Austrian Journal of Statistics
2023, Volume 52, 34–62.
http://www.ajs.or.at/
doi:10.17713/ajs.v52i5.1500

On the Type-I Half-logistic Distribution and Related Contributions: A Review

Mustapha Muhammad
Guangdong Univ. of Petrochemical Tech.
Bayero University Kano

M.H.Tahir
The Islamia University of Bahawalpur

Lixia Liu*
Hebei Normal University

Farrukh Jamal
The Islamia University of Bahawalpur

Christophe Chesneau
LMNO, University of Caen

Badamasi Abba
Central South University
Yusuf Maitama Sule University

Abstract

The half-logistic (HL) distribution is a widely considered statistical model for studying lifetime phenomena arising in science, engineering, finance, and biomedical sciences. One of its weaknesses is that it has a decreasing probability density function and an increasing hazard rate function only. Due to that, researchers have been modifying the HL distribution to have more functional ability. This article provides an extensive overview of the HL distribution and its generalization (or extensions). The recent advancements regarding the HL distribution have led to numerous results in modern theory and statistical computing techniques across science and engineering. This work extended the body of literature in a summarized way to clarify some of the states of knowledge, potentials, and important roles played by the HL distribution and related models in probability theory and statistical studies in various areas and applications. In particular, at least sixty-seven flexible extensions of the HL distribution have been proposed in the past few years. We give a brief introduction to these distributions, emphasizing model parameters, properties derived, and the estimation method. Conclusively, there is no doubt that this summary could create a consensus between various related results in both theory and applications of the HL-related models to develop an interest in future studies.

Keywords: generalized half logistic distribution, half logistic distribution, parameter estimation.

1. Introduction

There have been several classical distributions in probability and statistical studies, such as exponential, Weibull, Burr, logistic, normal, gamma, beta, half logistic, and Pareto distributions, among others. Some of these models were motivated by practical applications (derived from the data), while others were developed due to mathematical interest. Most of the related models have some closed-form and manageable properties and several computer packages to ease utilization. These models have been utilized in various studies such as reliability, life testing, physics, biomedical sciences, engineering, finance, and economics, among others.

The half Logistic (HL) distribution was proposed by Balakrishnan (1985). It may be defined as a continuous distribution derived from the absolute value of a random variable following the logistic distribution; if Y is a random variable with the logistic distribution, then $X = |Y|$ is a random variable with the HL distribution. The HL model is one of the probability models that has received considerable attention from practitioners from various statistical studies and application areas. Similar to other classical distributions, the HL distribution appears to have some weaknesses and limitations when dealing with complex data. For instance, the HL distribution can accommodate decreasing probability density function (PDF) and increasing hazard rate function (HRF). It is however unable to cover non-monotone PDF or HRF. This inability is a critical situation concerning the studies of a complex phenomenon that occurs in reliability studies, human mortality analysis, lifetime testing, engineering modeling, physics, electronics, communication, medical sciences, public health, and biomedical sciences. The HRF behavior can be an upside-down bathtub, a bathtub, or other non-monotone.

Practitioners addressed such issue by extending existing models and developing various techniques for combining two or more models to produce new flexible ones that produce more accurate results than the parent model. For instance, Tahir and Cordeiro (2016); Ahmad, Hamedani, and Butt (2019) provided a review regarding various techniques for generating new models and the list of models developed through several techniques. Tomy, Jose, and Veena (2020) gave a summary regarding exponential distribution-related studies and applications. Several researchers provided an in-depth examination of Weibull distributions, Weibull-related models, and applications. One can see Almalki and Nadarajah (2014); Luko (1999); Arthur and Hallinan (1993); Quinn and Quinn (2010); Lai, Murthy, and Xie (2006), among others. Alshkaki (2020) gave some review and tabulated the list of names of some generated models related to Kumaraswamy distributions and their generalizations. Selim (2020) provided a brief survey on several extensions and modifications of the Kumaraswamy and beta distributions proposed in the literature. Recently, Tomy and Satish (2021) provided an overview of recent developments in trigonometric transformation of statistical distributions. Rahman, Al-Zahrani, Shahbaz, and Shahbaz (2020) provided an overview of recent developments in transmuted statistical distributions, among other things.

The HL distribution was extended (or generalized) through different available techniques due to its importance in real-life studies. Concerning the techniques and mixture of models, in some cases, the constructed model may appear to have more complicated properties such as the closed-form expressions of the quantile function (Q), HRF, skewness and kurtosis, entropy, stress strength reliability parameter (R), structural properties, extreme value analysis, characterization of models from well-established theorems, dealing with different methods of parameter estimation, testing results numerically or by simulation, and goodness-of-fits test. These complications are no longer significant due to the availability and utilization of several advanced computational and mathematical ideas, power series, sampling techniques, and computer packages available in `R`, `Mathematica`, `Python`, `OX`, `Matlab`, and `Maple`, among others. Several extensions of the HL distribution are based on the exponentiation technique Gupta, Gupta, and Gupta (1998), beta-generator of distributions (beta-G) with $G(.)$ any valid CDF by Eugene, Lee, and Famoye (2002), Kumaraswamy-G Cordeiro and de Castro (2011), Topp-Leone-G Al-Shomrani, Arif, Shawky, Hanif, and Shahbaz (2016), Mc.Donald-G Alexander, Cordeiro, Ortega, and Sarabia (2012), T-X families Alzaatreh, Lee, and Famoye (2013), new extended beta Muhammad and Liu (2021a), exponentiated T-X families Alzaghal, Famoye, and Lee (2013), sine-G Kumar, Singh, and Singh (2015b), exponentiated sine-G Muhammad, Alshanbari, Alanzi, Liu, Sami, Chesneau, and Jamal (2021a), Gamma-G Type-III Torabi and

Montazeri Hedesh (2012), new extended Topp–Leone-G Mustapha, Lixia, Badamasi, Isyaku, Mouna, Hexin, and Sani (2022), transmuted-G Shaw and Buckley (2009), and other transformations, convolution, and compounding methods. Tahir and Cordeiro (2016) provided a comprehensive review regarding various techniques for generating new flexible models.

On the other hand, several researchers have studied different aspects of inferential procedures regarding the HL distribution. The aspect of parameter estimation is significant for any probability distribution. Therefore, various estimation methods are frequently provided and studied through various viewpoints, such as censored data, uncensored data, group data, and record values. The traditional estimation methods include the maximum likelihood estimation (MLE), method of moment estimation (MME), least-square estimation (LSE), and weighted least-squares estimation (WLSE). Other important methods include the Bayes estimation (BE) under different objective priors and different loss functions, maximum product spacing estimation (MPSE), percentile estimation (PE), approximate maximum likelihood estimation (AMLE), modified maximum likelihood estimation (MMLE), modified method of moments estimation (MMME), maximum product spacing estimation (MPSE), Anderson-Darling estimation (ADE) and Cramér-on Mises estimation (CVME), among others. In most of the inferential studies, the estimators' performances were examined by conducting Monte Carlo simulation studies via the bias, mean square error, and coverage probability with 95% confidence level, among others. The results are supported by the application of relevant real data for illustrative purposes. Here, we emphasize the techniques and some significant results presented in various articles through the HL distribution.

In this work, we aim to provide an overview regarding the HL distribution, as well as synthesize and extend the body of literature in a summarized way to clarify the state of knowledge, importance, and role of the HL distribution in probability and statistical studies. As a result, we hope to provide a summary that will help researchers identify the need for and develop interest in using the newly constructed models related to the HL distribution in their various fields of research. This review could create a consensus between various related results in both theory and applications of the HL distribution to develop interest in future studies. This can produce more accurate and reliable results, allowing for better real-life exploration and human progress.

2. The HL distribution and some related contributions

This section gives a brief review of the HL distribution and some extensions. The related contributions by researchers are extensively summarized.

2.1. The HL distribution

The HL distribution is an important model in reliability, biomedical studies, engineering, finance, computer science, communication, insurance, and life-testing. The distribution is defined on the positive real number and has a monotone increasing HRF. Some properties of the HL distribution and related works are discussed.

Definition 1. If X is a random variable following the HL distribution with a scale parameter $\alpha > 0$, then X has the CDF $(F(x))$ and corresponding PDF $(f(x))$ given by

$$F(x) = \frac{1-e^{-\alpha x}}{1+e^{-\alpha x}}, \quad f(x) = \frac{2\alpha e^{-\alpha x}}{(1+e^{-\alpha x})^2}, \ \ \alpha, x > 0,$$

respectively.

In some literature, the scale parameter α appears in the denominator of the variable. The survival function (SF) ($s(x)$) and (HRF) ($h(x)$) of the HL distribution are given by

$$s(x) = \frac{2\, e^{-\alpha x}}{1 + e^{-\alpha x}}, \quad h(x) = \frac{\alpha}{1 + e^{-\alpha x}}, \ x > 0,$$

respectively.

Figure 1 provided the plots of the PDF and HRF of the HL distribution. We can notice that the PDF is always a decreasing function since $\frac{d \log f(x)}{dx} = -\alpha\,(1 - e^{-\alpha x})\,/\,(1 + e^{-\alpha x}) < 0$ for all $\alpha > 0$. The HRF is always an increasing function since $\frac{d \log h(x)}{dx} = \alpha e^{-\alpha x}/\,(1 + e^{-\alpha x}) > 0$ for all $\alpha > 0$.

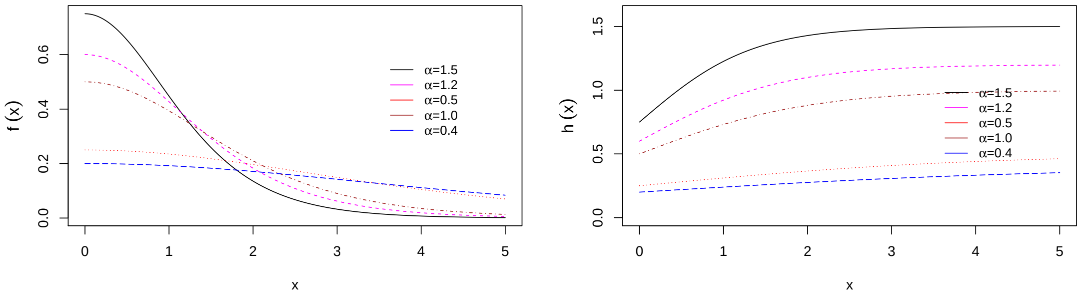

Figure 1: Plots of the PDF and HRF of the HL distribution for some parameter values

The quantile function of the HL distribution is given by

$$Q(u) = -\frac{1}{\alpha} \log\left(\frac{1-u}{1+u}\right), \quad u \in (0, 1).$$

The median of the HL distribution is $Med = 1.0986/\alpha$ and the mode is $Mode = \alpha/2$. Olapade (2003) provided some characterizations results based on some homogeneous differential equation and log-transformation from the HL distribution.

The following result is about several equivalent expressions of the moments of the HL distribution.

Theorem 2.1. *Let X be a random variable with the HL distribution with scale parameter $\alpha > 0$. Then the r^{th} moment of X can be expressed as*

1. *First expression (see Balakrishnan and Wong (1991)):*

$$E[X^r] = r!\, 2\, \eta(r), \quad \alpha = 1,$$

where $\eta(r)$ is the Dirichlet eta function. In particular, we have $E[X] = \log 4$ and $E[X^2] = \pi^2/3$.

2. *Second expression (see Oliveira, Santos, Xavier, Trindade, and Cordeiro (2016)):*

$$E[X^r] = r!\, 2\,(1 - 2^{1-r})\zeta(r), \quad \alpha = 1,$$

where $\zeta(r)$ is the Riemann zeta function.

3. *Third expression:*

$$E[X^r] = 2\left(-\frac{1}{\alpha}\right)^r \sum_{i=0}^{\infty} (i+1)\mathcal{B}_{r,0}(i+1, 1), \quad \alpha > 0,$$

where $\mathcal{B}_{p,r}(t, z) = \partial^{p+r}\mathcal{B}(t, z)/(\partial t^p\, \partial z^r)$ and $\mathcal{B}(t, z) = \int_0^1 (1 - v)^{z-1} v^{t-1} dv$ is a beta function, $t, z > 0$, available in `https://www.wolframalpha.com/input/?i=d`

In addition, according to Balakrishnan and Wong (1991), the moment generating function of X is given by

$$M_X(t) = 2J_1(1+t, 1-t), \;\; \alpha = 1,$$

where $J_p(a,b) = \int_0^p [u^{a-1}/(1+u)^{a+b}]du$ *is the type-II incomplete beta function.*

In some literature, some researchers studied the two-parameter HL distribution with scale and location parameters defined with the CDF given by

$$F(x) = \frac{1-e^{-\frac{x-\theta}{\alpha}}}{1+e^{-\frac{x-\theta}{\alpha}}}, \;\; x > \theta, \alpha, \theta > 0.$$

The shape parameter was added to the HL distribution by the exponentiation method of Gupta *et al.* (1998), and called the exponentiated (EHL) (or generalized) HL distribution (see Kantam, Ramakrishna, and Ravikumar (2013)). The CDF of the EHL distribution is generally defined by

$$F(x) = \left(\frac{1-e^{-\frac{x-\theta}{\alpha}}}{1+e^{-\frac{x-\theta}{\alpha}}}\right)^{\beta}, \;\; x > \theta, \alpha, \theta, \beta > 0,$$

In most cases, the location parameter $\theta = 0$, and the scale parameter is multiplied by the variable. If $\beta = 1$, it is the HL distribution. The moment of the EHL distribution is given in Oliveira *et al.* (2016) and Muhammad (2016). When $\alpha = 1$, some properties of the HL distribution can be deduced from the EHL distribution. Seo and Kang (2015b) demonstrated some relationships between the EHL distribution and the exponential distribution, type-I Pareto distribution, uniform distribution, Weibull distribution, and Gumbel distribution via log transformation.

