{\partial \alpha} = \sum_x \delta_x^{\mathcal{P}} \left(\frac{\beta^2 \mathrm{e}^{\beta x}}{(\alpha\sigma^2(\mathrm{e}^{\beta x} - 1) + \beta)^2} \right),$$

$$\frac{\partial \ell_{\mathcal{P}}(\boldsymbol{\theta}|\boldsymbol{D}, \boldsymbol{E})}{\partial \beta} = \sum_x \delta_x^{\mathcal{P}} \left(\frac{\alpha x \mathrm{e}^{\beta x}}{1 + \sigma^2 \frac{\alpha}{\beta}(\mathrm{e}^{\beta x} - 1)} - \frac{\mathrm{e}^{\beta x}\left[\beta x \mathrm{e}^{\beta x} - (\mathrm{e}^{\beta x} - 1)\right]}{\left[\frac{\beta}{\alpha\sigma}\left(1 + \sigma^2 \frac{\alpha}{\beta}(\mathrm{e}^{\beta x} - 1)\right)\right]^2} \right),$$

$$\frac{\partial \ell_{\mathcal{P}}(\boldsymbol{\theta}|\boldsymbol{D}, \boldsymbol{E})}{\partial \gamma} = \sum_x \delta_x^{\mathcal{P}},$$

$$\frac{\partial \ell_{\mathcal{P}}(\boldsymbol{\theta}|\boldsymbol{D}, \boldsymbol{E})}{\partial \sigma^2} = \sum_x \delta_x^{\mathcal{P}} \left(\frac{\alpha^2 \mathrm{e}^{\beta x}(\mathrm{e}^{\beta x} - 1)}{\beta\left(\sigma^2 \frac{\alpha}{\beta}(\mathrm{e}^{\beta x} - 1) + 1\right)^2} \right),$$

where $\delta_x^{\mathcal{P}} \equiv \delta_x^{\mathcal{P}}(\boldsymbol{\theta}) = [D_x/\mu_x - E_x]$. By equating these likelihood equations to zero and solving simultaneously the resulting system of equations, we can also obtain the ML estimator of $\boldsymbol{\theta} = (\alpha, \beta, \gamma, \sigma^2)'$. There is no closed-form expression for the ML estimator $\widehat{\boldsymbol{\theta}}$; therefore, its computation must be performed numerically using a nonlinear optimization algorithm. The maximization of the log-likelihood function can be performed, for example, using the R programming language (Team 2023), specifically applying the `optim` function or the Differential Evolution Optimization algorithm (Ardia, Boudt, Carl, Mullen, and Peterson 2011).

3.2. Bell distribution

In the presence of overdispersion, an alternative to the Poisson distribution is the Bell distribution (Castellares, Ferrari, and Lemonte 2018). A discrete random variable Z has a Bell distribution if its probability mass function is given by

$$\Pr_{\mathcal{B}}(Z = z) = \frac{\exp\{1 - \exp(W_0(\lambda))\} W_0(\lambda)^z B_z}{z!}, \quad z = 0, 1, \ldots,$$

where $\lambda > 0$, $W_0(\cdot)$ is the Lambert function (Corless, Gonnet, Hare, Jeffrey, and Knuth 1996), and B_z are the Bell numbers (Bell 1934a,b) defined by $B_z = (1/\mathrm{e}) \sum_{k=0}^{\infty} k^z/k!$, which is the z-th moment of a Poisson distribution with a parameter equal to 1. We have that $\mathbb{E}(Z) = \lambda < \lambda[1 + W_0(\lambda)] = \mathbb{VAR}(Z)$, which implies that the Bell distribution may be suitable for modeling count data with overdispersion, unlike the Poisson distribution.

Here, we assume that the number of deaths D_x are generated by a Bell distribution. Hence, the likelihood function for the parameter vector $\boldsymbol{\theta} = (\alpha, \beta, \gamma, \sigma^2)'$ assumes the form

$$L_{\mathcal{B}}(\boldsymbol{\theta}) \equiv L_{\mathcal{B}}(\boldsymbol{\theta}|\boldsymbol{D}, \boldsymbol{E}) = \prod_x \exp\{1 - \exp(W_0(\mu_x E_x))\} \frac{W_0(\mu_x E_x)^{D_x} B_{D_x}}{D_x!}.$$

The log-likelihood function, unless constant terms, can be expressed in the form

$$\ell_{\mathcal{B}}(\boldsymbol{\theta}) \equiv \ell_{\mathcal{B}}(\boldsymbol{\theta}|\boldsymbol{D}, \boldsymbol{E}) = \ln(L_{\mathcal{B}}(\boldsymbol{\theta})) = \sum_x [D_x \ln(W_0(\mu_x(\boldsymbol{\theta}) E_x)) - \exp\{W_0(\mu_x(\boldsymbol{\theta}) E_x)\}].$$

In the usual manner, the ML estimator $\widehat{\boldsymbol{\theta}} = (\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\sigma}^2)'$ is obtained by maximizing the log-likelihood function with respect to the model parameters, and the maximization can be performed by using the `optim` function or the Differential Evolution Optimization algorithm, for example. Using the result (see, for example, Corless *et al.* 1996)

$$\frac{dW_0(z)}{dz} = \frac{W_0(z)}{z[1+W_0(z)]}, \quad z \notin \left\{0, -\frac{1}{\mathrm{e}}\right\},$$

and the identity $W_0(z)\exp(W_0(z)) = z$ for $z \neq -1/\mathrm{e}$, the score functions can be expressed in the forms

$$\frac{\partial \ell_{\mathcal{B}}(\boldsymbol{\theta}|\boldsymbol{D},\boldsymbol{E})}{\partial \alpha} = \sum_x \delta_x^{\mathcal{B}} \left(\frac{\beta^2 \mathrm{e}^{\beta x}}{(\alpha\sigma^2(\mathrm{e}^{\beta x}-1)+\beta)^2} \right),$$

