

# On the Bimodal Gumbel Model with Application to Environmental Data

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## Abstract

The Gumbel model is a very popular statistical model due to its wide applicability for instance in the course of certain survival, environmental, financial or reliability studies. In this work, we have introduced a bimodal generalization of the Gumbel distribution that can be an alternative to model bimodal data. We derive the analytical shapes of the corresponding probability density function and the hazard rate function and provide graphical illustrations. Furthermore, We have discussed the properties of this density such as mode, bimodality, moment generating function and moments. Our results were verified using the Markov chain Monte Carlo simulation method. The maximum likelihood method is used for parameters estimation. Finally, we also carry out an application to real data that demonstrates the usefulness of the proposed distribution.

*Keywords:* Gumbel distribution, BG, MCMC.

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## 1. Introduction

Asymmetrical models for a real-valued random variable such as the Gumbel and generalized extreme value distributions have been extensively utilized for modeling various random phenomena encountered for instance in the course of certain environmental, financial or reliability studies.

Let  $Y_1, Y_2, \dots, Y_n$  be a series of independent random variables with common distribution function  $F$ , and  $M_n = \max\{Y_1, Y_2, \dots, Y_n\}$ . The Gumbel distribution is one of the extreme value distributions, characterized by Fisher and Tippett (1928) as the limit distribution for maxima. That is, if exist the normalization sequences  $a_n$  and  $b_n > 0$  such that  $\mathbb{P}[(M_n - a_n)/b_n]$  converges to a non-degenerate distribution  $G$ , then  $G$  is an extreme value distribution. The  $G$  distribution must be one of three types: Fréchet, Gumbel or negative Weibull. Among these three distributions the Gumbel distribution (Gumbel 1958) is the only one with a light tail. For this reason it can be considered as a good alternative for modeling extreme data whose tails are not heavy.

A random variable  $Y$  have a Gumbel distribution with location parameter  $\mu \in \mathbb{R}$  and scale parameter  $\sigma > 0$ , denoted by  $Y \sim F_G(\cdot; \mu, \sigma)$ , if its probability density function (PDF) and cumulated distribution function (CDF) are given, respectively, by

$$f_G(y; \mu, \sigma) = \frac{1}{\sigma} \exp \left\{ - \left( \frac{y - \mu}{\sigma} \right) - \exp \left[ - \left( \frac{y - \mu}{\sigma} \right) \right] \right\} \quad (1.1)$$

and

$$F_G(y; \mu, \sigma) = \exp \left\{ - \exp \left[ - \left( \frac{y - \mu}{\sigma} \right) \right] \right\}, \quad y \in \mathbb{R}. \quad (1.2)$$

Generalizations of the Gumbel distribution have been proposed by several authors. [Pinheiro and Ferrari \(2015\)](#) carried out a comprehensive review of the generalization of Gumbel distribution and after comparing them they conclude that some distributions suffer from over-parameterization. Another generalization of Gumbel is the Exponentiated Gumbel Type-2 by [Okorie, Akpanta, and Ohakwe \(2016\)](#) and the references in [Okorie, Akpanta, Ohakwe, Chikezie, and Obi \(2017\)](#) and [Pinheiro and Ferrari \(2015\)](#).

Despite its broad applicability in many fields the Gumbel is not suitable to model bimodal data. Furthermore, all the models cited above are not suitable for capturing this. In this context, in this paper, we propose the bimodal Gumbel (BG) distribution as an alternative model of extreme data with more than one mode. Our approach consists of introduce bimodality in (1.1) as [Elal-Olivero \(2010\)](#). Results involving bimodality in other related probabilistic models can be found, for example, in [Martinez, Achcar, Jacome, and Santos \(2013\)](#), [Çankaya, Bulut, Doğru, and Arslan \(2015\)](#); and more recently, in [Vila, Leao, Saulo, Shahzad, and Santos-Neto \(2020b\)](#); [Vila, Ferreira, Saulo, Prata, and Ortega \(2020a\)](#); [Vila and Çankaya \(2021\)](#); [Vila, Saulo, and Roldan \(2021\)](#); and [Gómez-Déniz, Sarabia, and Calderín-Ojeda \(2021\)](#). The advantage of our model in comparison to other generalizations of the Gumbel distribution is the number of parameters and the fact that it can be used to model extreme data with one or two modes.