2.2. HL and related contributions

Here, we discussed pioneers' contributions regarding the HL, GHL and some other related distributions in theoretical studies and practice.

Balakrishnan (1985) established certain recurrence relations for the moments and product moments of order statistics from the HL distribution. Later, the results were generalized by Saran and Pande (2012). Balakrishnan and Puthenpura (1986) included a table of values with the coefficients of the best linear unbiased estimators (BLUEs) of location and scale parameters of the HL distribution and the values of the variances and covariance of these estimators. Balakrishnan and Wong (1994) extended their previous results in Balakrishnan (1985); Balakrishnan and Puthenpura (1986) and tabulated the BLUEs when the samples are singly and doubly type-II censored from the HL distribution. Also, they presented the values of the variances and covariance of these estimators; and provided two examples in which tabulated values are applied. Sultan, Mahmoud, and Saleh (2006) derived the recurrence relations for the single and product moments of progressively type-II right censored order statistics from the HL distribution, and discussed the MLEs of its location and scale parameters by considering Balakrishnan, Balakrishnan, and Aggarwala (2000). Rao and Kantam (2012a) offered a chart of extreme values and a mean analysis based on the HL distribution. Liu and Balakrishnan (2020) developed an algorithm for computing single and product moments of order statistics from the HL distribution based on complete and type-II censored samples, as well as recurrence relations that enable the computation of means, variances and covariances of all order statistics for all sample sizes in a simple and efficient way. In addition, the findings demonstrated that the results are crucial in defining the BLUE of the scale parameter HL distribution, as well as the parameters of the more general location-scale.

Balakrishnan and Wong (1991) showed that the MLEs of the HL distribution parameters are not explicit for the scale parameter based on either complete or right censored samples.

The author next showed how to construct an explicit estimator by approximating the likelihood function. This estimator's bias and variance are explored, demonstrating that it is as efficient as the BLUE. By calculating the PDF of the scaled HL distribution based on progressively type-II censored samples, Balakrishnan and Asgharzadeh (2005) developed a straightforward method for constructing an explicit estimator. Some wide range of sample sizes and progressive censoring schemes were considered. Then the bias and variance of the estimator were examined. It is shown that the estimator is as efficient as the MLE. Later, it was shown that the asymptotic normality-based coverage probability of pivotal values is insufficient, especially when the effective sample size is small. As a result, it was suggested that, for the generation of confidence intervals, unconditional simulated percentage points of the pivotal values can be used. The bias reduction for the MLEs of the parameters in the HL distribution was described in Giles (2012), and it was found that the bias-corrected estimator of the location parameter outperforms its bootstrapped-based counterpart significantly. By estimating the likelihood function and obtaining the asymptotic variances of MLE and the approximate MLE, Kim and Han (2010) calculated the scale parameter of the HL distribution under progressively type-II censored sample and derived an explicit estimator. Also, they used the Laplace approximation and significance sampling approaches to calculate the BE.

In the setting of the HL distribution, the MLEs and BEs of the scale parameter under the multiple type-II hybrid censoring scheme were investigated in Jeon and Kang (2020a). The least variance unbiased estimators, simple estimators, and ranked-set sample estimators were studied in Adatia (2000). Based on a progressively type-II censored sample, AL-Hussaini, Abdel-Hamid, and Hashem (2015b) investigated Bayesian interval prediction under progressive stress accelerated life testing. The prediction bounds of future order statistics are determined by assuming that the lifetime of a unit under usage condition stress follows the HL distribution with a scale parameter obeying the inverse power law. Seo and Kang (2015c) provided a new approach for estimating the scale parameter of the HL distribution based on progressively type-II censored samples that is based on a pivotal quantity, and the proposed method gives a simpler estimate equation as the likelihood equation. Kang, Cho, and Han (2009) established a double hybrid censoring technique and derived some AMLEs of the scale parameter for the HL distribution. The scale parameter was calculated using two different methods of Taylor series expansion and the approximate MLE method. The suggested approach additionally discusses the MLE and LSE of the scale parameter.

The HL distribution was used to develop the reliability test strategies in Rosaiah, Kantam, and Rao (2009). According to the amount of failures from each group, Rao and Rao (2016) considered a two-stage group acceptance sampling strategy based on life tests that follow the HL distribution, where the choice on lot acceptance can be taken in the first or second stage. When group size and test length are established, the suggested plan's design parameters are determined to fulfill the consumer's risk at the stated unreliability, such as the number of groups necessary and the acceptance number for each of two stages. In terms of average sample quantity and operating parameters, a special case of the proposed plan is compared to the proposed plan. Seo and Kang (2015a) calculated the entropy of the HL distribution using the scale parameter and type-II censored samples, as well as providing a MLE and an approximate confidence interval for entropy. The hierarchical structure of the gamma prior distribution, which creates a noninformative prior, has been used to construct hierarchical BE methods. Panichkitkosolkul and Saothayanun (2012) used a conventional bootstrap confidence interval, percentile bootstrap confidence interval, and bias-corrected percentile bootstrap confidence interval to create the bootstrap confidence intervals for the process capability index. Prasad, Sowmya, and Mahesh (2016) addressed a control mechanism based on the mean value function of the HL distribution based on a non-homogeneous Poisson process that was planted on the cumulative observations of interval domain failure data and utilized MLE to estimate the model's unknown parameters. For skewed distributions, Rao and Kantam (2012b) mentioned variable control charts for the average and range of subgroups. These are constructed using the percentiles of sampling distributions of sample mean and sample interval, as well as the skewness correction process based on the coefficient of skewness, and the coverage probabilities in both ways are compared using simulation.

The Bayesian prediction boundaries of order statistics based on progressively type-II censored competing risks data from a general class of distributions were investigated in AL-Hussaini, Abdel-Hamid, and Hashem (2015a), and the results were subsequently applied to the HL distribution. Future order statistics prediction intervals were also calculated using three alternative progressive type-II censoring strategies. Based on a gradually type-II censored sample, Wang (2008) provided an exact confidence interval and an exact test for the scale parameter of the HL distribution. Kang and Park (2005) used several type-II censored samples to construct AMLEs of this scale parameter. Adatia (1997) gave an approximation of the size and location parameters of the HL distribution based on a few order statistics such as (k) for reasonably large doubly censored samples. Also, for sampling sizes of $N = 10(1)40$ and $k = 2(1)4$, tabulated the ranks, coefficients, variances, and relative efficiencies of the BLUEs and approximate BLUEs of the parameters based on a selected few order statistics (k). The variances obtained were then compared to those of the AMLEs. Seo and Kang (2017) introduced a new approach for estimating the scale parameters of the two-parameter HL distribution based progressively type-II censoring scheme based on a pivotal quantity based on an unbiased estimator of the location parameter based on a pivotal quantity. In comparison to the maximum likelihood equation, the proposed method gave a simpler estimate equation. Using a pivotal quantity, the shortest-length prediction intervals for the censored failure times are then calculated.

On the basis of a doubly type-II censored sample, Jang, Park, and Kim (2011) investigated the MLE and BE of the scale parameter of the HL distribution. The MLE and the Bayes estimator appear to be implicit. To discuss the asymptotic variances and AMLE, a simple approach of approximating the likelihood function to produce an explicit estimator was utilized. In Tierney and Kadane (1986), the Bayes estimator was calculated using the Laplace approximation. Seo and Kang (2016b) proposed an objective Bayesian approach for estimating the parameter under a gradually type-II censoring scheme that took into account the Jeffreys and reference priors, which were also evaluated to see if they matched probability-matching criteria. Due to the nature of their posterior PDFs, the Metropolis Hasting technique was used to create Markov chain Monte Carlo samples. Under noninformative priors, Kang, Kim, and Lee (2014) proposed Bayesian hypothesis testing procedures for the equality of the scale parameters of the HL distribution. As a result, the Bayes factor is generally incorrect and results in a calibration challenge, requiring the Bayes factor to be defined up to a multiplicative constant. As a remedy to the problem, default Bayesian hypothesis testing procedures based on the fractional Bayes factor and the intrinsic Bayes factors under the reference priors were presented. The scale parameter of the HL distribution was estimated using a type-I progressively hybrid censoring strategy in Wang and Liu (2017). The highest posterior confidence intervals for the relevant highest posterior confidence intervals were produced. On the basis of the generalized adaptive progressive hybrid censored sample, Cho and Lee (2021) proposed the generalized adaptive progressive hybrid censoring scheme and the estimate of the parameter of the HL distribution. The MLE was used to estimate the parameter, and Taylor series expansion was used to approximate the MLE. The Bayes estimator is based on the squared error loss function, and it is calculated using the Tierney and Kadane approximations. The noninformative priors for the ratio of the scale parameters in the HL distribution were established by Kang, Kim, and Lee (2012). Furthermore, the second order matching prior matches the alternative coverage probabilities and the highest posterior PDF matching prior when the first and second order matching priors are provided. The one-at-a-time reference prior and Jefereys' prior are then revealed to be second-order matching priors. Later, using a simulation analysis and real data, the proposed reference prior matches the intended coverage probability in a frequentist sense.

Under a multiply progressive censoring method, Park and Lee (2020) used the MLE of the scale parameter of the HL distribution using Taylor series expansion. In the estimation of parameters of the HL distribution, Adatia and Saleh (2020) described the generalized ranked-set sampling technique. The novel estimators are based on ranked-set sample estimators and linear minimum variance unbiased estimators. Also tabulated are the coefficients, variances, and relative efficiencies. Based on multiple type-II censored data, Kang, Cho, Han, and Sakong (2010) constructed four modified empirical distribution function type tests for the HL distribution using AMLEs. Modified normalized sample Lorenz curve plot and new test statistics for the model were also proposed. On the basis of a future sample from the HL distribution, AL-Hussaini, Abdel-Hamid, and Hashem (2020) addressed the point and interval predictions of the sth order statistic. The informative and future samples based on gradually type-II censored under competing risks model and assumption were discussed. In addition, six distinct progressive filtering techniques are subjected to numerical calculations. Furthermore, biases, mean squared prediction errors of maximum likelihood predictors, coverage probabilities, and average interval lengths of Bayesian prediction intervals were all taken into account. The estimate of the scale parameter for the HL distribution using nearly best unbiased estimators utilizing Blom's approach, least square method for the BLUEs, MLE, and approximate maximum likelihood were covered in Jamjoom (2003). The BLUEs were also explored using two optimally determined order statistics. Based on the asymptotic distribution of the MLEs, log-transformed MLE, pivotal quantity, and generalized pivotal quantity, Potdar and Shirke (2017) computed the confidence interval for the scale parameter of the HL distribution. The EM algorithm is used to generate the MLEs. Also, using coverage probability, length, and the coverage-to-length ratio, they compared the performance of the confidence intervals suggested by Balakrishnan and Asgharzadeh (2005).