$$\frac{\partial \ell_{\mathcal{B}}(\boldsymbol{\theta}|\boldsymbol{D},\boldsymbol{E})}{\partial \beta} = \sum_x \delta_x^{\mathcal{B}} \left(\frac{\alpha x \mathrm{e}^{\beta x}}{1+\sigma^2\frac{\alpha}{\beta}(\mathrm{e}^{\beta x}-1)} - \frac{\mathrm{e}^{\beta x}\left[\beta x \mathrm{e}^{\beta x} - (\mathrm{e}^{\beta x}-1)\right]}{\left[\frac{\beta}{\alpha\sigma}\left(1+\sigma^2\frac{\alpha}{\beta}(\mathrm{e}^{\beta x}-1)\right)\right]^2} \right),$$

$$\frac{\partial \ell_{\mathcal{B}}(\boldsymbol{\theta}|\boldsymbol{D},\boldsymbol{E})}{\partial \gamma} = \sum_x \delta_x^{\mathcal{B}},$$

$$\frac{\partial \ell_{\mathcal{B}}(\boldsymbol{\theta}|\boldsymbol{D},\boldsymbol{E})}{\partial \sigma^2} = \sum_x \delta_x^{\mathcal{B}} \left(\frac{\alpha^2 \mathrm{e}^{\beta x}(\mathrm{e}^{\beta x}-1)}{\beta\left(\sigma^2\frac{\alpha}{\beta}(\mathrm{e}^{\beta x}-1)+1\right)^2} \right),$$

where $\delta_x^{\mathcal{B}} \equiv \delta_x^{\mathcal{B}}(\boldsymbol{\theta}) = [D_x/\mu_x - E_x]/[1+W_0(\mu_x E_x)]$. We can also obtain the ML estimator $\widehat{\boldsymbol{\theta}} = (\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\sigma}^2)'$ by equating these likelihood equations to zero and solving simultaneously the resulting system of equations. Since there is no closed-form expression for the ML estimator, the computation has to be performed numerically using a nonlinear optimization algorithm.

4. Human mortality database analysis

In this section, we apply the gamma–Makeham mortality model in a real database. To estimate the gamma-Gompertz-Makeham model parameters, we apply the ML method presented in the previous section under both discrete Poisson and Bell distributions. We estimate the model parameters from the Human Mortality Database (http://www.mortality.org) using raw death counts and exposures after age 30. Based on the ML estimates obtained from these two discrete distributions, we compute the remaining life expectancy, single life annuity and single assurance at ages 30, 55, and 80 in France, Italy, Japan, and Sweden, from 1947 to 2020 males and females separately. To perform the estimation procedure, we use the R programming language (Team 2023) with the optimization of the log-likelihood functions obtained by using the Differential Evolution Optimization algorithm (Ardia *et al.* 2011) through the package `DEoptim` (Mullen, Ardia, Gil, Windover, and Cline 2011).

4.1. Parameter estimates

The ML estimates of α, β, γ and σ^2 and their respective 95% confidence intervals are presented in Figures 1, 2, 3, and 4, respectively. The assumption of the Bell and Poisson distributions for the death counts leads to similar results regarding the ML estimates of the parameter α (Figure 1) and parameter β (Figure 2). From these figures, note that the Bell distribution provides wider confidence intervals for the parameters α and β than the Poisson distribution, mainly to Sweden. Both distributions lead to similar conclusions on the over-time pattern of the parameters α and β. The initial risk of senescence, represented by the parameter α, presents a log-linear trend for the female populations. We observe the same pattern for the male populations after year 1980. The rate of deterioration with age, represented by the parameter β, seems to be increasing with time in all the populations, except the French female and the Japanese male populations, where it seems to be constant since the years 2000 and 2010, respectively.

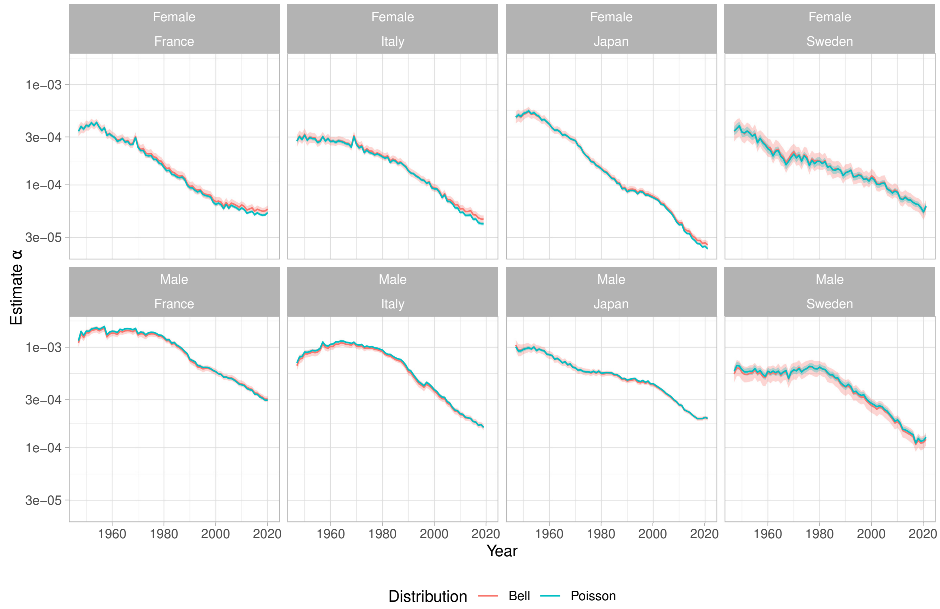

Figure 1: Estimates using the Bell and Poisson distributions of the parameter α for the gamma-Gompertz-Makeham model, applied to the populations of France, Italy, Japan, and Sweden, are presented in the columns, ordered from left to right. The first row represents estimates for females, while the second row corresponds to males. The analysis focuses on the post-1947 period and considers the population aged 30 and above.