This paper is organized as follows. In Section 2, we define the BG distribution. In Section 3, we provide general properties of the BG distribution including the cumulative distribution function, hazard rate function, mode, bimodality, moment generating function and moments, and stochastic representation. In Section 4, we provide graphical illustrations. Estimation of the parameters by maximum likelihood is investigated in Section 5. In Section 6, we discuss an application to real data. Some conclusions are addressed in Section 7.

## 2. The bimodal Gumbel model

We said that a real-valued random variable  $X$  has a bimodal Gumbel (BG) distribution with parameters  $\mu \in \mathbb{R}$ ,  $\sigma > 0$  and  $\delta \in \mathbb{R}$ , denoted by  $X \sim F_{BG}(\cdot; \mu, \sigma, \delta)$ , if its PDF is given by

$$f_{BG}(x; \mu, \sigma, \delta) = \frac{[(1 - \delta x)^2 + 1] \exp \left\{ - \left( \frac{x - \mu}{\sigma} \right) - \exp \left[ - \left( \frac{x - \mu}{\sigma} \right) \right] \right\}}{\sigma [1 + \delta^2 \sigma^2 \frac{\pi^2}{6} + (\delta \mu + \delta \sigma \gamma - 1)^2]}, \quad x \in \mathbb{R}, \quad (2.1)$$

where  $\gamma$  is the Euler's constant.

Let

$$Z_\delta = 1 + \delta^2 \sigma^2 \frac{\pi^2}{6} + (\delta \mu + \delta \sigma \gamma - 1)^2 \quad (2.2)$$

be the normalization constant of the BG distribution (2.1). Using this notation, note that  $f_{BG}(x; \mu, \sigma, \delta) = [(1 - \delta x)^2 + 1] f_G(x; \mu, \sigma) / Z_\delta$  and that  $f_{BG}(x; \mu, \sigma, 0) = f_G(x; \mu, \sigma)$ . In other words, we introduce bimodality in the Gumbel distribution (1.1) through of a quadratic transformation  $(1 - \delta x)^2 + 1$ .

The BG distribution function (2.1) is well defined, because

$$\begin{aligned} \int_{-\infty}^{\infty} f_{\text{BG}}(x; \mu, \sigma, \delta) dx &= \frac{1}{Z_\delta} [1 + \mathbb{E}(1 - \delta Y)^2], \quad Y \sim F_G(\cdot; \mu, \sigma) \\ &= \frac{1}{Z_\delta} \{1 + \delta^2 \text{Var}(Y) + [\delta \mathbb{E}(Y) - 1]^2\} = 1, \end{aligned}$$

where  $\mathbb{E}(Y) = \mu + \sigma\gamma$  and  $\text{Var}(Y) = \sigma^2\pi^2/6$ .

The CDF of a BG random variable  $X \sim F_{\text{BG}}(\cdot; \mu, \sigma, \delta)$ , defined for  $x \in \mathbb{R}$ , is given by (see Subsection 3.1 for more details)

$$\begin{aligned} F_{\text{BG}}(x; \mu, \sigma, \delta) &= \frac{[2 - \delta\mu(2 - \delta\mu)] \exp\left\{-\exp\left[-\left(\frac{x - \mu}{\sigma}\right)\right]\right\} + \delta^2\sigma^2 I\left(2; \exp\left[-\left(\frac{x - \mu}{\sigma}\right)\right], +\infty\right)}{1 + \delta^2\sigma^2\frac{\pi^2}{6} + (\delta\mu + \delta\sigma\gamma - 1)^2} \\ &\quad + \frac{2\delta(1 - \delta\mu) \left\{ (x - \mu) \exp\left\{-\exp\left[-\left(\frac{x - \mu}{\sigma}\right)\right]\right\} - \sigma\Gamma\left(0, \exp\left[-\left(\frac{x - \mu}{\sigma}\right)\right]\right) \right\}}{1 + \delta^2\sigma^2\frac{\pi^2}{6} + (\delta\mu + \delta\sigma\gamma - 1)^2}, \end{aligned} \quad (2.3)$$

where  $\Gamma(a, b)$  is the upper incomplete gamma function and

$$I(k; a, b) = (-1)^k \int_a^b \ln^k(v) \exp(-v) dv, \quad k \in \mathbb{N} \cup \{0\}, \quad 0 \leq a < b \leq +\infty \quad (2.4)$$

is the incomplete moments of the random variable  $V \sim F_G(\cdot; 0, 1)$ . For  $k > 1$ , closed form solutions for the definite integral  $I(k; a, b)$  are not available in terms of commonly used functions.