The generalized upper (k) record values and generalized lower (k) record values coming from the HL and inverse HL (IHL) distributions were considered by Thomas and Paul (2019). In addition, based on some moment relations of generalized upper (k) record values and those of generalized lower (k) record values, some characterization results of the HL distribution were derived, and diagnostic tools were devised to identify the HL distribution as a model to the distribution of a population. For the IHL distribution, similar characterization theorems and diagnostic tools were developed. In order to estimate the quantiles of the HL distribution, Adatia and Saleh (2004) used the generalized ranked-set sampling technique. Also taken into account are the lowest variance unbiased estimator, simple estimator, and ranked-set sample estimator. The relative efficiencies, variances, and coefficients were tabulated. The performance of the estimators of the HL distribution was compared in Asgharzadeh, Rezaie, and Abdi (2011). MLEs, AMLEs, technique of MMEs, PEs, LSEs, WLSEs, and estimators based on linear combinations of order statistics were used in this study. Awodutire, Olapade, Kolawole, and Ilori (2018) used the type-I generalized HL distribution to estimate breast cancer patients' survival periods. The three-parameter type-I generalized HL survival model was applied to breast cancer survival data using MLEs by Awodutire, Olapade, and Kolawole (2016). In Seo and Kang (2015b), the exponentiated HL (EHL) distribution's moment estimators and MLEs were discussed. The estimate of the entropy was determined. The approximate confidence intervals for unknown parameters are calculated using the exact expression of Fisher information. The log transformation was used to provide a relationship with the EHL and some other models. Chaturvedi, Kang, and Pathak (2016) used uniformly minimum variance unbiased estimators and MLEs to construct estimate and testing techniques for the generalized HL (GHL) distribution's SF under type-II censoring and sampling strategy. The MLEs for the GHL distribution were provided by Seo, Lee, and Kang (2012) based on upper records values. Seo and Kang (2014) used BE and type-II censored samples, the entropy estimators of the GHL distribution were obtained. For the single and product moments of increasingly type-II right-censored order statistics from the GHL distribution, Balakrishnan and Saleh (2013) found many recurrence relations. Later, these relations were used in a logical recursive fashion, allowing all of the means, variances, and covariances of progressively type-II right-censored order statistics from the GHL distribution to be computed for all sample sizes and progressive censoring methods. Balakrishnan and Aggarwala (1996) proposed that the single and product moments for order statistics from the right truncated GHL distribution satisfy several recurrence relations, and that the relationships can be used in a simple recursive manner to compute the single and product moments of all order statistics for any sample size and truncation parameter. The MLEs and AMLEs of the scale parameter of the GHL and HL distributions based on progressively type-II censored data were discussed in Kang and Seo (2011); Kang, Cho, and Han (2008). The proposed estimators were utilized to calculate the SF's estimates.

For the EHL distribution, Rastogi and Tripathi (2014) looked at estimating the SF and HRF using progressive type-II censoring. Under Linex and entropy loss functions, MLEs and BEs are examined. The Lindley method is used to get BE. The BEs are computed using Markov Chain Monte Carlo and the importance sampling strategy. For the unknown parameters, credible intervals, asymptotic confidence intervals, bootstrap-p, and bootstrap-t are built. Arora, Bhimani, and Patel (2010) investigated the MLEs of the GHL distribution parameters under type-I progressive censoring with changing failure rates. For the type-I GHL distribution, Kumar, Jain, and Gupta (2015a) derived some new explicit expressions and recurrence relations for marginal and joint moment generating functions of upper record values. Also, utilizing the recurrence relation for a single moment and conditional anticipation of upper record values, the characterization result of this distribution was produced. The MLEs of the top record values were discussed, as well as their confidence intervals. Gui (2017) investigated the problem of using MLEs, inverse moment and modified inverse moment estimation, and Monte Carlo method to estimate the unknown parameters of the EHL distribution. The performance of two approaches for constructing joint confidence regions was discussed.

When the life test is shortened at a pre-specified time, Rao and Naidu (2014) established acceptance sampling plans for the EHL distribution percentiles. To ensure that the stated life percentile is obtained under a given customer's risk, a minimum sample size is required. The operational characteristic values (and curves) of the sample plans, as well as the producer's risk, were then given. Liu, Shi, Bai, and Zhan (2018) investigated the reliability estimation of a multicomponent system, dubbed the N-M-cold-standby redundancy system, using a progressive type-II censoring sample for the GHL distribution with various shape parameters. There are N subsystems in the system, each made up of M statistically independent distributed strength components, and only one of these subsystems works under the influence of the stresses at any given moment, with the others acting as standbys. When one of the working subsystems fails, one of the backups takes over. When all of the subsystems fail, the system fails. The uniformly minimum variance unbiased estimator, MLE, BE under squared error loss function utilizing the Gauss hypergeometric function were used to measure the system's dependability. The approximate greatest PDF credible interval and asymptotic confidence interval were created. On the basis of increasingly type-II censored data under various loss functions, Kim, Kang, and Seo (2011b) presented BE of the shape parameter and SF in the GHL distribution. Based on a progressively type-II censored sample from the GHL distribution, Azimi, Yaghmaei, and Fasihi (2013) investigated the Bayesian and E-Bayesian (expectation of the BE) estimators for the parameter and SF under the LINEX and squared-error loss functions. The moment of order statistics and MLE based on complete observations of the five parameter type-I GHL distribution were explored in Bello, Awodutire, Sule, and Lawal (2020). On the basis of comprehensive and censored data, Torabi and Bagheri (2010) estimated the parameters for an extended GHL distribution.

Under the type-II hybrid censoring, Seo, Kim, and Kang (2013) calculated the MLEs and AMLEs of unknown parameters in the GHL distribution, as well as the estimated confidence. If record values have an EHL distribution, Seo and Kang (2016a) developed more efficient methods for estimating shape parameters in the presence of a scale parameter from Bayesian and non-Bayesian viewpoints. Also offered are estimate methods based on pivotal quantities, and a robust estimation method was established in the Bayesian approach by constructing some hierarchical structure of the parameter of interest. Chaturvedi and Kumari (2017) explored robust Bayesian analyses for the shape parameter of the GHL distribution using a $\epsilon-$contamination class of priors, taking into account type-II censoring and the sampling technique in Bartholomew (1963). To ensure the product's quality over time, Rao and Naidu (2016) created a resubmitted lot for group acceptance sampling plan. By assuming that the product lifetime follows the EHL distribution, the plan parameters were determined by setting the experiment termination time and the number of testers to meet both the producer and customer risks at the same time. If the lifetime is shortened at a pre-assigned time with a known index parameter, Rao and Naidu (2015) considered group acceptance sampling plans for the EHL distribution. When both the producer's and consumer's risks correspond to the specified quality standards, the number of groups and acceptance number of the suggested sampling plan are determined. The percentiles were used to calculate the termination time and the number of components in each group. Azimi, Yaghmaei, and Babanezhad (2012) developed BEs for the parameters of the GHL distribution under progressively type-II censored samples using the LINEX loss function, precautionary loss function, and entropy loss function. The median ranks approach (Benard's approximation) was used to estimate the parameters of the GHL distribution in Mohan, Rajasekharam, and Anjaneyulu (2016). Kim, Kang, Han, and Seo (2011a) analyzed the parameters of the GHL distribution using profile likelihood estimation and approximated the maximum profile likelihood estimates for the scale parameter using progressively type-II censored data. Because of the complex form of their usual MLE, Hu and Gui (2019) estimated the scale parameter of the EHL distribution under progressively type-II censoring and general progressively type-II censoring. The scale parameter was approximated by the MLE based on Taylor expansion and estimation based on pivotal quantities. After that, based on pivotal quantities, asymptotic confidence intervals were calculated.

The estimation of the stress-strength reliability using paired observations with ties from bivariate EHL distribution was studied in Xavier and Jose (2022). The significance sampling technique is used to assess MLE and BE with a squared error loss function and a gamma prior. For the EHL distribution, Raqab, Bdair, Rastogi, and Al-Aboud (2021) looked at frequentist and Bayesian ways of estimating unknown parameters. Point estimators and accompanying confidence intervals for the parameters are obtained using the expectation-Maximization, Lindley's approximation, and Metropolis-Hastings algorithms. If the lifetime of the units follows the EHL distribution, Rao, Rosaiah, and Naidu (2020a) established a multiple postponed state sampling plan for a time-truncated life test. The ideal parameters of the proposed plan, such as the number of subsequent lots required to decide whether to accept or reject the current lot, sample size, and rejection and acceptance numbers, were determined using the two-point technique. The suggested plan's execution was demonstrated with examples, and a table is created for various combinations of consumer and producer risks. Kumar and Farooqi (2017) used the type-I GHL distribution to establish some clear formulas and recurrence relations for the marginal and joint moment generating functions of generalized order statistics. The conditional expectation of generalized order statistics was also used to characterize the distribution. Rao, Rosaiah, and Naidu (2020b) calculated the number of groups required for each of two steps of EHL distribution in order to reduce the average sample size while fulfilling the producer and consumer risks. In addition, single-stage group sampling plans were evaluated as special cases of the stated plan and were compared to the suggested plan in terms of average sample size and operating characteristics. The likelihood approach, parametric percentile bootstrap, and bootstrap-t confidence intervals were explained in Almarashi (2020) when the lifetime of units under normal conditions follows the GHL lifetime distribution based on progressive type-II censored schemes. The ideal parameters and control limits are determined using an attribute np control chart built utilizing re-sampling techniques for monitoring non-conforming items under the EHL distribution described by Tanveer, Azam, Aslam, and Shujaat Navaz (2020). The control constants were established by considering the goal in-control ARL at a normal process, and the operational formulas for in-control and out-of-control average run lengths (ARLs) were derived.

The stress-strength reliability parameter estimation from the GHL distribution under type-I and type-II censoring was studied in Chaturvedi and Nandchahal (2017). The same aim was discussed using a generalization of the power transformed HL (PTHL) distribution in Xavier and Jose (2021b). Jeon and Kang (2020b) used the multiple type-I hybrid censoring approach to estimate the scale parameter of the EHL distribution, assuming the shape parameter was known. The maximum likelihood was calculated using two alternative Taylor series expansions, as well as BEs for the squared error and general entropy loss functions. The asymptotic confidence interval and the highest posterior density interval were obtained. The highest posterior density interval and the asymptotic confidence interval were calculated. The MLEs of the parameters of the Poisson HL (PHL) distribution were investigated using progressive type-II censoring and the predictive of future order statistics using one and two-sample Bayesian prediction techniques in Abdel-Hamid (2016). The prediction intervals' greatest conditional density and coverage probabilities were also estimated. Furthermore, Muhammad and Liu (2021b) investigated the characterization of the PHL distribution using truncated moments of a random variable function and its link to other models using log transformation. The stress-strength reliability parameter based on complete samples based on random variables with the PHL distribution was extensively explored in Muhammad, Wang, Li, Yan, and Chang (2020). Under various loss functions of the reliability parameter, MLEs and BEs utilizing gamma prior were investigated. Also determined were the approximation asymptotic confidence interval, non-parametric percentile bootstrap confidence interval, student's bootstrap confidence interval, and greatest posterior density credible interval. Lee and Cho (2017) used the MLEs and BEs under squared error loss, general entropy, linex loss functions, and balanced loss functions to estimate the parameters, reliability, and HRF of the inverted exponentiated HL (IEHL) distribution from progressive type-II censored data. To get BEs, the Lindley approximation and importance sampling approach were used. The highest posterior density credible intervals and approximate confidence intervals of the parameters are computed. In addition, Zhang and Gui (2020) investigated the inferences of unknown parameters, as well as the SF and HRF of the IEHL distribution by using progressive first-failure censoring as an extension and refinement of progressive censoring. Under unbalanced and balanced losses, such as general entropy loss function, squared error loss function, and Linex loss function, MLEs and BEs were produced. The Lindley approximation approach was used to establish an approximate explicit representation of BE. Coverage probability is taken into account as well as credible intervals and asymptotic confidence intervals. The reliability of the multicomponent stress strength model based on the PTHL distribution was investigated in Xavier and Jose (2021a). With either the Weibull or the PTHL distribution, the strength components susceptible to common stress were considered to be independent. The maximum likelihood approach was used to calculate the multicomponent stress-strength reliability and its asymptotic confidence interval. Furthermore, BEs of multicomponent stress-strength reliability were produced using gamma priors and a squared error loss function. The BE and one-sample Bayes prediction of future observables from the HL generated Weibull (HLGW) distribution were explored in AL-Hussaini and Abdel-Hamid (2017).

Aldahlan (2019) considered the entire sample when discussing MLE, LSE, WLSE, PE, and CVME procedures for the shape and scale parameters of the HL Lomax (HLL) distribution. For the HL-truncated exponential (HLTE) distribution, Gul and Mohsin (2021) derived recurrence relations for the single moments and product moments of order statistics for various values of scale parameter. The recurrence relations are used to tabulate the means, variances, and covariances in samples of size $n = 1(1)10$ and to determine the first k moments of all order statistics quickly and easily. Swathi and Anjaneyulu (2020) proposed utilizing the median rank approach (Benard's approximation) to estimate the location and scale parameters of the HL Rayleigh (HLR) distribution. For both parameters in the whole sample, the average estimate, variance, standard deviation, mean absolute deviation, mean square error, simulated error, and relative absolute bias were also computed. Swathi, Lakshmi, and Anjaneyulu (2020) examined the performance of the change point technique in the case of the HLR distribution, taking average run lengths into account, and determined that the HLR model is good at identifying tiny shifts.