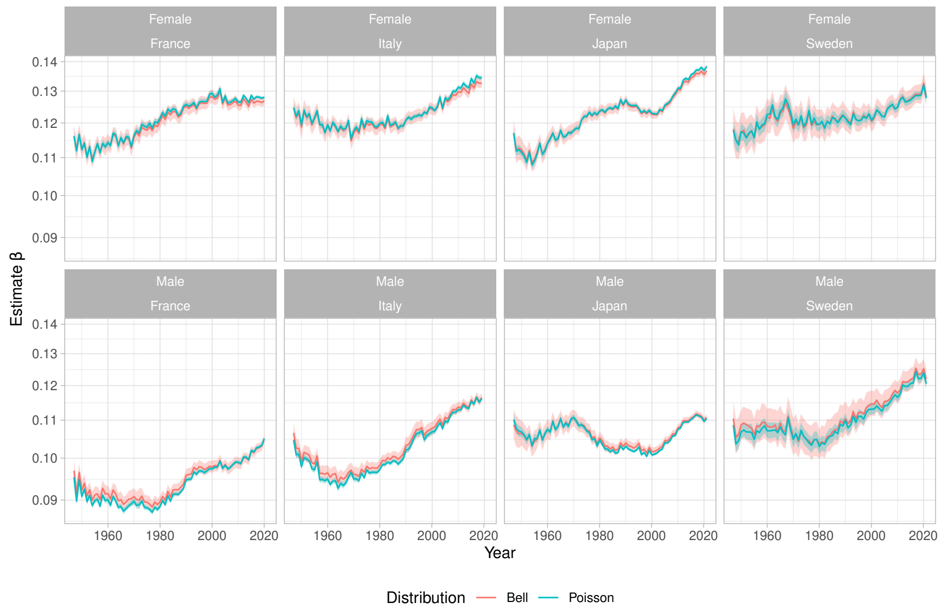

Figure 2: Column-wise presentation of β estimates in the gamma-Gompertz model using the Bell and Poisson distributions, applied to populations of France, Italy, Japan, and Sweden. The first row features estimates for females, while the second row pertains to males. The analysis concentrates on the post-1947 period and includes the population aged 30 and above.

Note that the Bell and Poisson distributions also lead to similar results regarding the parameter γ (Figure 3), with a slight difference for Italy males from 1970 to 1990. However, for the parameter σ^2 (Figure 4), it does not, especially in recent years and among the male populations. In the gamma-Gompertz-Makeham setting, the parameter σ^2 captures the unobserved individual heterogeneity (Vaupel, Manton, and Stallard 1979), which implies in a levelling-off on the population's mortality rates at the oldest ages (see, for example, Missov and Vaupel 2015; Barbi, Lagona, Marsili, Vaupel, and Wachter 2018). Therefore, the difference between the estimates of σ^2 under the Bell and the Poisson assumptions may be due to three main aspects of the mortality data: (i) the number of people alive at the oldest ages is small; therefore, the mortality rates at those ages present a high variability around the mortality plateau (Barbi *et al.* 2018), which the Poisson distribution cannot accommodate; (ii) the number of males alive at the oldest ages is much smaller than the number of females (see, for example, Alvarez, Villavicencio, Strozza, and Camarda 2021; Dang, Camarda, Meslé, Ouellette, and Vallin 2023), which also increases uncertainty on $\widehat{\sigma}^2$; and (iii) the postponement of mortality (Vaupel 2010) implicates on a postponement of the mortality deceleration, and since the deaths after age 110 are grouped at age 110, the leveling-off of the risk of dying cannot be seen in all the recent populations (Dang *et al.* 2023). Finally, the difference between the estimates for σ^2 observed in Figure 4 also leads to different mortality plateaus. As the estimates of β and γ are similar under the assumption of both distributions, estimating a lower σ^2 assuming the Poisson distribution than when we assume the Bell distribution implicates in different mortality plateau levels, i.e. $\gamma + \beta/\sigma^2$.

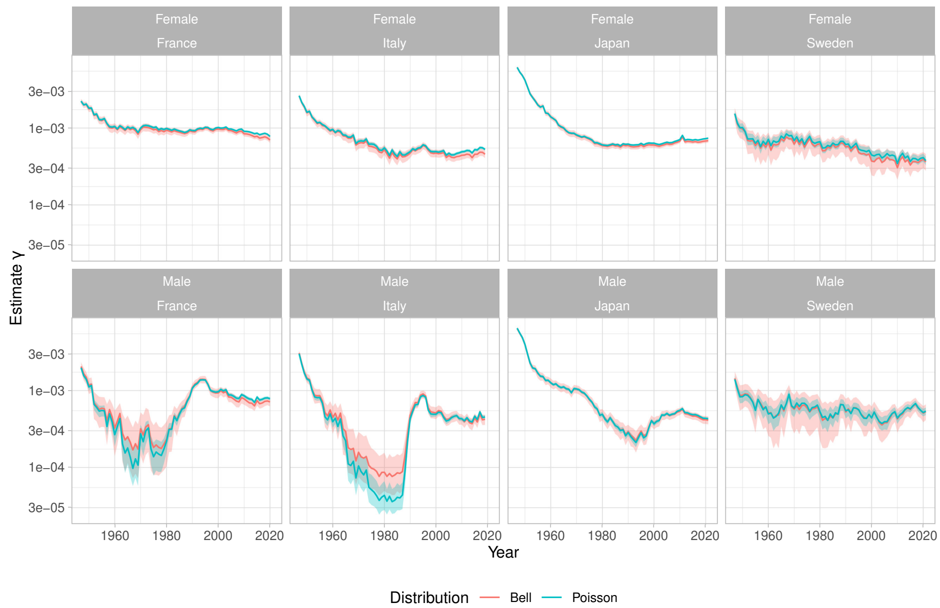

Figure 3: Estimates of the γ parameter, quantifying extrinsic mortality, in the gamma-Gompertz-Makeham model using the Bell and Poisson distributions for populations in France, Italy, Japan, and Sweden, which are organized in columns from left to right. The first row details estimates for females, and the second row corresponds to males. The analysis centers on the post-1947 period, covering the population aged 30 and above.

As above-mentioned, the mortality data presents overdispersion at the oldest ages. Furthermore, Hilbe (2011) and Dean and Lundy (2016) have shown that the standard errors of ML estimates are underestimated under the Poisson distributions in the presence of overdispersion. It explains the smaller standard errors under the Poisson distribution, and also smaller confidence intervals. On the other hand, the Bell distribution "corrects" the standard errors in the presence of overdispersion, providing more reliable results. Remembering that the Bell distribution is suitable to deal with overdispersion. Consequently, these results confirm that the Bell distribution can be an interesting alternative to the Poisson distribution in the context of ML estimation of the gamma-Gompertz-Makeham model parameters.