From the formula in (2.3), when  $\delta = 0$ ,  $F_{\text{BG}}(x; \mu, \sigma, 0) = \exp\{-\exp[-(\frac{x-\mu}{\sigma})]\} = F_G(x; \mu, \sigma)$ ,  $x \in \mathbb{R}$ , what was known in (1.2).

**Remark 1.** Let  $X \sim F_{\text{BG}}(\cdot; \mu, \sigma, \delta)$  and let  $g(\cdot)$  be a real-valued Borel measurable function. From definition of expectation and by using the PDF of the BG distribution (2.1), we have

$$\begin{aligned} \mathbb{E}[g(X)] &= \frac{1}{Z_\delta} \mathbb{E}[g(Y)(1 - \delta Y)^2 + g(Y)] \\ &= \frac{1}{Z_\delta} \{2\mathbb{E}[g(Y)] - 2\delta\mathbb{E}[Yg(Y)] + \delta^2\mathbb{E}[Y^2g(Y)]\}, \end{aligned}$$

where  $Y \sim F_G(\cdot; \mu, \sigma)$  and  $Z_\delta$  is as in (2.2).

### 3. Some properties of the BG distribution

In this section, some mathematical properties as closed expression for the CDF, rate, modes, bimodality, hazard function and moments of the BG distribution are discussed.

#### 3.1. Cumulative distribution function

In this subsection, we derive in detail the closed expression in (2.3) for the CDF of a BG random variable  $X \sim F_{\text{BG}}(\cdot; \mu, \sigma, \delta)$ .

Indeed, from Remark 1 with  $g(X) = \mathbb{1}_{\{X \leq x\}}$ , we obtain

$$F_{\text{BG}}(x; \mu, \sigma, \delta) = \frac{1}{Z_\delta} [2F_G(y; \mu, \sigma) - 2\delta\mathbb{E}(Y\mathbb{1}_{\{Y \leq x\}}) + \delta^2\mathbb{E}(Y^2\mathbb{1}_{\{Y \leq x\}})], \quad (3.1)$$

where  $Y \sim F_G(\cdot; \mu, \sigma)$  and  $Z_\delta$  is as in 2.2.

Taking the following change of variables  $z = \exp[-(y - \mu)/\sigma]$ ,  $dz = -(z/\sigma)dy$ ,  $y = \mu - \sigma \ln(z)$ , and using a binomial expansion we have

$$\begin{aligned} \mathbb{E}(Y^k \mathbf{1}_{\{Y \leq x\}}) &= \int_{\exp[-(\frac{x-\mu}{\sigma})]}^{+\infty} [\mu - \sigma \ln(z)]^k \exp(-z) dz \\ &= \sum_{i=0}^k \binom{k}{i} (-1)^i \mu^{k-i} \sigma^i \int_{\exp[-(\frac{x-\mu}{\sigma})]}^{+\infty} \ln^i(z) \exp(-z) dz \\ &= \sum_{i=0}^k \binom{k}{i} \mu^{k-i} \sigma^i I\left(i; \exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right], +\infty\right), \end{aligned} \quad (3.2)$$

where  $k \in \mathbb{N} \cup \{0\}$  and  $I(i; a, b)$  is as in (2.4). By combining (3.1) and (3.2), and by using the relations

$$\begin{aligned} I(0; a, +\infty) &= \exp(-a), \\ I(1; a, +\infty) &= -\exp(-a) \ln(a) - \Gamma(0, a), \end{aligned}$$

we get formula (2.3).