3. HL and related models

As mentioned in Section 1, practitioners and statisticians proposed a variety of models and model generators that allow the use of existing classical models to provide more flexible ones capable of accommodating variations of an assumed random phenomenon. Thus, it leads to several explorations of mathematical ideas, techniques, algorithms, and simulations in various fields of study. The generated models possess some advantages in modeling non-monotonic HRFs and skewed PDFs; thus, they explain better how the lifetime phenomenon occurs in various studies. Many authors in the literature consider HL-type models to be extended or generalized through various techniques. In most of the studies, several important properties of the newly proposed models are explored, and the estimation of the unknown parameters of the model is also discussed. Finally, the applications of the model in other fields of study are presented by fitting the model to real data and comparing it to other existing models.

This section focused on the number of parameters, some valuable properties, and estimation methods regarding the model discussed by the authors when introducing the distribution. We provide some abbreviations for some terms discussed in this work in Table 1. In Tables 2 and 3, we also present the list of some available probability models related to the HL distribution proposed in the literature.

Table 1: Abbreviations of some important properties and estimation methods

Moments (M)	Maximum likelihood estimation (MLE)
Quantile (Q)	Maximum likelihood for censored data (MLEc)
Vector of parameter (ξ)	Maximum likelihood for group data (MLEg)
Order statistics (Os)	Modified maximum likelihood estimation (MMLE)
Mean deviation (Md)	Bayes estimation (BE)
Bonferroni curve (Bc)	Bayes estimation for censored data (BEc)
Lorenz curve (Lc)	Least-square estimation (LSE)
Percentile estimation for censored data (PEc)	Least-square estimation for censored data (LSEc)
Percentile estimation (PE)	Weighted least-squares estimation (WLSE)
Residual entropy (ReE)	Weighted least-squares estimation for censored data (WLSEc)
Shannon entropy (SE)	Renyi entropy (RE)
q-Entropy (qE)	Mellin transform (Mt)
Havrda and Charvát entropy (HCE)	Arimoto entropy (AE)
Tsallis entropy (TE)	Kullback-Leibler divergence (KLD)
Log transformation (Lt)	Method of moments estimation (MME)
Moment of residual life (MRL)	Modified method of moments estimation (MMME)
Characterization (Ch)	Maximum product of spacings estimation (MPSE)
Stress–strength reliability parameter (R)	Anderson–Darling estimation (ADE)
Probability weighted moments (PWM)	Crammer–von Mises estimation (CVME)
Stochastic ordering (So)	

Table 2: Some probability models related to the HL distribution

	Model	Year	No. of parameters	Properties	Estimation method
1	Type-III generalized HL distribution Olapade (2008)	2008	1	M, Q, Os	
2	Four-parameter type-I generalized HL distribution Olapade (2011)	2011	4	M, Q, Lt, Ch	
3	Generalized (or exponentiated) HL distribution Kantam *et al.* (2013)	2013	2		MLE, MMLE
4	Exponential-half logistic Rao, Nagendram, and Rosaiah (2013)	2013	2		MLE, MLEg
5	HL generated-G distribution AL-Hussaini and Hussein (2014)	2014	ξ	M, Q, Os	MLE, BE
6	Type-I generalized HL distribution Olapade (2014)	2014	1	M, Q, Os, Ch, Lt	MLE
7	Exponentiated HL-G family of distributions Cordeiro, Alizadeh, and Ortega (2014)	2014	2, ξ	M, Q, Md, SE, RE, Os	MLE
8	Power HL distribution Krishnarani (2016)	2016	2	M, Q, Lt, Ch	MLE, LSE, MME
9	Beta HL distribution Jose and Manoharan (2016)	2016	3	Q	MLE
10	Mc Donald HL distribution Oliveira *et al.* (2016)	2016	3	M, Md, RE	MLE
11	Poisson HL distribution Abdel-Hamid (2016)	2016	2	M, Ch Muhammad and Liu (2021b)	MLEc, BEc
12	Type-I HL-G family of distributions Cordeiro, Alizadeh, and Diniz Marinho (2016)	2016	1, ξ	M, Q, Md, SE, RE, Os	MLE
13	Inverse HL distribution Rao, Mamidi, and Kantam (2016)	2016	1		MLE
14	Poisson odd generalize exponential HL distribution Muhammad (2016)	2016	3	M, Q, Os, SE, RE, R	MLE
15	Kumaraswamy HL distribution Usman, ul Haq, and Talib (2017)	2017	3	M, Q, Md, Os, RE, qE,	MLE
16	Generalized HL Poisson distribution Muhammad (2017)	2017	3	M, Q, SE, RE	MLE
17	Exponential generalised HL additive failure rate model Raoa, Rosaiahb, and Srinivasb (2017)	2017	2		MLE
18	HL-Poisson distribution Muhammad and Yahaya (2017)	2017	2	M, Q, Os, Ch Muhammad and Liu (2019), SE	MLE
19	Type-I HL Gompertz distribution Ogunde, Oseghale, and Audu (2017)	2017	2	M, Q, Os	MLE
20	HL generated Weibull distribution AL-Hussaini and Abdel-Hamid (2017)	2017	3	M, Q, Os	BE
21	Exponentiated generalized standardized HL distribution Cordeiro, de Andrade, Bourguignon, and Gomes-Silva (2017)	2017	2	M, Q, Os, R, RE, MRL	MLE
22	Extended HL distribution Altun, Khan, Alizadeh, Ozel, and Butt (2018)	2018	3	M, Md, Bc, Lc	MLE, SLE, WSLE
23	Five-parameter type-I generalized HL distribution Sule, Braimah, and Ahmed (2018)	2018	5	M, Q, Lt	
24	HL Lomax distribution Anwar and Zahoor (2018)	2018	2	M, Q, Os, Md, SE, RE, MRL	MLE
25	HL generalized Weibull distribution Anwar and Bibi (2018)	2018	3	M, Q, Os	MLE
26	Marshall-Olkin HL distribution Yegen and Gamze (2018)	2018	2	Q, Os	MLE
27	Odd exponentiated HL Burr XII distribution Aldahlan and Afify (2018)	2018	2	M, Q, Os, MRL	MLE
28	Exponentiated HL exponential distribution Almarashi, Khalil, Elgarhy, and ElSehetry (2018)	2018	3	M, Q, Os, MRL, PWM	MLE
29	Transmuted type-I generalized HL distribution Rabaiah and Al-Zou'bi (2018)	2018	3	M, Q, Os	MLE
30	Transmuted HL distribution Samuel and Kehinde (2019b)	2019	1	M, Q, MRL, Os, R	
31	Transmuted type-I generalized HL distribution Samuel and Kehinde (2019a)	2019	2	M, Q, Os	MLE
32	Poisson generalized HL distribution Muhammad and Liu (2019)	2019	3	M, Q, Md, Os, Ch, Lt, R, MRL, Bc, Lc, SE, RE, PWM, KLD	MLE
33	Type-II HL exponential distribution Elgarhy, ul Haq, and Perveen (2019)	2019	2	M, Q, Md, Os, RE, PWM	MLE
34	Type-I HL inverted Kumaraswamy distribution ZeinEldin, Chesneau, Jamal, and Elgarhy (2019)	2019	3	M, Q, Md, Os, R	MLE, LSE, WLSE, CVME, ADE

Table 3: Some probability models related to the HL distribution

	Model	Year	No. of parameters	Properties	Estimation method
35	Type-I HL Burr X distribution Shrahili, Elbatal, and Muhammad (2019)	2019	3	M, Q, Md, Bc, Lc, SE, RE, MRL	MLE
36	Gamma power HLdistribution Arshad, Chesneau, Ghazali, Jamal, and Mansoor (2019)	2019	3	M, Q, Os, Lt, So, Bc, Lc, R	MLE
37	Kumaraswamy type-I HL G family of distributions Elsehetry and EL-Sherpieny (2019)	2019	3, ξ	M, Q, Md, Os, PWM	MLE
38	Type-II Kumaraswamy HL G family of distributions El-Sherpieny and Elsehetry (2019)	2019	2, ξ	M, Q, Md, Os, PWM	MLE
39	Exponentiated HL Lomax distribution Jamal, Reyad, Ahmed, Shah, and Altun (2019)	2019	3	M, Q, Os, R, RE, Bc, Lc, PWM	MLE
40	Generalizated power transformed HL distribution Xavier and Jose (2021b)	2020	3	M, Q, Mt, R	MLE, PE, MME
41	Bivariate exponentiated HL distribution Alotaibi, Rezk, Ghosh, and Dey (2020)	2020	4	Ch	MLE
42	Type-II HL Lomax distribution Hassan, Elgarhy, and Mohamed (2020)	2020	3	M, Q, So, RE, Bc, Lc, MRL, PWM	MLE, SLE, WSLE, ADE
43	Generalized odd HL-G family of distributions Al-Marzouki and Alrajhi (2020)	2020	2, ξ	M, Q, Os, PWM	MLE
44	HL X-gamma distribution Bantan, Hassan, Elsehetry, and Kibria (2020)	2020	2	M, Q, Lc, Bc, MRL, So,	MLE, MLEc
45	Type-I HL Rayleigh distribution Al-Babtain (2020)	2020	2	M, Q, MRL	MLE
46	Beta type-I generalized HL distribution Awodutire, Nduka, and Ijomah (2020)	2020	5		MLE
47	Type-I HLModified Weibull Kyurkchiev (2020)	2020	4	Q	
48	HL modified exponential distribution Chaudhary and Kumar (2020)	2020	3	Q	MLE, SLE, CVME
49	HL odd Weibull-Topp-Leone-G family of distribution Chipepa, Wanduku, and Oluyede (2020)	2020	2, ξ	M, Q, Os, RE, PWM	MLE
50	New type-I HLG family of distributions Alizadeh, Nematollahi, Altun, and Rasekhi (2020)	2020	2, ξ	M, Q, Md, Lt	MLE
51	Odd exponentiated HLexponential distribution Aldahlan and Afify (2020)	2020	3	Q	MLE, LSE, WLSE, PE, CVME, ADE, MPSE
52	Discrete HL distribution Barbiero and Hitaj (2020)	2020	1	M, Q	MLE
53	HL truncated exponential distribution Gul and Mohsin (2021)	2020	2	Os	
54	Complementary Poisson generalized half logistic distribution Muhammad and Liu (2021b)	2021	3	M, Q, Md, Os, Ch, Lt, MRL, Bc, Lc, SE, RE	MLE, MLEc, LSE, PE
55	New two-parameter modified HL distribution Mohammad (2021)	2021	2	M, Q, Os, ReE	MLE, SLE, WSLE, CVME, ADE, MPSE
56	HL inverse Lomax distribution Al-Marzouki, Jamal, Chesneau, and Elgarhy (2021)	2021	3	M, Q, Os, Md, RE, qE, R, So	MLE, LSE, WLSE, PE, CVME, ADE, MPSE
57	Marshall-Olkin odd exponential HLG family of distributions Oluyede and Chipepa (2021)	2021	2, ξ	M, Q, Os, RE	MLE
58	Exponentiated HL power generalized Weibull-G distributions Oluyede, Chipepa, and Wanduku (2021)	2021	3, ξ	M, Q, Os, RE, PWM	MLE
59	Type-I HLNadarajah-Haghighi distribution Shrahili, Muhammad, Elbatal, Muhammad, Bouchane, and Abba (2023)	2021	3	M, Q, Md, Bc, Lc, SE, RE, KLD	MLE
60	Type-I HL odd Weibull generator of distributions El-Morshedy, Alshammari, Tyagi, Elbatal, Hamed, and Eliwa (2021)	2021	2, ξ	M, Q, Md, Os, Bc, Lc, RE, HCE, AE, TE, MRL, PWM	MLE, SLE, WSLE, CVM, BE
61	Topp-Leone odd exponential HL-G family of distributions Chipepa and Oluyede (2021)	2021	2, ξ	M, Q, RE, PWM	MLE
62	Exponentiated HL odd Lindley-G family of distributions Sengweni, Oluyede, and Makubate (2021)	2021	2, ξ	M, Q, Os, RE, PWM	MLE
63	Odd Lindley HL distribution Eliwa, Altun, Alhussain, Ahmed, Salah, Ahmed, and El-Morshedy (2021)	2021	1	Q, Lt	MLE, LSE, WLSE, CVME
64	Extended cosine generalized HL distribution Muhammad, Bantan, Liu, Chesneau, Tahir, Jamal, and Elgarhy (2021b)	2021	3	M, Q, Md, Os, R, MRL	MLE, LSE, PE, BE
65	Sine HL inverse Rayleigh distribution Shrahili, Elbatal, and Elgarhy (2021)	2021	2	M, Q, RE, HCE, AE, TE, Bc, Lc	MLE
66	New weighted HL distribution Hashempour and Alizadeh (2021)	2021	2	M, Q, Os	MLE, LSE, WLSE, CVME,ADE, MPSE
67	Extended Bakouch HL distribution Nikolay, Anton, Asen, and Todorka (2021)	2021	2		
68	New extended beta HL distribution Muhammad and Liu (2021a)	2021	4	M, Q, Os, Md, Bc, Lc, RE	MLE, BE
69	Discrete type-II HL exponential distribution ul Haq, Babar, Hashmi, Alghamdi, and Afify (2021)	2021	2	M, Q	MLE
70	Marshall-Olkin HL-G family of distributions Makubate, Chipepa, Oluyede, Peter *et al.* (2021)	2021	1, ξ	M, Q, Os, RE	MLE
71	Poisson–logarithmic half-logistic distribution Hashem, Kuş, Pekgor, and Abdel-Hamid (2022)	2022	3	M, Q, Os, SE, RE	MLEc, LSEc, WLSEc, PEc

4. Final remarks

In this paper, we have reviewed some recent developments and contributions by researchers with respect to the HL distribution and their extensions. The information used in this work was gathered from the available resources in the various literature. Several results in probability theory, statistics, and applications are summarized. We briefly introduced several extensions of the HL model, emphasizing the number of model parameters, properties derived, and estimation methods. Several generators have been recently developed in the literature; therefore, several new extensions of HL can be developed in the future. It is clear that most of the newly constructed models are yet to be discussed under various estimation techniques in various viewpoints and in reliability studies, among others. This review could identify the need and develop researchers' interest in utilizing the newly constructed models related to the HL distribution in their various fields of studies to provide more accurate and reliable results for better real-life exploration and progress.