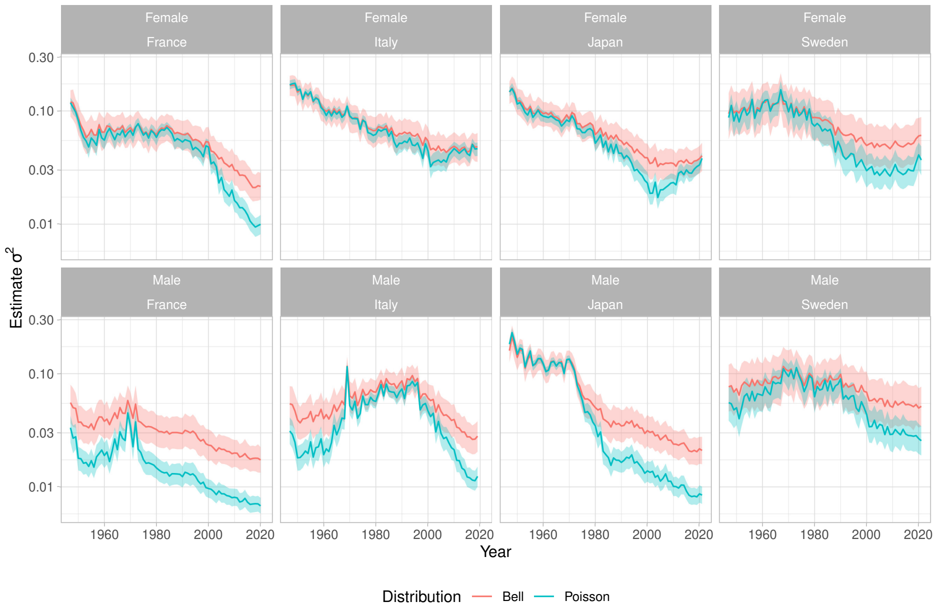

Figure 4: Unobserved heterogeneity estimates using both the Bell and Poisson distributions in the gamma-Gompertz-Makeham model (i.e. estimates of σ^2) are presented for populations in France, Italy, Japan, and Sweden, which are arranged in columns from left to right. The first row presents the estimates for females, while the second row pertains to males. The analysis centers on the post-1947 period and for the population aged 30 and above.

4.2. Single life annuity, remaining life expectancy, and single assurance

The closed-form expressions for the single life annuity at age x ($\bar{a}_x$), remaining life expectancy at age x ($\mathring{e}_x$), and single assurance at age x ($\bar{A}_x$) are provided in Proposition 5, Corollary 5 and Proposition 7, respectively. In the following, we shall compute these measures based on the ML estimates of the gamma-Gompertz-Makeham model parameters under the discrete Poisson and Bell distributions; that is, after computing the ML estimates, we obtain these quantities by replacing the unknown parameters with their respective ML estimates. The force of interest used to compute these actuarial values was $\delta = 0.05$.

The estimated remaining life expectancies at ages 30, 55, and 80 are presented in Figures 5, 6, and 7, respectively. Also, the actual remaining life expectancies in these figures were obtained from the life tables provided by the Human Mortality Database (http://www.mortality.org). These figures show that the remaining life expectancies obtained from the Poisson and Bell distributions are similar. It was expected since the ML estimates obtained from these two discrete distributions are (approximately) the same in most cases. Over time, the estimated remaining life expectancies follow the same pattern as the actual remaining life expectancy. However, in some cases, the estimated $\mathring{e}_x$ presents a lower value than the observed one. The estimated values of $\bar{a}_x$ and $\bar{A}_x$ (with $x = 30, 55$ and 80) are provided in Appendix for the years 1950, 1960, 1970, 1980, 1990, 2000 and 2010; see Tables 1, 2, 3 and 4. As both Poisson and Bell distributions provide similar ML estimates for the gamma–Makeham model parameters, the estimated values for $\bar{a}_x$ and $\bar{A}_x$ are similar under both distributions.

Finally, to measure how close the actual and estimated remaining life expectancies under the gamma-Gompertz-Makeham law are, we present the difference between them. Figures 8, 9, and 10 show this difference at ages 30, 55, and 80, respectively. The confidence intervals are also presented. At ages 30 and 50 (Figures 8 and 9), both Poisson and Bell distributions provide an estimated $\mathring{e}_x$ about 0.5 year lower than the actual. At age 80, however, this difference is smaller, and, for some populations, the actual remaining life expectancies are within the confidence boundaries under the Bell distribution assumption (Figure 10). In short, the gamma–Makeham model, especially under the Bell distribution, provides proper results to deal with human mortality in practice, and so the recently introduced Bell distribution can be an interesting alternative to the usual Poisson distribution in estimating the gamma-

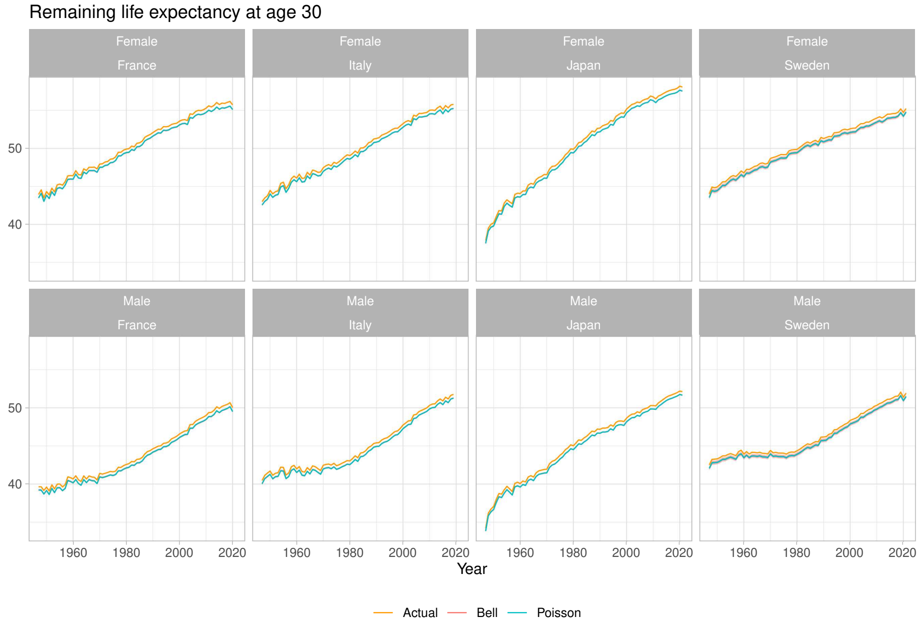

Figure 5: Comparison of the actual and estimated remaining life expectancy at age 30 using Bell and Poisson distributions after the year 1947. Results are presented for populations in France, Italy, Japan, and Sweden, organized in columns from left to right. The first row provides estimates for females, while the second row for males.