### 3.2. Stochastic representation

Suppose  $Y_k$  has a weighted Gumbel distribution with parameters  $\mu \in \mathbb{R}$  and  $\sigma > 0$ . That is, if  $Y \sim F_G(\cdot; \mu, \sigma)$ , by (3.2),  $Y_k$  has CDF given by, for each  $x \in \mathbb{R}$  and  $k = 0, 1, 2, \dots$ ,

$$F_{Y_k}(x) = \frac{\mathbb{E}(Y^k \mathbf{1}_{\{Y \leq x\}})}{\mathbb{E}(Y^k)} = \frac{\sum_{i=0}^k \binom{k}{i} \mu^{k-i} \sigma^i I\left(i; \exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right], +\infty\right)}{\sum_{i=0}^k \binom{k}{i} \mu^{k-i} \sigma^i I(i; 0, +\infty)}, \quad (3.3)$$

where  $I(k; a, b)$  is the incomplete moments defined in (2.4). Note that  $F_{Y_0}(x) = F_G(x; \mu, \sigma)$ . Let  $W$  be a discrete random variable, so that  $W = 1$  or  $W = 2$  or  $W = 3$ , each with probability

$$p_1 = \frac{2}{Z_\delta}, \quad p_2 = -\frac{2(\mu + \sigma\gamma)\delta}{Z_\delta}, \quad p_3 = \frac{[\sigma^2 \frac{\pi^2}{6} + (\mu + \sigma\gamma)^2] \delta^2}{Z_\delta},$$

respectively, where  $[\delta > 0$  and  $\mu + \sigma\gamma < 0]$  or  $[\delta < 0$  and  $\mu + \sigma\gamma > 0]$ , and  $Z_\delta$  is as in (2.2). It is straightforward see that  $p_1 + p_2 + p_3 = 1$ .

Assume that

$$T = \sum_{k=1}^3 \frac{1}{k} Y_{k-1} \delta_{W,k} \delta_{W,l}, \quad l = 1, 2, 3,$$

and that  $W$  is independent of  $Y_l$ , for each  $l = 1, 2, 3$ . Here  $\delta_{x,y}$  is the Kronecker delta function, i.e., this function is 1 if the variables are equal, and 0 otherwise.

**Proposition 3.1.** *The following holds*

$$X = WT \quad \text{if and only if} \quad X \sim F_{BG}(\cdot; \mu, \sigma, \delta). \quad (3.4)$$

*Proof.* By Law of total probability and by independence, we get

$$\begin{aligned}\mathbb{P}(X \leq x) &= \mathbb{P}(WT \leq x) = \sum_{l=1}^3 \mathbb{P}(WT \leq x | W = l) \mathbb{P}(W = l) \\ &= \sum_{l=1}^3 \mathbb{P}(Y_{l-1} \leq x | W = l) \mathbb{P}(W = l) = \sum_{l=1}^3 F_{Y_{l-1}}(x) p_l.\end{aligned}$$

By using the CDF of the weighted Gumbel distribution, given in (3.3), and by definitions of  $p_l$ 's and  $Z_\delta$ , the above expression is

$$\begin{aligned}&= \frac{2}{Z_\delta} F_{Y_0}(x) - \frac{2(\mu + \sigma\gamma)\delta}{Z_\delta} F_{Y_1}(x) + \frac{[\sigma^2 \frac{\pi^2}{6} + (\mu + \sigma\gamma)^2] \delta^2}{Z_\delta} F_{Y_2}(x) \\ &= \frac{2}{Z_\delta} F_G(x; \mu, \sigma) - \frac{2(\mu + \sigma\gamma)\delta}{Z_\delta} \frac{\sum_{i=0}^1 \binom{1}{i} \mu^{1-i} \sigma^i I\left(i; \exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right], +\infty\right)}{\sum_{i=0}^1 \binom{1}{i} \mu^{1-i} \sigma^i I(i; 0, +\infty)} \\ &\quad + \frac{[\sigma^2 \frac{\pi^2}{6} + (\mu + \sigma\gamma)^2] \delta^2}{Z_\delta} \frac{\sum_{i=0}^2 \binom{2}{i} \mu^{2-i} \sigma^i I\left(i; \exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right], +\infty\right)}{\sum_{i=0}^2 \binom{2}{i} \mu^{2-i} \sigma^i I(i; 0, +\infty)} \\ &= \frac{[2 - \delta\mu(2 - \delta\mu)] \exp\left\{-\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\} + \delta^2 \sigma^2 I\left(2; \exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right], +\infty\right)}{1 + \delta^2 \sigma^2 \frac{\pi^2}{6} + (\delta\mu + \delta\sigma\gamma - 1)^2} \\ &\quad + \frac{2\delta(1 - \delta\mu) \left\{ (x - \mu) \exp\left\{-\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\} - \sigma \Gamma\left(0, \exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right) \right\}}{1 + \delta^2 \sigma^2 \frac{\pi^2}{6} + (\delta\mu + \delta\sigma\gamma - 1)^2} \\ &\stackrel{(2.3)}{=} F_{BG}(x; \mu, \sigma, \delta),\end{aligned}$$