Acknowledgements

This work was supported by the Key Fund of the Department of Education of Hebei Province (ZD2018065).
The authors are thankful to the Editorial Board and the reviewers for their valuable comments and suggestions.

References

Abdel-Hamid AH (2016). "Properties, Estimations and Predictions for a Poisson-half-logistic Distribution Based on Progressively Type-II Censored Samples." *Applied Mathematical Modelling*, **40**(15-16), 7164–7181. doi:10.1016/j.apm.2016.03.007.

Adatia A (1997). "Approximate BLUEs of the Parameters of the Half Logistic Distribution Based on Fairly Large Doubly Censored Samples." *Computational Statistics & Data Analysis*, **24**(2), 179–191. doi:https://doi.org/10.1016/S0167-9473(96)00058-8.

Adatia A (2000). "Estimation of Parameters of the Half-logistic Distribution Using Generalized Ranked Set Sampling." *Computational Statistics & Data Analysis*, **33**(1), 1–13. doi:https://doi.org/10.1016/S0167-9473(99)00035-3.

Adatia A, Saleh AKME (2004). "Estimation of Quantiles of Half Logistic Distribution Using Generalized Ranked-Set Sampling." *International Journal of Statistical Sciences*, **3**(Special Issue), 127–138.

Adatia A, Saleh EAKMD (2020). "Sample Design and Estimation of Parameters of Half Logistic Distribution Using Generalized Ranked-Set Sampling." *Journal of Statistical Theory and Applications*, **19**(1), 109–117. doi:https://doi.org/10.2991/jsta.d.200303.001.

Ahmad Z, Hamedani GG, Butt NS (2019). "Recent Developments in Distribution Theory: A Brief Survey and Some New Generalized Classes of Distributions." *Pakistan Journal of Statistics and Operation Research*, **15**(1), 87–110. doi:https://doi.org/10.18187/pjsor.v15i1.2803.

Al-Babtain AA (2020). "A New Extended Rayleigh Distribution." *Journal of King Saud University – Science*, **32**(5), 2576–2581. ISSN 1018-3647. doi:https://doi.org/10.1016/j.jksus.2020.04.015. URL https://www.sciencedirect.com/science/article/pii/S1018364720301622.

AL-Hussaini EK, Abdel-Hamid AH (2017). "Bayes Inference Using Half-logistic Generated Weibull Model Based on Type-II Censoring." *Communications in Statistics-Theory and Methods*, **46**(5), 2103–2122. doi:http://dx.doi.org/10.1080/03610926.2015.1032427.

AL-Hussaini EK, Abdel-Hamid AH, Hashem AF (2015a). "Bayesian Prediction Intervals of Order Statistics Based on Progressively Type-II Censored Competing Risks Data from the Half-logistic Distribution." *Journal of the Egyptian Mathematical Society*, **23**(1), 190–196. doi:http://dx.doi.org/10.1016/j.joems.2014.01.008.

AL-Hussaini EK, Abdel-Hamid AH, Hashem AF (2015b). "One-sample Bayesian Prediction Intervals Based on Progressively Type-II Censored Data from the Half-logistic Distribution under Progressive Stress Model." *Metrika*, **78**(7), 771–783. doi:10.1007/s00184-014-0526-4.

AL-Hussaini EK, Abdel-Hamid AH, Hashem AF (2020). "Prediction for Progressively Type-II Censored Competing Risks Data from the Half-Logistic Distribution." *Journal of Statistical Theory and Applications*, **19**(1), 36–48. doi:https://doi.org/10.2991/jsta.d.200224.004.

AL-Hussaini EK, Hussein M (2014). "On a Class of Half-Logistic Generated Distributions." *Wulfenia*, **21**(54), 171–186.

Al-Marzouki S, Alrajhi S (2020). "A New-Flexible Generated Family of Distributions Based on Half-Logistic Distribution." *Journal of Computational and Theoretical Nanoscience*, **17**(11), 4813–4818. doi:https://doi.org/10.1166/jctn.2020.9332.

Al-Marzouki S, Jamal F, Chesneau C, Elgarhy M (2021). "Half Logistic Inverse Lomax Distribution with Applications." *Symmetry*, **13**(2), 309. doi:https://doi.org/10.3390/sym13020309.

Al-Shomrani A, Arif O, Shawky A, Hanif S, Shahbaz MQ (2016). "Topp–Leone Family of Distributions: Some Properties and Application." *Pakistan Journal of Statistics and Operation Research*, **12**(3), 443–451. doi:https://doi.org/10.18187/pjsor.v12i3.1458.

Aldahlan M, Afify AZ (2018). "The Odd Exponentiated Half-logistic Burr XII Distribution." *Pakistan Journal of Statistics and Operation Research*, **14**(12), 305–317. doi:https://doi.org/10.18187/pjsor.v14i2.2285.

Aldahlan MA (2019). "Different Methods of Estimation for the Parameters of Half Logistic Lomax Distribution." *Applied Mathematical Sciences*, **13**(4), 201–208. doi:https://doi.org/10.12988/ams.2019.9116.

Aldahlan MAD, Afify AZ (2020). "The Odd Exponentiated Half-logistic Exponential Distribution: Estimation Methods and Application to Engineering Data." *Mathematics*, **8**(10), 1684. doi:10.3390/math8101684.

Alexander C, Cordeiro GM, Ortega EMM, Sarabia JM (2012). "Generalized Beta-generated Distributions." *Computational Statistics & Data Analysis*, **56**(6), 1880–1897. doi:https://doi.org/10.1016/j.csda.2011.11.015.

Alizadeh M, Nematollahi A, Altun E, Rasekhi M (2020). "A Study on A New Type I Half-Logistic Family of Distributions and Its Applications." *Statistics, Optimization & Information Computing*, **8**(4), 934–949. doi:10.19139/soic-2310-5070-674.

Almalki SJ, Nadarajah S (2014). "Modifications of the Weibull Distribution: A Review." *Reliability Engineering & System Safety*, **124**, 32–55. doi:http://dx.doi.org/10.1016/j.ress.2013.11.010.

Almarashi AM (2020). "Parameters Estimation for Constant-Stress Partially Accelerated Life Tests of Generalized Half-Logistic Distribution Based on Progressive Type-II Censoring." *REVSTAT–Statistical Journal*, **18**(4), 437–452. doi:https://doi.org/10.57805/revstat.v18i4.310.

Almarashi AM, Khalil MG, Elgarhy M, ElSehetry MM (2018). "Exponentiated Half Logistic Exponential Distribution with Statistical Properties and Applications." *Advances and Applications in Statistics*, **53**(4), 423–440. doi:http://dx.doi.org/10.17654/AS053040423.

Alotaibi RM, Rezk HR, Ghosh I, Dey S (2020). "Bivariate Exponentiated Half Logistic Distribution: Properties and Application." *Communications in Statistics-Theory and Methods*, **50**(24), 6099–6121. doi:https://doi.org/10.1080/03610926.2020.1739310.

Alshkaki R (2020). "A Generalized Modification of the Kumaraswamy Distribution for Modeling and Analyzing Real-life Data." *Statistics, Optimization & Information Computing*, **8**(2), 521–548. doi:https://doi.org/10.19139/soic-2310-5070-869.

Altun E, Khan MN, Alizadeh M, Ozel G, Butt NS (2018). "Extended Half-logistic Distribution with Theory and Lifetime Data Application." *Pakistan Journal of Statistics & Operations Research*, **14**(2), 319–331. doi:https://doi.org/10.18187/pjsor.v14i2.2396.

Alzaatreh A, Lee C, Famoye F (2013). "A New Method for Generating Families of Continuous Distributions." *Metron*, **71**(1), 63–79. doi:https://doi.org/10.1007/s40300-013-0007-y.

Alzaghal A, Famoye F, Lee C (2013). "Exponentiated TX Family of Distributions with Some Applications." *International Journal of Statistics and Probability*, **2**(3), 31–49. doi:10.5539/ijsp.v2n3p31.

Anwar M, Bibi A (2018). "The Half-logistic Generalized Weibull Distribution." *Journal of Probability and Statistics*, **2018**, 1–12. doi:https://doi.org/10.1155/2018/8767826.

Anwar M, Zahoor J (2018). "The Half-logistic Lomax Distribution for Lifetime Modeling." *Journal of Probability and Statistics*, **2018**, 1–12. doi:https://doi.org/10.1155/2018/3152807.

Arora SH, Bhimani GC, Patel MN (2010). "Some Results on Maximum Likelihood Estimators of Parameters of Generalized Half Logistic Distribution under Type-I Progressive Censoring with Changing Failure Rate." *International Journal of Contemporary Mathematical Sciences*, **5**(14), 685–698.

Arshad RMI, Chesneau C, Ghazali S, Jamal F, Mansoor M (2019). "The Gamma Power Half-Logistic Distribution: Theory and Applications." doi:https://hal.archives-ouvertes.fr/hal-02306651.

Arthur J, Hallinan J (1993). "A Review of the Weibull Distribution." *Journal of Quality Technology*, **25**(2), 85–93. doi:https://doi.org/10.1080/00224065.1993.11979431.

Asgharzadeh A, Rezaie R, Abdi M (2011). "Comparisons of Methods of Estimation for the Half-logistic Distribution." *Selcuk Journal of Applied Mathematics*, **Special Issue**, 93–108.

Awodutire PO, Nduka EC, Ijomah MA (2020). "The Beta Type I Generalized Half Logistic Distribution: Properties and Application." *Asian Journal of Probability and Statistics*, **6**(2), 27–41. doi:10.9734/AJPAS/2020/v6i230156.

Awodutire PO, Olapade AK, Kolawole OA (2016). "The Type I Generalized Half Logistic Survival Model." *International Journal of Theoretical and Applied Mathematics*, **2**(2), 74–78. doi:10.11648/j.ijtam.20160202.17.

Awodutire PO, Olapade AK, Kolawole OA, Ilori OR (2018). "Assessing Survival Times of Breast Cancer Patients Using Type I Generalized Half Logistic Survival Model." *Journal of Advances in Medicine and Medical Research*, **25**(2), 1–7. doi:10.9734/JAMMR/2018/38556.

Azimi R, Yaghmaei F, Babanezhad M (2012). "Bayesian Estimation of Generalized Half Logistic Distribution under Progressive Type-II Censored." *Applied Mathematical Sciences*, **6**(106), 5253–5261.

Azimi R, Yaghmaei F, Fasihi B (2013). "E-Bayesian Estimation Based on Generalized Half Logistic Progressive Type-II Censored Data." *International Journal of Advanced Mathematical Science*, **1**(2), 56–63.

Balakrishnan N (1985). "Order Statistics from the Half Logistic Distribution." *Journal of Statistical Computation and Simulation*, **20**(4), 287–309. doi:http://dx.doi.org/10.1080/00949658508810784.

Balakrishnan N, Aggarwala R (1996). "Relationships for Moments of Order Statistics from the Right-truncated Generalized Half Logistic Distribution." *Annals of the Institute of Statistical Mathematics*, **48**(3), 519–534. doi:https://doi.org/10.1007/BF00050852.

Balakrishnan N, Asgharzadeh A (2005). "Inference for the Scaled Half-logistic Distribution Based on Progressively Type-II Censored Samples." *Communications in Statistics-Theory and Methods*, **34**(1), 73–87. doi:http://dx.doi.org/10.1081/STA-200045814.