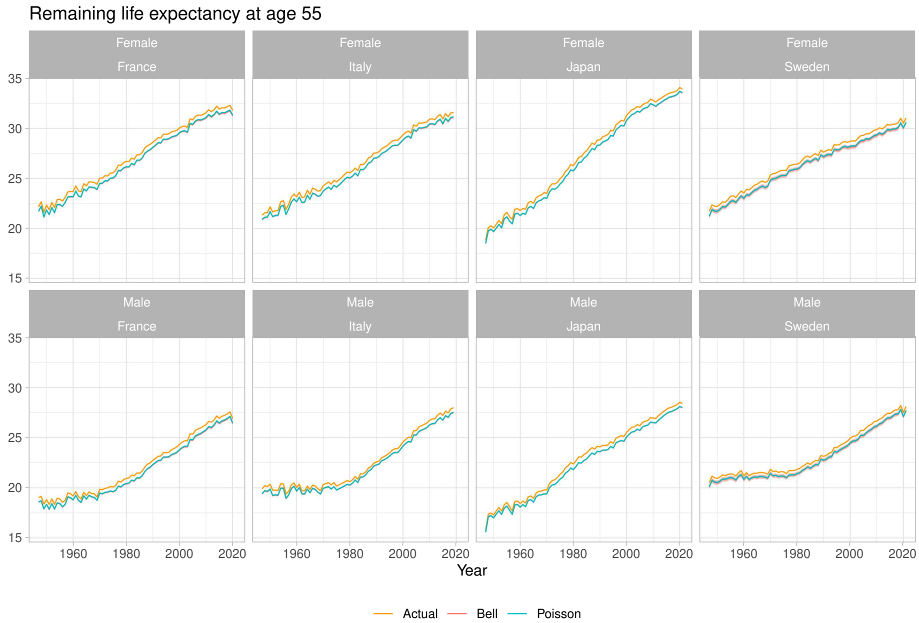

Figure 6: Comparison of the actual and estimated remaining life expectancy at age 55, utilizing Bell and Poisson distributions post-1947. Results are shown for France, Italy, Japan, and Sweden, organized in columns from left to right. The first row presents estimates for females, while the second row provides for males.

Gompertz-Makeham model parameters.

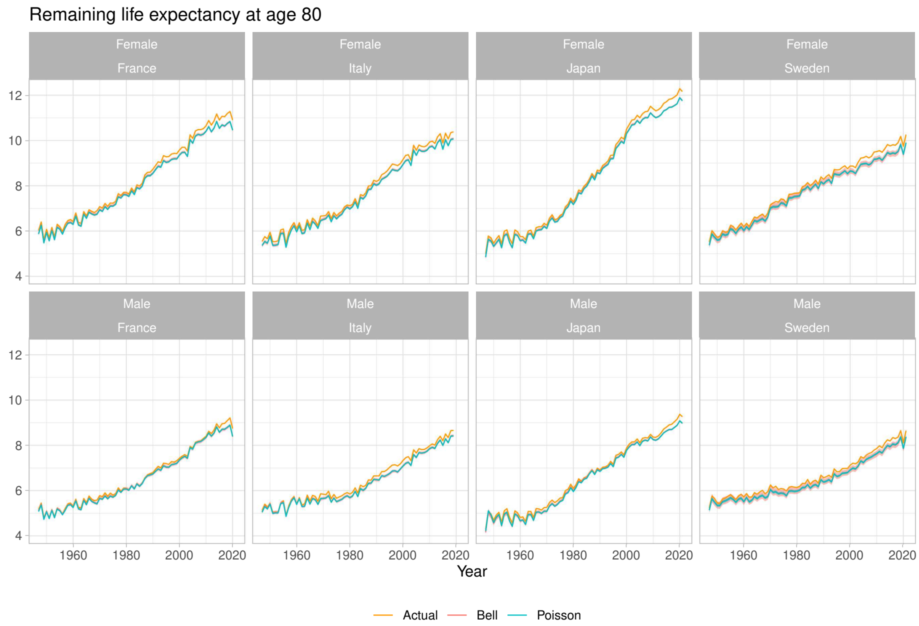

Figure 7: Analysis of the actual and estimated remaining life expectancy at age 80, employing Bell and Poisson distributions post-1947. Results are displayed for populations in France, Italy, Japan, and Sweden, organized in columns from left to right. The first row provides estimates for females, while the second row details estimates for males.