where  $\Gamma(a, b)$  is the upper incomplete gamma function.

Then the statement in (3.4) follows.  $\square$

### 3.3. Rate of a random variable with BG distribution

Following Klugman, Panjer, and Willmot (1998), for a continuous random variable  $X$  with density function  $f_X(x)$ , the rate of a random variable is given by

$$\tau_X = - \lim_{x \rightarrow \infty} \frac{d \ln [f_X(x)]}{dx}.$$

A simple computation shows that  $\tau_{BG(\mu, \sigma, \delta)} = 1/\sigma$ .

In what follows we present some comparisons between the rates of random variables with known distributions: Inverse-gamma, Log-normal, Generalized-Pareto, BWeibull (Vila and Çankaya 2021), BGamma (Vila et al. 2020a), BG, exponential and Normal;

$$\begin{aligned}\tau_{\text{InvGamma}(\alpha, \sigma)} &= \tau_{\text{LogNorm}(\mu, \kappa^2)} = \tau_{\text{GenPareto}(\alpha, \sigma, \zeta)} = \tau_{\text{BWeibull}(\alpha < 1, \sigma, \delta)} = 0 \\ &< \tau_{BG(\mu, \sigma, \delta)} = \tau_{\text{BWeibull}(\alpha = 1, \sigma, \delta)} = \tau_{\text{BGamma}(\alpha, 1/\sigma, \delta)} = \tau_{\text{exp}(1/\sigma)} = 1/\sigma \\ &< \tau_{\text{BWeibull}(\alpha > 1, \sigma, \delta)} = \tau_{\text{Normal}(\mu, \kappa^2)} = +\infty.\end{aligned}$$

In other words, far enough out in the tail, every BG distribution looks like an exponential distribution.

### 3.4. Bimodality

**Proposition 3.2.** *A point  $x \in \mathbb{R}$  is a mode of the BG density (2.1) if it is a root of the following non-polynomial function:*

$$g(x) = \frac{1}{\sigma} \left\{ \exp \left[ - \left( \frac{x - \mu}{\sigma} \right) \right] - 1 \right\} - \frac{2\delta(1 - \delta x)}{(1 - \delta x)^2 + 1}. \quad (3.5)$$

*Proof.* Taking the derivative of  $f_{\text{BG}}(x; \mu, \sigma)$  with respect to  $x$ , we have

$$f'_{\text{BG}}(x; \mu, \sigma, \delta) = f_{\text{BG}}(x; \mu, \sigma, \delta)g(x). \quad (3.6)$$

Hence, the proof follows.  $\square$

Let  $\mathcal{C}$  be the set formed for all  $(\mu, \sigma, \delta) \in \mathbb{R} \times (0, +\infty) \times \mathbb{R}$  such that the following hold:

$$\delta > \max \left\{ 1, \frac{1}{\sigma} \left[ \exp \left( \frac{\mu}{\sigma} \right) - 1 \right] \right\}, \quad (3.7)$$

$$\frac{2\delta(1 + \delta)}{(1 + \delta)^2 + 1} < \frac{1}{\sigma} \left\{ \exp \left( \frac{1 + \mu}{\sigma} \right) - 1 \right\}, \quad (3.8)$$

$$\frac{2\delta(1 - 2\delta)}{(1 - 2\delta)^2 + 1} < \frac{1}{\sigma} \left\{ \exp \left[ - \left( \frac{2 - \mu}{\sigma} \right) \right] - 1 \right\}, \quad (3.9)$$

$$\frac{2\delta(1 - 3\delta)}{(1 - 3\delta)^2 + 1} > \frac{1}{\sigma} \left\{ \exp \left[ - \left( \frac{3 - \mu}{\sigma} \right) \right] - 1 \right\}. \quad (3.10)$$