Balakrishnan N, Balakrishnan N, Aggarwala R (2000). *Progressive Censoring: Theory, Methods, and Applications*. Springer Science & Business Media. doi:https://doi.org/10.1007/978-1-4612-1334-5.

Balakrishnan N, Puthenpura S (1986). "Best Linear Unbiased Estimators of Location and Scale Parameters of the Half Logistic Distribution." *Journal of Statistical Computation and Simulation*, **25**(3-4), 193–204. doi:http://dx.doi.org/10.1080/00949658608810932.

Balakrishnan N, Saleh HM (2013). "Recurrence Relations for Single and Product Moments of Progressively Type-II Censored Order Statistics from a Generalized Half-logistic Distribution with Application to Inference." *Journal of Statistical Computation and Simulation*, **83**(9), 1704–1721. doi:http://dx.doi.org/10.1080/00949655.2012.668698.

Balakrishnan N, Wong KHT (1991). "Approximate MLEs for the Location and Scale Parameters of the Half-logistic Distribution with Type-II Right-censoring." *IEEE Transactions on Reliability*, **40**(2), 140–145. doi:10.1109/24.87114.

Balakrishnan N, Wong KHT (1994). "Best Linear Unbiased Estimation of Location and Scale Parameters of the Half-logistic Distribution Based on Type II Censored Samples." *American Journal of Mathematical and Management Sciences*, **14**(1-2), 53–101. doi:http://dx.doi.org/10.1080/01966324.1994.10737370.

Bantan R, Hassan AS, Elsehetry M, Kibria BM (2020). "Half-logistic xgamma Distribution: Properties and Estimation under Censored Samples." *Discrete Dynamics in Nature and Society*, **2020**, 1–18. doi:https://doi.org/10.1155/2020/9136513.

Barbiero A, Hitaj A (2020). "A Discrete Analogue of the Half-Logistic Distribution." In *2020 International Conference on Decision Aid Sciences and Application (DASA)*, volume 2020, pp. 64–67. IEEE. doi:10.1109/DASA51403.2020.9317237.

Bartholomew DJ (1963). "The Sampling Distribution of an Estimate Arising in Life Testing." *Technometrics*, **5**(3), 361–374. doi:https://doi.org/10.2307/1266339.

Bello OA, Awodutire PO, Sule I, Lawal HO (2020). "Some Properties and Estimation of Parameters for the Five Parameter Type I Generalized Half Logistic Distribution under Complete Observation." *Asian Research Journal of Mathematics*, **16**(4), 39–46. doi:10.9734/ARJOM/2020/v16i430185.

Chaturvedi A, Kang SB, Pathak A (2016). "Estimation and Testing Procedures for the Reliability Functions of Generalized Half Logistic Distribution." *Journal of The Korean Statistical Society*, **45**(2), 314–328. doi:http://dx.doi.org/10.1016/j.jkss.2015.11.007.

Chaturvedi A, Kumari T (2017). "Robust Bayesian Analysis of Generalized Half Logistic Distribution." *Statistics, Optimization & Information Computing*, **5**(2), 158–178. doi:10.19139/soic.v5i2.268.

Chaturvedi A, Nandchahal S (2017). "Shrinkage Estimators of the Reliability Characteristics of Generalized Half Logistic Distribution." *International Journal of Linguistics and Computational Applications*, **1**, 29–36.

Chaudhary AK, Kumar V (2020). "Half-Logistic Modified Exponential Distribution: Properties and Applications." *EPRA International Journal of Multidisciplinary Research*, **6**(12), 276–287. doi:https://doi.org/10.36713/epra3213.

Chipepa F, Oluyede B (2021). "The Topp-Leone Odd Exponential Half Logistic-G Family of Distributions: Model Properties and Applications." *Pakistan Journal of Statistics*, **37**(3), 253–277.

Chipepa F, Wanduku D, Oluyede B (2020). "Half-Logistic Odd Weibull-Topp-Leone-G Family of Distributions: Model, Properties and Applications." *Afrika Statistika*, **15**(4), 2481–2510. doi:http://dx.doi.org/10.16929/as/2020.2481.169.

Cho S, Lee K (2021). "Estimation for Half-logistic Distribution Based on Generalized Adaptive Progressive Hybrid Censored Sample." *The Korean Data & Information Science Society*, **32**(2), 405–416. doi:http://dx.doi.org/10.7465/jkdi.2021.32.2.405.

Cordeiro GM, Alizadeh M, Diniz Marinho PR (2016). "The Type I Half-logistic Family of Distributions." *Journal of Statistical Computation and Simulation*, **86**(4), 707–728. doi:http://dx.doi.org/10.1080/00949655.2015.1031233.

Cordeiro GM, Alizadeh M, Ortega EMM (2014). "The Exponentiated Half-logistic Family of Distributions: Properties and Applications." *Journal of Probability and Statistics*, **2014**, 1–21. doi:http://dx.doi.org/10.1155/2014/864396.

Cordeiro GM, de Andrade TAN, Bourguignon M, Gomes-Silva F (2017). "The Exponentiated Generalized Standardized Half-logistic Distribution." *International Journal of Statistics and Probability*, **6**(3), 24–42. doi:10.5539/ijsp.v6n3p24.

Cordeiro GM, de Castro M (2011). "A New Family of Generalized Distributions." *Journal of Statistical Computation and Simulation*, **81**(7), 883–898. doi:https://doi.org/10.1080/00949650903530745.

El-Morshedy M, Alshammari FS, Tyagi A, Elbatal I, Hamed YS, Eliwa MS (2021). "Bayesian and Frequentist Inferences on a Type I Half-logistic Odd Weibull Generator with Applications in Engineering." *Entropy*, **23**(4), 446. doi:https://doi.org/10.3390/e23040446.

El-Sherpieny ESA, Elsehetry MM (2019). "Type II Kumaraswamy Half Logistic Family of Distributions with Applications to Exponential Model." *Annals of Data Science*, **6**(1), 1–20. doi:https://doi.org/10.1007/s40745-018-00187-z.

Elgarhy M, ul Haq MA, Perveen I (2019). "Type II Half Logistic Exponential Distribution with Applications." *Annals of Data Science*, **6**(2), 245–257. doi:https://doi.org/10.1007/s40745-018-0175-y.

Eliwa MS, Altun E, Alhussain ZA, Ahmed EA, Salah MM, Ahmed HH, El-Morshedy M (2021). "A New One-parameter Lifetime Distribution and Its Regression Model with Applications." *Plos One*, **16**(2), e0246969. doi:https://doi.org/10.1371/journal.pone.0246969.

Elsehetry M, EL-Sherpieny Es (2019). "Kumaraswamy Type I Half Logistic Family of Distributions with Applications." *Gazi University Journal of Science*, **32**(1), 333–349. URL https://dergipark.org.tr/en/pub/gujs/issue/43586/312217.

Eugene N, Lee C, Famoye F (2002). "Beta-normal Distribution and Its Applications." *Communications in Statistics-Theory and Methods*, **31**(4), 497–512. doi:https://doi.org/10.1081/STA-120003130.

Giles DE (2012). "Bias Reduction for the Maximum Likelihood Estimators of the Parameters in the Half-logistic Distribution." *Communications in Statistics-Theory and Methods*, **41**(2), 212–222. doi:http://dx.doi.org/10.1080/03610926.2010.521278.

Gui W (2017). "Exponentiated Half Logistic Distribution: Different Estimation Methods and Joint Confidence Regions." *Communications in Statistics-Simulation and Computation*, **46**(6), 4600–4617. doi:http://dx.doi.org/10.1080/03610918.2015.1122053.

Gul A, Mohsin M (2021). "Recurrence Relations for Moments of Order Statistics from Half Logistic-truncated Exponential Distribution." *Communications in Statistics-Theory and Methods*, **50**(17), 3889–3902. doi:10.1080/03610926.2019.1710753.

Gupta RC, Gupta PL, Gupta RD (1998). "Modeling Failure Time Data by Lehman Alternatives." *Communications in Statistics-Theory and Methods*, **27**(4), 887–904. doi:https://doi.org/10.1080/03610929808832134.

Hashem AF, Kuş C, Pekgor A, Abdel-Hamid AH (2022). "Poisson–logarithmic Half-logistic Distribution with Inference under a Progressive-stress Model Based on Adaptive Type-II Progressive Hybrid Censoring." *Journal of the Egyptian Mathematical Society*, **30**(1), 1–33. doi:https://doi.org/10.1186/s42787-022-00148-9.

Hashempour M, Alizadeh M (2021). "A New Weighted Half-Logistic Distribution: Properties, Applications and Different Method of Estimations." *Statistics, Optimization & Information Computing*, **10**, 0–15. doi:10.19139/soic-2310-5070-1314.

Hassan A, Elgarhy M, Mohamed R (2020). "Statistical Properties and Estimation of Type II Half Logistic Lomax Distribution." *Thailand Statistician*, **18**(3), 290–305.

Hu X, Gui W (2019). "Statistical Inference for the Exponentiated Half Logistic Distribution Based on Censored Data." *Journal of Computational and Applied Mathematics*, **362**, 219–244. doi:https://doi.org/10.1016/j.cam.2019.05.017.

Jamal F, Reyad HM, Ahmed SO, Shah MAA, Altun E (2019). "Exponentiated Half Logistic Lomax Distribution with Properties and Applications." *NED University Journal of Research*, **16**(2), 1–11.

Jamjoom AA (2003). "Estimation of the Scale Parameter of the Half Logistic Distribution by Blom‘s Method." *JKAU: Science*, **15**(1), 15–29.

Jang DH, Park J, Kim C (2011). "Estimation of the Scale Parameter of the Half-logistic Distribution with Multiply Type II Censored Sample." *Journal of the Korean Statistical Society*, **40**(3), 291–301. doi:10.1016/j.jkss.2010.12.001.

Jeon YE, Kang SB (2020a). "Estimation for the Half-logistic Distribution Based on Multiply Type-II Hybrid Censoring." *Physica A: Statistical Mechanics and its Applications*, **550**, 124501. doi:https://doi.org/10.1016/j.physa.2020.124501.

Jeon YE, Kang SB (2020b). "Estimation of the Exponentiated Half-logistic Distribution Based on Multiply Type-I Hybrid Censoring." *Communications for Statistical Applications and Methods*, **27**(1), 47–64. doi:https://doi.org/10.29220/CSAM.2020.27.1.047.

Jose JK, Manoharan M (2016). "Beta Half Logistic Distribution–A New Probability Model for Lifetime Data." *Journal of Statistics and Management Systems*, **19**(4), 587–604. doi:http://dx.doi.org/10.1080/09720510.2015.1103457.

Kang SB, Cho YS, Han JT (2008). "Estimation for the Half Logistic Distribution under Progressive Type-II Censoring." *Communications for Statistical Applications and Methods*, **15**(6), 815–823. doi:https://doi.org/10.5351/CKSS.2008.15.6.815.

Kang SB, Cho YS, Han JT (2009). "Estimation for the Half Logistic Distribution Based on Double Hybrid Censored Samples." *Communications for Statistical Applications and Methods*, **16**(6), 1055–1066. doi:https://doi.org/10.5351/CKSS.2009.16.6.1055.

Kang SB, Cho YS, Han JT, Sakong J (2010). "Goodness-of-fit Test for the Half Logistic Distribution Based on Multiply Type-II Censored Samples." *Journal of the Korean Data and Information Science Society*, **21**(2), 317–325.

Kang SB, Park YK (2005). "Estimation for the Half-logistic Distribution Based on Multiply Type-II Censored Samples." *Journal of the Korean Data and Information Science Society*, **16**(1), 145–156.

Kang SB, Seo JI (2011). "Estimation in an Exponentiated Half Logistic Distribution under Progressively Type-II Censoring." *Communications for Statistical Applications and Methods*, **18**(5), 657–666. doi:http://dx.doi.org/10.5351/CKSS.2011.18.5.657.

Kang SG, Kim DH, Lee WD (2012). "Noninformative Priors for the Ratio of the Scale Parameters in the Half Logistic Distributions." *Journal of the Korean Data and Information Science Society*, **23**(4), 833–841. doi:http://dx.doi.org/10.7465/jkdi.2012.23.4.833.

Kang SG, Kim DH, Lee WD (2014). "Default Bayesian Hypothesis Testing for the Scale Parameters in the Half Logistic Distributions." *Journal of the Korean Data and Information Science Society*, **25**(2), 465–472. doi:http://dx.doi.org/10.7465/jkdi.2014.25.2.465.

Kantam RRL, Ramakrishna V, Ravikumar MS (2013). "Estimation and Testing in Type I Generalized Half Logistic Distribution." *Journal of Modern Applied Statistical Methods*, **12**(1), 22. doi:10.22237/jmasm/1367382060.

Kim C, Han K (2010). "Estimation of the Scale Parameter of the Half-logistic Distribution under Progressively Type II Censored Sample." *Statistical Papers*, **51**(2), 375–387. doi:10.1007/s00362-009-0197-9.