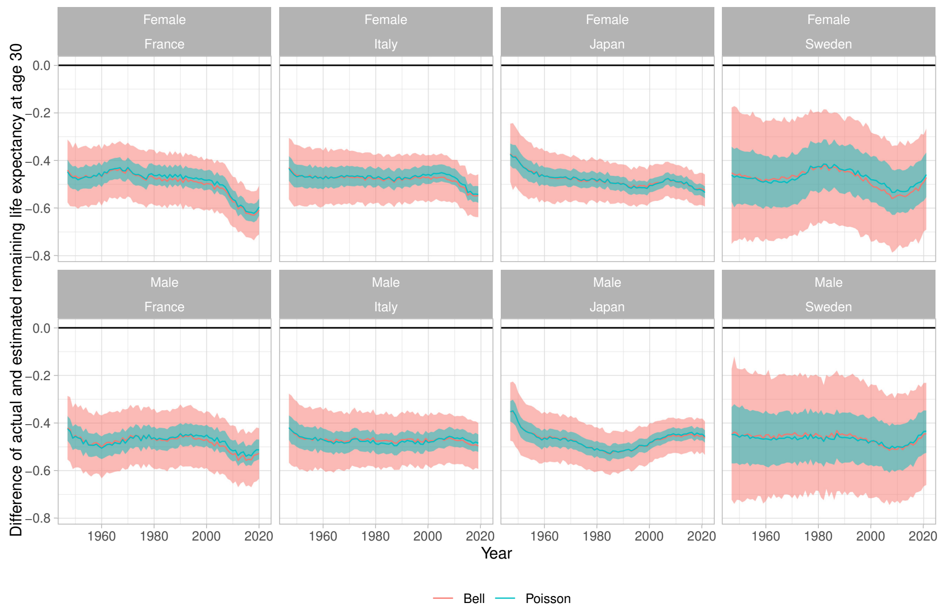

Figure 8: Discrepancy and its associated confidence interval (in years) between the actual and estimated remaining life expectancy at age 30 for the Bell and Poisson distributions post-1947. Values close to zero signify accurate estimations of the remaining life expectancies at age 30 ($\mathring{e}_{30}$). Results are presented for populations in France, Italy, Japan, and Sweden, arranged in columns from left to right. The first row presents estimates for females, while the second-row details estimates for males.

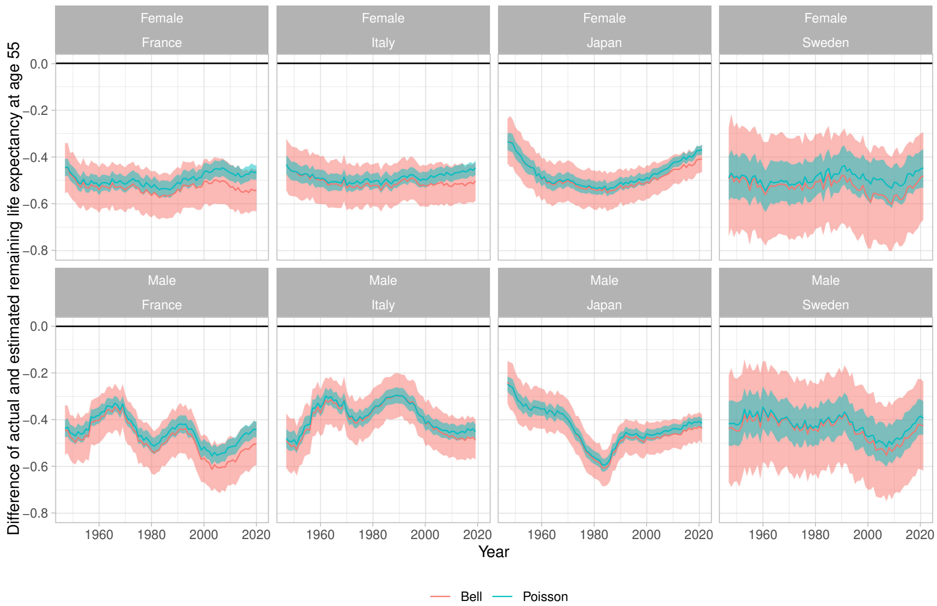

Figure 9: Discrepancy and its corresponding confidence interval (in years) between the actual and estimated remaining life expectancy at age 55 for the Bell and Poisson distributions after the year 1947. Values close to zero suggest accurate estimations of the remaining life expectancies at age 55 ($\mathring{e}_{55}$). Results are displayed for populations in France, Italy, Japan, and Sweden, organized in columns from left to right. The first row provides estimates for females, while the second row details estimates for males.

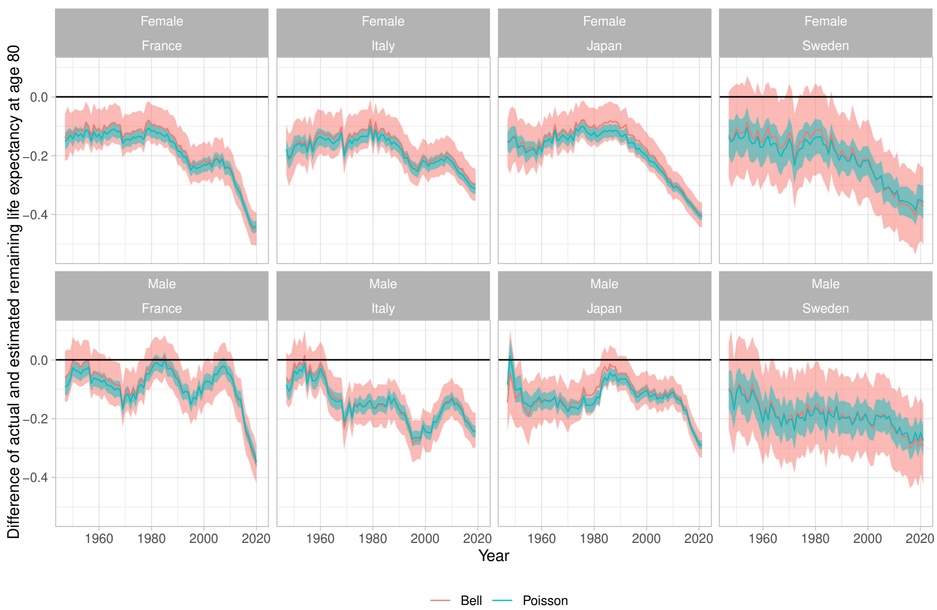

Figure 10: Disparity and its corresponding confidence interval (in years) between the actual and estimated remaining life expectancy at age 80 for the Bell and Poisson distributions post-1947. Values approaching zero indicate precise estimations of the remaining life expectancies at age 80 ($\mathring{e}_{80}$). Findings are presented for populations in France, Italy, Japan, and Sweden, organized in columns from left to right. The first row shows estimates for females, while the second-row presents the estimates for males.