**Remark 2.** By considering  $\mu = \sigma = 1$  and  $\delta > e - 1$ , we have that  $(\mu, \sigma, \delta) \in \mathcal{C}$ . That is, the set  $\mathcal{C}$  is non-empty.

**Lemma 3.3.** *If  $(\mu, \sigma, \delta) \in \mathcal{C}$  then the function  $g(x)$  has at least three distinct real roots.*

*Proof.* Since  $(\mu, \sigma, \delta) \in \mathcal{C}$ , a simple observation in the definition (3.5) of  $g(x)$  shows that

- $g(-1) = \frac{1}{\sigma} \left\{ \exp \left( \frac{1 + \mu}{\sigma} \right) - 1 \right\} - \frac{2\delta(1 + \delta)}{(1 + \delta)^2 + 1} > 0$ , because of condition (3.8);
- $g(0) = \frac{1}{\sigma} \left\{ \exp \left( \frac{\mu}{\sigma} \right) - 1 \right\} - \delta < 0$ , because of condition (3.7);
- $g(2) = \frac{1}{\sigma} \left\{ \exp \left[ - \left( \frac{2 - \mu}{\sigma} \right) \right] - 1 \right\} - \frac{2\delta(1 - 2\delta)}{(1 - 2\delta)^2 + 1} > 0$ , because of condition (3.9);
- $g(3) = \frac{1}{\sigma} \left\{ \exp \left[ - \left( \frac{3 - \mu}{\sigma} \right) \right] - 1 \right\} - \frac{2\delta(1 - 3\delta)}{(1 - 3\delta)^2 + 1} < 0$ , because of condition (3.10).

Since  $g(x)$  is a continuous real-valued function, by Intermediate Value Theorem, there are some points  $r_1, r_2, r_3$ , with  $-1 < r_1 < 0$ ,  $0 < r_2 < 2$  and  $2 < r_3 < 3$ , so that  $g(r_i) = 0$ ,  $i = 1, 2, 3$ .  $\square$

Let  $\mathcal{D}$  be the set formed for all  $x \in \mathbb{R}$  such that

$$2\delta^2 \frac{(1 - \delta x)^2 - 1}{(1 - \delta x)^2 + 1} < -\frac{1}{\sigma^2} \exp \left[ - \left( \frac{x - \mu}{\sigma} \right) \right] \quad \text{where } (\mu, \sigma, \delta) \in \mathcal{C}.$$

**Remark 3.** The set  $\mathcal{D}$  is non-empty. To see this just take  $\mu = \sigma = 1$  and  $\delta = 2 (> e - 1)$ . By Remark 2,  $(\mu, \sigma, \delta) \in \mathcal{C}$ . In this case we have  $\mathcal{D} = (0.132178, 0.937349)$ .

**Lemma 3.4.** If  $(\mu, \sigma, \delta) \in \mathcal{C}$  then the function  $g(x)$  has no root outside the interval  $(-1, 3)$ .

Furthermore, if  $-1 < r_1 < 0$ ,  $0 < r_2 < 2$  and  $2 < r_3 < 3$  are the roots of  $g(x)$  found in Lemma 3.3, so that  $r_2 \in \mathcal{D} \subset (0, 2)$ , then these roots are unique.