Kim YK, Kang SB, Han SH, Seo JI (2011a). "Prole Likelihood Estimation of Generalized Half Logistic Distribution under Progressively Type-II Censoring." *Journal of the Korean Data and Information Science Society*, **22**(3), 597–603.

Kim YK, Kang SB, Seo JI (2011b). "Bayesian Estimation in the Generalized Half Logistic Distribution under Progressively Type-II Censoring." *Journal of the Korean Data and Information Science Society*, **22**(5), 977–989. URL http://koreascience.or.kr/article/JAKO201134036356130.page.

Krishnarani SD (2016). "On a Power Transformation of Half-logistic Distribution." *Journal of Probability and Statistics*, **2016**, 1–10. doi:http://dx.doi.org/10.1155/2016/2084236.

Kumar D, Farooqi MS (2017). "The Type I Generalized Half Logistic Distribution Based on Generalized Order Statistics." *International Journal of Agricultural and Statistical Sciences*, **13**(1), 337–344.

Kumar D, Jain N, Gupta S (2015a). "The Type I Generalized Half-logistic Distribution Based on Upper Record Values." *Journal of Probability and Statistics*, **2015**, 1–11. doi:http://dx.doi.org/10.1155/2015/393608.

Kumar D, Singh U, Singh SK (2015b). "A New Distribution Using Sine Function–Its Application to Bladder Cancer Patients Data." *Journal of Statistics Applications & Probability*, **4**(3), 417–427. doi:10.12785/jsap/040309.

Kyurkchiev NV (2020). "On a"Type I Half-logistic Modified Weibull"Model: Some Extended Models and Applications." *Biomath Communications*, **7**(1), 4–13. doi:https://doi.org/10.11145/bmc.2020.03.029.

Lai CD, Murthy D, Xie M (2006). "Weibull Distributions and Their Applications." In *Springer Handbooks*, pp. 63–78. Springer. doi:https://doi.org/10.1007/978-1-84628-288-$_{-}$3.

Lee K, Cho Y (2017). "Bayesian and Maximum Likelihood Estimations of the Inverted Exponentiated Half Logistic Distribution under Progressive Type II Censoring." *Journal of Applied Statistics*, **44**(5), 811–832. doi:http://dx.doi.org/10.1080/02664763.2016.1183602.

Liu K, Balakrishnan N (2020). "Recurrence Relations for Moments of Order Statistics from Half Logistic-geometric Distribution and Their Applications." *Communications in Statistics-Simulation and Computation*, **51**(11), 6537–6555. doi:https://doi.org/10.1080/03610918.2020.1805464.

Liu Y, Shi Y, Bai X, Zhan P (2018). "Reliability Estimation of a NM-cold-standby Redundancy System in a Multicomponent Stress–strength Model with Generalized Half-logistic Distribution." *Physica A: Statistical Mechanics and its Applications*, **490**, 231–249. doi:http://dx.doi.org/10.1016/j.physa.2017.08.028.

Luko SN (1999). "A Review of the Weibull Distribution and Selected Engineering Applications." *SAE Transactions*, **108**(2), 398–412. doi:https://doi.org/10.4271/1999-01-2859.

Makubate B, Chipepa F, Oluyede B, Peter PO, *et al.* (2021). "The Marshall-Olkin Half Logistic-G Family of Distributions With Applications." *International Journal of Statistics and Probability*, **10**(2), 120–120. doi:https://doi.org/10.5539/ijsp.v10n2p119.

Mohammad SG (2021). "A New Two-parameter Modified Half-logistic Distribution: Properties and Applications." *Statistics, Optimization & Information Computing*, **10**(2), 0–16. doi:10.19139/soic-2310-5070-1210.

Mohan RC, Rajasekharam OV, Anjaneyulu GVSR (2016). "Estimation of Scale (σ) and Shape (θ) parameters of Type I Generalized Half Logistic Distribution Using Median Ranks Method." *International Journal of Statistics and Systems*, **11**(1), 9–18.

Muhammad I, Wang X, Li C, Yan M, Chang M (2020). "Estimation of the Reliability of a Stress–Strength System from Poisson Half Logistic Distribution." *Entropy*, **22**(11), 1307. doi:10.3390/e22111307.

Muhammad M (2016). "Poisson-odd Generalized Exponential Family of Distributions: Theory and Applications." *Hacettepe Journal of Mathematics and Statistics*, **47**(6), 1652–1670. doi:10.15672/HJMS.2016.393.

Muhammad M (2017). "Generalized Half-logistic Poisson Distributions." *Communications for Statistical Applications and Methods*, **24**(4), 353–365. doi:https://doi.org/10.5351/CSAM.2017.24.4.353.

Muhammad M, Alshanbari HM, Alanzi ARA, Liu L, Sami W, Chesneau C, Jamal F (2021a). "A New Generator of Probability Models: The Exponentiated Sine-G Family for Lifetime Studies." *Entropy*, **23**(11), 1394. doi:10.3390/e23111394.

Muhammad M, Bantan RA, Liu L, Chesneau C, Tahir MH, Jamal F, Elgarhy M (2021b). "A New Extended Cosine—G Distributions for Lifetime Studies." *Mathematics*, **9**(21), 2758. doi:https://doi.org/10.3390/math9212758.

Muhammad M, Liu L (2019). "A New Extension of the Generalized Half Logistic Distribution with Applications to Real Data." *Entropy*, **21**(4), 339. doi:10.3390/e21040339.

Muhammad M, Liu L (2021a). "A New Extension of the Beta Generator of Distributions." *Mathematica Slovaca*, **72**(5), 1319–1336. doi:10.1515/ms-2022-0090.

Muhammad M, Liu L (2021b). "A New Three Parameter Lifetime Model: The Complementary Poisson Generalized Half Logistic Distribution." *IEEE Access*, **9**, 60089–60107. doi:10.1109/ACCESS.2021.3071555.

Muhammad M, Yahaya MA (2017). "The Half Logistic-Poisson Sistribution." *Asian Journal of Mathematics and Applications*, **2017**, 1–15. URL https://scienceasia.asia/volume-2017.

Mustapha M, Lixia L, Badamasi A, Isyaku M, Mouna B, Hexin Z, Sani M (2022). "A New Extension of the Topp–Leone-Family of Models with Applications to Real Data." *Annals of Data Science*. doi:https://doi.org/10.1007/s40745-022-00456-y.

Nikolay K, Anton I, Asen R, Todorka T (2021). "On the Extended Half–logistic Model by H. Bakouch with "Polynomial Variable Transfer". Application to Approximate the Specific "Data BG COVID–19"." *AIP Conference Proceedings 2321, 030021 (2021)*. doi:https://doi.org/10.1063/5.0040122.

Ogunde AA, Oseghale OI, Audu AT (2017). "Performance Rating of the Type I Half Logistic Gompertz Distribution: An Analytical Approach." *American Journal of Mathematics and Statistics*, **7**(3), 93–98. doi:10.5923/j.ajms.20170703.01.

Olapade AK (2003). "On Characterizations of the Half Logistic Distribution." *InterStat*, **2**, 1–7.

Olapade AK (2008). "On Type III Generalized Half Logistic Distribution." URL arXivpreprintarXiv:0806.1580.

Olapade AK (2011). "On a Four-parameter Type I Generalized Half Logistic Distribution." *Proceedings of the Jangjeon PIIathernatical Society*, **14**(2), 189–198. URL http://hdl.handle.net/10019.1/16998.

Olapade AK (2014). "The Type I Generalized Half Logistic Distribution." *Journal of the Iranian Statistical Society*, **13**(1), 69–82.

Oliveira J, Santos J, Xavier C, Trindade D, Cordeiro GM (2016). "The McDonald Half-logistic Distribution: Theory and Practice." *Communications in Statistics-Theory and Methods*, **45**(7), 2005–2022.

Oluyede B, Chipepa F (2021). "The Marshall-Olkin Odd Exponential Half Logistic-G Family of Distributions: Properties and Applications." *Statistics, Optimization & Information Computing*, **10**, 0–24. doi:10.19139/soic-2310-5070-938.

Oluyede B, Chipepa F, Wanduku D (2021). "Exponentiated Half Logistic-Power Generalized Weibull-G Family of Distributions: Model, Properties and Appli-cations." *Eurasian Bulletin of Mathematics*, **3**(3), 134.

Panichkitkosolkul W, Saothayanun L (2012). "Bootstrap Confidence Intervals for the Process Capability Index under Half-logistic Distribution." *Maejo International Journal of Science and Technology*, **6**(2), 272–281.

Park S, Lee K (2020). "Estimation of Half Logistic Distribution under Multiply Progressive Censoring." *The Korean Data & Information Science Society*, **31**(5), 839–850. doi:http://dx.doi.org/10.7465/jkdi.2020.31.5.839.

Potdar KG, Shirke DT (2017). "Confidence Intervals for the Scaled Half-Logistic Distribution under Progressive Type-II Censoring." *Journal of Modern Applied Statistical Methods*, **16**(1), 324–349. doi:10.22237/jmasm/1493597880.

Prasad SR, Sowmya K, Mahesh R (2016). "Monitoring Software Failure Process Using Half Logistic Distribution." *International Journal of Computer Applications*, **145**(4), 1–8. doi:10.5120/ijca2016910535.

Quinn JB, Quinn GD (2010). "A Practical and Systematic Review of Weibull Statistics for Reporting Strengths of Dental Materials." *Dental Materials*, **26**(2), 135–147. doi:https://doi.org/10.1016/j.dental.2009.09.006.

Rabaiah FS, Al-Zou'bi LMA (2018). *Generalizations of Power Function and Type-I Half Logistic Distributions Using Quadratic Transmutation Map.*

Rahman MM, Al-Zahrani B, Shahbaz SH, Shahbaz MQ (2020). "Transmuted Probability Distributions: A Review." *Pakistan Journal of Statistics and Operation Research*, **16**(1), 83–94. doi:https://doi.org/10.18187/pjsor.v16i1.3217.

Rao RS, Mamidi PL, Kantam R (2016). "Modified Maximum Likelihood Estimation: Inverse Half Logistic Distribution." *i-Manager's Journal on Mathematics*, **5**(4), 1–11.

Rao SB, Kantam RRL (2012a). "Extreme Value Charts and Analysis of Means Based on Half Logistic Distribution." *International Journal of Quality & Reliability Management*, **29**(5), 501–511. doi:http://dx.doi.org/10.1108/02656711211230490.

Rao SB, Kantam RRL (2012b). "Mean and Range Charts for Skewed Distributions–A Comparison Based on Half Logistic Distribution." *Pakistan Journal of Statistics*, **28**(4), 437–444.

Rao SB, Nagendram S, Rosaiah K (2013). "Exponential—half Logistic Additive Failure Rate Model." *International Journal of Scientific and Research Publications*, **3**(5), 1–10.

Rao SB, Rao SG (2016). "A Two-stage Group Acceptance Sampling Plan Based on Life Tests for Half Logistic Distribution." *Model Assisted Statistics and Applications*, **11**(3), 203–211. doi:10.3233/MAS-160368.

Rao SG, Naidu RC (2014). "Acceptance Sampling Plans for Percentiles Based on the Exponentiated Half Logistic Distribution." *Applications and Applied Mathematics: An International Journal (AAM)*, **9**(1), 39–53.

Rao SG, Naidu RC (2015). "An Exponentiated Half Logistic Distribution to Develop a Group Acceptance Sampling Plans with Truncated Time." *Journal of Statistics and Management Systems*, **18**(6), 519–531. doi:http://dx.doi.org/10.1080/09720510.2014.968376.

Rao SG, Naidu RC (2016). "Group Acceptance Sampling Plans for Resubmitted Lots under Exponentiated Half Logistic Distribution." *Journal of Industrial and Production Engineering*, **33**(2), 114–122. doi:http://dx.doi.org/10.1080/21681015.2015.1111265.

Rao SG, Rosaiah K, Naidu RC (2020a). "Design of Multiple-deferred State Sampling Plans for Exponentiated Half Logistic Distribution." *Cogent Mathematics & Statistics*, **7**(1), 1857915. doi:https://doi.org/10.1080/25742558.2020.1857915.

Rao SG, Rosaiah K, Naidu RCH (2020b). "A Two-Stage Group Sampling Plan Based on Truncated Life Tests for Exponentiated Half Logistic Distribution." *Computational Methods in Science and Technology*, **26**(2), 55–64. doi:10.12921/cmst.2020.0000017.

Raoa SB, Rosaiahb K, Srinivasb IL (2017). "Exponential-Generalised Half Logistic Additive Failure Rate Model: An Inferential Study." *ProbStat Forum*, **10**, 63–70.