5. Concluding remarks

We have derived several structural properties of the gamma-Gompertz-Makeham model in statistics, demography, and actuarial sciences. All the structural properties we have derived are expressed in closed form, depending only on special mathematical functions like hypergeometric functions. The estimation of the gamma-Gompertz-Makeham model parameters was performed by using the maximum likelihood procedure based on the discrete Poisson and Bell distributions. The latter is useful to deal with overdispersion, unlike the Poisson distribution. We have considered both discrete distributions for estimating the gamma-Gompertz-Makeham model parameters using actual mortality data from the Human Mortality Database (HMD 2012). We have noted that the two discrete distributions work almost equally well, the ML estimates obtained from the Bell distribution having a slight advantage, mainly in the presence of overdispersion. As a result, the Bell distribution we propose here to estimate the gamma-Gompertz-Makeham model parameters can be an interesting alternative to the Poisson distribution in practice.

Acknowledgments

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Appendix: Single life annuity and assurance

Tables 1 and 2 presents the estimated single life annuity for females and males, respectively, and Tables 3 and 4 presents the estimated single assurance for females and males, respectively.

Table 1: Single life annuity at age x for females

		Poisson distribution			Bell distribution		
Country	Year	$\bar{a}_{30}$	$\bar{a}_{55}$	$\bar{a}_{80}$	$\bar{a}_{30}$	$\bar{a}_{55}$	$\bar{a}_{80}$
France	1950	17.1022	12.4887	4.8155	17.1044	12.4850	4.8248
France	1960	17.4587	12.9585	5.0693	17.4604	12.9523	5.0845
France	1970	17.6417	13.3852	5.4943	17.6438	13.3738	5.5113
France	1980	17.8577	13.9035	5.9257	17.8600	13.8871	5.9453
France	1990	18.0751	14.4536	6.5261	18.0736	14.4351	6.5326
France	2000	18.1899	14.8346	7.0179	18.1917	14.8135	7.0296
France	2010	18.3232	15.1744	7.6283	18.3299	15.1501	7.6307
Italy	1950	17.1765	12.4601	4.7012	17.1777	12.4585	4.7024
Italy	1960	17.4557	12.8237	4.8413	17.4557	12.8204	4.8513
Italy	1970	17.6350	13.1634	5.2030	17.6354	13.1564	5.2150
Italy	1980	17.8334	13.5881	5.5237	17.8329	13.5784	5.5349
Italy	1990	18.0594	14.1676	6.1954	18.0605	14.1618	6.2080
Italy	2000	18.2400	14.6568	6.7644	18.2400	14.6443	6.7761
Italy	2010	18.3977	15.0814	7.2309	18.4005	15.0604	7.2458
Japan	1950	16.3081	11.6664	4.3816	16.3084	11.6653	4.3819
Japan	1960	17.1321	12.3221	4.5669	17.1326	12.3196	4.5744
Japan	1970	17.5192	12.9327	4.9448	17.5184	12.9280	4.9533
Japan	1980	17.9245	13.8233	5.6633	17.9243	13.8163	5.6767
Japan	1990	18.1805	14.5408	6.5222	18.1819	14.5298	6.5347
Japan	2000	18.3551	15.1093	7.5550	18.3578	15.0982	7.5662
Japan	2010	18.4783	15.4943	8.0131	18.4822	15.4814	8.0185
Sweden	1950	17.3003	12.4925	4.5913	17.2972	12.4800	4.5928
Sweden	1960	17.5757	12.9577	4.8778	17.5779	12.9557	4.9099
Sweden	1970	17.7652	13.5003	5.5415	17.7694	13.4947	5.5638
Sweden	1980	17.8942	13.8316	5.8803	17.8918	13.8141	5.8931
Sweden	1990	18.0450	14.1945	6.2452	18.0427	14.1702	6.2536
Sweden	2000	18.1778	14.4679	6.6005	18.1768	14.4352	6.5962
Sweden	2010	18.3213	14.7877	6.9270	18.3219	14.7703	6.9262

Table 2: Single life annuity at age x for males

		Poisson distribution			Bell distribution		
Country	Year	$\bar{a}_{30}$	$\bar{a}_{55}$	$\bar{a}_{80}$	$\bar{a}_{30}$	$\bar{a}_{55}$	$\bar{a}_{80}$
France	1950	16.3721	11.1269	4.1926	16.3711	11.1206	4.1968
France	1960	16.5934	11.3065	4.3242	16.5947	11.3069	4.3245
France	1970	16.7008	11.5404	4.6052	16.7009	11.5401	4.6044
France	1980	16.8693	11.8850	4.8866	16.8694	11.8783	4.8874
France	1990	17.1136	12.5576	5.3396	17.1125	12.5485	5.3397
France	2000	17.3764	13.0487	5.7413	17.3793	13.0350	5.7413
France	2010	17.6580	13.6442	6.3599	17.6622	13.6232	6.3536
Italy	1950	16.7213	11.7376	4.4404	16.7206	11.7292	4.4426
Italy	1960	16.8323	11.6674	4.4272	16.8344	11.6690	4.4341
Italy	1970	16.9202	11.7513	4.6091	16.9186	11.7501	4.6054
Italy	1980	17.0268	11.8787	4.6344	17.0299	11.8816	4.6338
Italy	1990	17.3282	12.5927	5.1688	17.3291	12.5979	5.1632
Italy	2000	17.6297	13.2351	5.5788	17.6292	13.2338	5.5749
Italy	2010	17.9443	13.9083	6.0877	17.9448	13.9023	6.0930
Japan	1950	15.8293	10.6028	3.8437	15.8330	10.6136	3.8306
Japan	1960	16.5496	11.0855	3.9095	16.5481	11.0828	3.9050
Japan	1970	16.8629	11.6113	4.2092	16.8632	11.6118	4.2112
Japan	1980	17.3016	12.4781	4.8315	17.3020	12.4743	4.8467
Japan	1990	17.5713	13.0536	5.4248	17.5711	13.0493	5.4303
Japan	2000	17.7061	13.4755	6.0147	17.7061	13.4693	6.0150
Japan	2010	17.8947	13.9294	6.3231	17.8950	13.9199	6.3274
Sweden	1950	17.0598	12.0322	4.4037	17.0592	12.0285	4.4030
Sweden	1960	17.1777	12.1503	4.5177	17.1752	12.1499	4.5108
Sweden	1970	17.2043	12.3250	4.8779	17.2049	12.3313	4.8820
Sweden	1980	17.1971	12.3133	4.8376	17.1952	12.3095	4.8385
Sweden	1990	17.4620	12.7992	5.1111	17.4634	12.8010	5.1140
Sweden	2000	17.7413	13.3796	5.4833	17.7405	13.3731	5.4862
Sweden	2010	17.9564	13.8978	5.9601	17.9559	13.8912	5.9583