Raqab MZ, Bdair OM, Rastogi MK, Al-Aboud FM (2021). "Inference for an Exponentiated Half Logistic Distribution with Application to Cancer Hybrid Censored Data." *Communications in Statistics-Simulation and Computation*, **50**(4), 1178–1201. doi:10.1080/03610918.2019.1580724.

Rastogi MK, Tripathi YM (2014). "Parameter and Reliability Estimation for an Exponentiated Half-logistic Distribution under Progressive Type II Censoring." *Journal of Statistical Computation and Simulation*, **84**(8), 1711–1727. doi:http://dx.doi.org/10.1080/00949655.2012.762366.

Rosaiah K, Kantam RRL, Rao SB (2009). "Reliability Test Plan for Half Logistic Distribution." *Calcutta Statistical Association Bulletin*, **61**(1-4), 183–196. doi:https://doi.org/10.1177/00080683200901.

Samuel AF, Kehinde OA (2019a). "On the Flexibility of a Transmuted Type I Generalized Half Logistic Distribution with Application." *Engineering Mathematics*, **3**(1), 13–18. doi:10.11648/j.engmath.20190301.14.

Samuel AF, Kehinde OA (2019b). "A Study on Transmuted Half Logistic Distribution: Properties and Application." *International Journal of Statistical Distributions and Applications*, **5**(3), 54–59. doi:10.11648/j.ijsd.20190503.12.

Saran J, Pande V (2012). "Recurrence Relations for Moments of Progressively Type-II Right Censored Order Statistics from Half Logistic Distribution." *Journal of Statistical Theory and Applications*, **11**, 87–96. URL http://www.mscs.mu.edu/~jsta/issues/11(1).

Selim MA (2020). "The Distributions of Beta-Generated and Kumaraswamy-Generalized Families: A Brief Survey." *Available online.*

Sengweni W, Oluyede B, Makubate B (2021). "The Exponentiated Half-logistic Odd Lindley-G Family of Distributions with Applications." *Journal of Nonlinear Sciences & Applications (JNSA)*, **14**(5), 287–309. doi:10.22436/jnsa.014.05.01.

Seo JI, Kang SB (2014). "Entropy Estimation of Generalized Half-logistic Distribution (GHLD) Based on Type-II Censored Samples." *Entropy*, **16**(1), 443. doi:10.3390/e16010443.

Seo JI, Kang SB (2015a). "Bayesian Estimation of the Entropy of the Half-logistic Distribution Based on Type-II Censored Samples." *International Journal of Applied Mathematics and Statistics*, **53**(6), 58–66. URL http://www.ceser.in/ceserp/index.php/ijamas/article/view/3745.

Seo JI, Kang SB (2015b). "Notes on the Exponentiated Half Logistic Distribution." *Applied Mathematical Modelling*, **39**(21), 6491–6500. doi:http://dx.doi.org/10.1016/j.apm.2015.01.039.

Seo JI, Kang SB (2015c). "Pivotal Inference for the Scaled Half Logistic Distribution Based on Progressively Type-II Censored Samples." *Statistics & Probability Letters*, **104**, 109–116. doi:http://dx.doi.org/10.1016/j.spl.2015.05.011.

Seo JI, Kang SB (2016a). "More Efficient Approaches to the Exponentiated Half-logistic Distribution Based on Record Values." *SpringerPlus*, **5**(1), 1–18. doi:10.1186/s40064âĂŚ016âĂŚ3047âĂŚy.

Seo JI, Kang SB (2016b). "An Objective Bayesian Analysis of the Two-parameter Half-logistic Distribution Based on Progressively Type-II Censored Samples." *Journal of Applied Statistics*, **43**(12), 2172–2190. doi:http://dx.doi.org/10.1080/02664763.2015.1134446.

Seo JI, Kang SB (2017). "Inference for the Two-parameter Half-logistic Distribution Using Pivotal Qantities under Progressively Type-II Censoring Schemes." *Communications in Statistics-Simulation and Computation*, **46**(7), 5462–5478. doi:http://dx.doi.org/10.1080/03610918.2016.1161795.

Seo JI, Kim Y, Kang SB (2013). "Estimation on the Generalized Half Logistic Distribution under Type-II Hybrid Censoring." *Communications for Statistical Applications and Methods*, **20**(1), 63–75. doi:http://dx.doi.org/10.5351/CSAM.2013.20.1.063.

Seo JI, Lee HJ, Kang SB (2012). "Estimation for Generalized Half Logistic Distribution Based on Records." *Journal of the Korean Data and Information Science Society*, **23**(6), 1249–1257. doi:http://dx.doi.org/10.7465/jkdi.2012.23.6.1249.

Shaw WT, Buckley IRC (2009). "The Alchemy of Probability Distributions: Beyond Gram-Charlier Expansions, and a Skew-kurtotic-normal Distribution from a Rank Transmutation Map." *arXiv preprint arXiv:0901.0434*. doi:https://doi.org/10.48550/arXiv.0901.0434.

Shrahili M, Elbatal I, Elgarhy M (2021). "Sine Half-Logistic Inverse Rayleigh Distribution: Properties, Estimation, and Applications in Biomedical Data." *Journal of Mathematics*, **2021**, 1–10. doi:https://doi.org/10.1155/2021/4220479.

Shrahili M, Elbatal M, Muhammad M (2019). "The Type I Half-logistic Burr X Distribution: Theory and Practice." *Journal of Nonlinear Sciences and Applications*, **12**(5), 262–277. doi:10.22436/jnsa.012.05.01.

Shrahili M, Muhammad M, Elbatal I, Muhammad I, Bouchane M, Abba B (2023). "Properties and Applications of the Type I Half-logistic Nadarajah-Haghighi Distribution." *Austrian Journal of Statistics*, **52**(2), 1–21. doi:10.17713/ajs.v52i2.1363.

Sule I, Braimah OJ, Ahmed I (2018). "Generation of Type I Generalized Half Logistic Distribution." *Available online*.

Sultan KS, Mahmoud MR, Saleh HM (2006). "Moments and Estimation From Progressively Censored Data of Half Logistic Distribution." *International Journal of Reliability and Applications*, **7**(2), 187–201. URL https://koreascience.kr/article/JAKO200625522698382.page.

Swathi KS, Anjaneyulu GVSR (2020). "Estimation of Location (θ) and Scale (λ) of Two-parameter Half Logistic-Rayleigh Distribution (HLRD) Using Median Rank Regression Method." *International Journal of Multidisciplinary Educational Research*, **9**(4), 57–60.

Swathi SK, Lakshmi VM, Anjaneyulu GVSR (2020). "Change Point Method with Half Logistic Rayleigh Distribution (HLRD)." *International Journal for Research in Applied Science & Engineering Technology (IJRASET)*, **8**(1), 580–586.

Tahir MH, Cordeiro GM (2016). "Compounding of Distributions: A Survey and New Generalized Classes." *Journal of Statistical Distributions and Applications*, **3**(1), 1–35. doi:https://doi.org/10.1186/s40488-016-0052-1.

Tanveer A, Azam M, Aslam M, Shujaat Navaz M (2020). "Attribute np Control Charts Using Resampling Systems for Monitoring Non-conforming Items Using Exponentiated Half Logistic Distribution." *Operations Research and Decisions*, **2**, 115–143. doi:10.37190/ord200207.

Thomas PY, Paul J (2019). "On Diagnostic Devices for Proposing Half-logistic and Inverse Half-logistic Models Using Generalized (k) Record Values." *Communications in Statistics-Theory and Methods*, **48**(5), 1073–1091. doi:https://doi.org/10.1080/03610926.2018.1423697.

Tierney L, Kadane JB (1986). "Accurate Approximations for Posterior Moments and Marginal Densities." *Journal of the American Statistical Association*, **81**(393), 82–86. doi:10.1080/01621459.1986.10478240.

Tomy L, Jose M, Veena G (2020). "A Review on Recent Generalizations of Exponential Distribution." *Biometrics & Biostatistics International Journal*, **9**(4), 152–156. doi:10.15406/bbij.2020.09.00313.

Tomy L, Satish G (2021). "A Review Study on Trigonometric Transformations of Statistical Distributions." *Biometrics & Biostatistics International Journal*, **10**(4), 130–136. doi:10.15406/bbij.2021.10.00339.

Torabi H, Bagheri FL (2010). "Estimation of Parameters for an Extended Generalized Half Logistic Distribution Based on Complete and Censored Data." *Journal of the Iranian Statistical Society (JIRSS)*, **9**(2), 171–195.

Torabi H, Montazeri Hedesh N (2012). "The Gamma-uniform Distribution and Its Applications." *Kybernetika*, **48**(1), 16–30. URL http://eudml.org/doc/246973.

ul Haq MA, Babar A, Hashmi S, Alghamdi AS, Afify AZ (2021). "The Discrete Type-II Half-Logistic Exponential Distribution with Applications to COVID-19 Data." *Pakistan Journal of Statistics and Operation Research*, **17**(4), 921–932. doi:http://dx.doi.org/10.18187/pjsor.v17i4.3772.

Usman RM, ul Haq MA, Talib J (2017). "Kumaraswamy Half-logistic Distribution: Properties and Applications." *Journal of Statistics Applications and Probability*, **6**(3), 597–609. doi:http://dx.doi.org/10.18576/jsap/060315.

Wang B (2008). "Interval Estimation for the Scaled Half-logistic Distribution under Progressive Type-II Censoring." *Communications in Statistics-Theory and Methods*, **38**(3), 364–371. doi:http://dx.doi.org/10.1080/03610920802213681.

Wang C, Liu H (2017). "Estimation for the Scaled Half-logistic Distribution under Type-I Progressively Hybrid Censoring Scheme." *Communications in Statistics-Theory and Methods*, **46**(24), 12045–12058. doi:http://dx.doi.org/10.1080/03610926.2017.1291968.

Xavier T, Jose JK (2021a). "Estimation of Reliability in a Multicomponent Stress–Strength Model Based on Power Transformed Half-Logistic Distribution." *International Journal of Reliability, Quality and Safety Engineering*, **28**(02). doi:https://doi.org/10.1142/S0218539321500091.

Xavier T, Jose JK (2021b). "A Study of Stress-strength Reliability Using a Generalization of Power Transformed Half-logistic Distribution." *Communications in Statistics-Theory and Methods*, **50**(18), 1–17. doi:https://doi.org/10.1080/03610926.2020.1716250.

Xavier T, Jose JK (2022). "Stress–strength Reliability Estimation Involving Paired Observation with Ties Using Bivariate Exponentiated Half-logistic Model." *Journal of Applied Statistics*, **49**(5), 1049–1064.

Yegen D, Gamze Ö (2018). "Marshall-Olkin Half Logistic Distribution with Theory and Applications." *The Journal of Operations Research, Statistics, Econometrics and Management Information Systems*, **6**(2), 407–416. doi:10.17093/alphanumeric.409992.

ZeinEldin RA, Chesneau C, Jamal F, Elgarhy M (2019). "Statistical Properties and Different Methods of Estimation for Type I Half Logistic Inverted Kumaraswamy Distribution." *Mathematics*, **7**(10), 1002. doi:10.3390/math7101002.

Zhang F, Gui W (2020). "Parameter and Reliability Inferences of Inverted Exponentiated Half-Logistic Distribution under the Progressive First-Failure Censoring." *Mathematics*, **8**(5), 1–29. doi:10.3390/math8050708.

Affiliation:

Mustapha Muhammad
1. Department of Mathematics
Guangdong University of Petrochemical Technology
Maoming 525000
P.R.China.
2. Department of Mathematical Sciences
Faculty of Physical Sciences
Bayero University Kano (BUK)
Nigeria
E-mail: mmmahmoud12@sci.just.edu.jo; mmuhammad.mth@buk.edu.ng

M. H. Tahir
Department of Statistics
The Islamia University of Bahawalpur
Bahawalpur-63100
Pakistan
E-mail:mtahir.stat@gmail.com

Lixia Liu
School of Mathematical Sciences
Hebei Normal University
P.R.China
E-mail: liulixia@hebtu.edu.cn

Farrukh Jamal
Department of Statistics
The Islamia University of Bahawalpur
Bahawalpur-63100
Pakistan
E-mail: farrukh.jamal@iub.edu.pk

Christophe Chesneau
Department of Mathematics
LMNO, University of Caen, 14032 Caen
France
E-mail: christophe.chesneau@unicaen.fr

Badamasi Abba
1. School of Mathematics and Statistics
Central South University
Changsha, China
2. Department of Mathematics
Yusuf Maitama Sule University
Kano, Nigeria
E-mail: babba@yumsuk.edu.ng;badamasiabba@csu.edu.cn

Austrian Journal of Statistics http://www.ajs.or.at/
published by the Austrian Society of Statistics http://www.osg.or.at/
Volume 52 *Submitted:* 2022-03-12
2023 *Accepted:* 2023-01-23