Table 3: Single assurance at age x for females

		Poisson distribution			Bell distribution		
Country	Year	$\bar{A}_{30}$	$\bar{A}_{55}$	$\bar{A}_{80}$	$\bar{A}_{30}$	$\bar{A}_{55}$	$\bar{A}_{80}$
France	1950	0.1367	0.3641	0.7523	0.1367	0.3645	0.7519
France	1960	0.1225	0.3453	0.7423	0.1225	0.3459	0.7417
France	1970	0.1137	0.3243	0.7210	0.1138	0.3253	0.7204
France	1980	0.1031	0.2986	0.6993	0.1033	0.2998	0.6986
France	1990	0.0927	0.2715	0.6692	0.0929	0.2727	0.6690
France	2000	0.0868	0.2520	0.6439	0.0869	0.2534	0.6436
France	2010	0.0808	0.2360	0.6138	0.0807	0.2377	0.6142
Italy	1950	0.1339	0.3667	0.7589	0.1339	0.3670	0.7589
Italy	1960	0.1229	0.3526	0.7542	0.1230	0.3529	0.7538
Italy	1970	0.1153	0.3375	0.7371	0.1155	0.3381	0.7366
Italy	1980	0.1060	0.3170	0.7214	0.1062	0.3177	0.7210
Italy	1990	0.0951	0.2885	0.6879	0.0951	0.2889	0.6874
Italy	2000	0.0862	0.2641	0.6593	0.0863	0.2649	0.6589
Italy	2010	0.0784	0.2429	0.6359	0.0785	0.2443	0.6355
Japan	1950	0.1651	0.3910	0.7661	0.1651	0.3910	0.7661
Japan	1960	0.1369	0.3749	0.7665	0.1370	0.3751	0.7662
Japan	1970	0.1204	0.3481	0.7495	0.1205	0.3484	0.7492
Japan	1980	0.1013	0.3049	0.7142	0.1014	0.3054	0.7136
Japan	1990	0.0886	0.2691	0.6709	0.0887	0.2698	0.6704
Japan	2000	0.0800	0.2407	0.6189	0.0801	0.2415	0.6186
Japan	2010	0.0737	0.2211	0.5954	0.0737	0.2220	0.5954
Sweden	1950	0.1303	0.3688	0.7667	0.1307	0.3697	0.7667
Sweden	1960	0.1181	0.3475	0.7533	0.1181	0.3478	0.7519
Sweden	1970	0.1088	0.3206	0.7200	0.1089	0.3213	0.7191
Sweden	1980	0.1026	0.3042	0.7030	0.1029	0.3053	0.7026
Sweden	1990	0.0954	0.2866	0.6850	0.0958	0.2881	0.6848
Sweden	2000	0.0896	0.2741	0.6680	0.0899	0.2761	0.6685
Sweden	2010	0.0828	0.2586	0.6520	0.0829	0.2597	0.6523

Table 4: Single assurance at age x for males

		Poisson distribution			Bell distribution		
Country	Year	$\bar{A}_{30}$	$\bar{A}_{55}$	$\bar{A}_{80}$	$\bar{A}_{30}$	$\bar{A}_{55}$	$\bar{A}_{80}$
France	1950	0.1764	0.4375	0.7870	0.1764	0.4378	0.7868
France	1960	0.1695	0.4336	0.7832	0.1692	0.4333	0.7830
France	1970	0.1638	0.4215	0.7689	0.1637	0.4214	0.7688
France	1980	0.1558	0.4048	0.7551	0.1559	0.4053	0.7551
France	1990	0.1396	0.3656	0.7287	0.1398	0.3662	0.7288
France	2000	0.1271	0.3417	0.7088	0.1272	0.3428	0.7091
France	2010	0.1136	0.3126	0.6780	0.1138	0.3142	0.6788
Italy	1950	0.1574	0.4047	0.7731	0.1575	0.4052	0.7731
Italy	1960	0.1568	0.4146	0.7775	0.1565	0.4143	0.7770
Italy	1970	0.1537	0.4120	0.7693	0.1536	0.4118	0.7693
Italy	1980	0.1486	0.4060	0.7683	0.1484	0.4058	0.7682
Italy	1990	0.1316	0.3676	0.7398	0.1313	0.3670	0.7399
Italy	2000	0.1164	0.3351	0.7189	0.1163	0.3351	0.7191
Italy	2010	0.1011	0.3020	0.6937	0.1012	0.3024	0.6935
Japan	1950	0.1886	0.4458	0.7952	0.1885	0.4455	0.7960
Japan	1960	0.1661	0.4378	0.8004	0.1662	0.4380	0.8007
Japan	1970	0.1517	0.4127	0.7859	0.1517	0.4127	0.7858
Japan	1980	0.1327	0.3730	0.7566	0.1327	0.3733	0.7559
Japan	1990	0.1203	0.3457	0.7277	0.1203	0.3459	0.7274
Japan	2000	0.1132	0.3240	0.6977	0.1133	0.3244	0.6977
Japan	2010	0.1031	0.3001	0.6813	0.1032	0.3008	0.6812
Sweden	1950	0.1433	0.3934	0.7770	0.1432	0.3935	0.7770
Sweden	1960	0.1388	0.3893	0.7723	0.1389	0.3893	0.7727
Sweden	1970	0.1369	0.3799	0.7538	0.1367	0.3793	0.7534
Sweden	1980	0.1377	0.3810	0.7561	0.1380	0.3814	0.7562
Sweden	1990	0.1244	0.3565	0.7422	0.1243	0.3564	0.7421
Sweden	2000	0.1112	0.3284	0.7241	0.1112	0.3287	0.7239
Sweden	2010	0.1005	0.3025	0.7002	0.1006	0.3029	0.7003

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