*Proof.* Taking the derivative of  $g(x)$  with respect to  $x$ , we have

$$g'(x) = -\frac{1}{\sigma^2} \exp \left[ -\left( \frac{x - \mu}{\sigma} \right) \right] - 2\delta^2 \frac{(1 - \delta x)^2 - 1}{(1 - \delta x)^2 + 1}.$$

Since, for  $x \leq 0$  or  $x \geq 2/\delta$ ,  $2\delta^2 [(1 - \delta x)^2 - 1] / [(1 - \delta x)^2 + 1] \geq 0$ , we obtain

$$g'(x) \leq -\frac{1}{\sigma^2} \exp \left[ -\left( \frac{x - \mu}{\sigma} \right) \right] < 0 \quad \text{for } x \leq 0 \text{ ou } x \geq 2/\delta.$$

In other words, outside of the interval  $(-1, 3)$  the function  $g(x)$  has no zeros because this one is decreasing on  $(-\infty, 0] \cup [2/\delta, +\infty)$  with  $2/\delta < 2$  (by condition (3.7)). This implies that the roots  $-1 < r_1 < 0$  and  $2 < r_3 < 3$ , of  $g(x)$ , are unique in at their respective intervals. Finally, since  $r_2 \in \mathcal{D}$  we have that the function  $g(x)$  crosses the abscissa axis at the only point  $r_2$ . Therefore, we conclude that the root  $r_2$  is also unique in  $\mathcal{D} \subset (0, 2)$ .  $\square$

**Remark 4.** Let  $\mu = \sigma = 1$  and  $\delta = 2 (> e - 1)$ . By Remark 3,  $\mathcal{D} = (0.132178, 0.937349)$ . Computationally it can be verified that  $-1 < r_1 = -0.0896138 < 0$ ,  $0 < r_2 = 0.389792 < 2$  and  $2 < r_3 = 2.79117 < 3$  are the only roots of  $g(x)$ , with  $r_2 \in \mathcal{D}$ .

**Theorem 1 (Bimodality).** If  $r_2 \in \mathcal{D}$  then the BG distribution (2.1) is bimodal.

*Proof.* By Lemma 3.4 the function  $g(x)$  in (3.5) has exactly three roots  $r_1, r_2, r_3$  so that  $r_1 < r_2 < r_3$ . Since  $f_{\text{BG}}(x; \mu, \sigma) \rightarrow 0$  as  $x \rightarrow \pm\infty$ , it follows that the BG distribution (2.1) increases on the intervals  $(-\infty, r_1)$  and  $(r_2, r_3)$ , and decreases on  $(r_1, r_2)$  and  $(r_3, +\infty)$ . That is,  $r_1$  and  $r_3$  are two maximum points and  $r_2$  is the unique minimum point. Therefore, the bimodality of  $f_{\text{BG}}(x; \mu, \sigma)$  is guaranteed.  $\square$

### 3.5. The hazard function

The survival and hazard functions, denoted by SF and HR of the BG is given by  $S_{\text{BG}}(x; \mu, \sigma, \delta) = 1 - F_{\text{BG}}(x; \mu, \sigma, \delta)$  and  $H_{\text{BG}}(x; \mu, \sigma, \delta) = f_{\text{BG}}(x; \mu, \sigma, \delta) / [1 - F_{\text{BG}}(x; \mu, \sigma, \delta)]$ .

Considering  $r_1, r_2, r_3$  the three distinct real roots, of function  $g(x)$ , found in Lemma 3.3, we state the following result.

**Proposition 3.5.** If  $r_2 \in \mathcal{D}$  then the HR  $H_{\text{BG}}(x; \mu, \sigma, \delta)$  of the BG distribution (2.1) has the following monotonicity properties:

- 1) It is increasing for each  $x < r_1$  or  $r_2 < x < r_3$ ,
- 2) It is decreasing for each  $x \in \mathcal{D}$ .

*Proof.* As a sub-product of the proof of Theorem 1, note that the density  $f_{\text{BG}}(x; \mu, \sigma, \delta)$  is increasing on the intervals  $x < r_1$  or  $r_2 < x < r_3$ . Since  $1 - F_{\text{BG}}(x; \mu, \sigma, \delta)$  is a decreasing function, we have that the hazard function on  $x < r_1$  or  $r_2 < x < r_3$  is the product of two increasing and nonnegative functions, then the proof of first item follows.

By Glaser (1980), for to prove the second item, it is enough to prove that the function  $G_X(x)$ , defined as

$$G_X(x) = -\frac{f'_{\text{BG}}(x; \mu, \sigma, \delta)}{f_{\text{BG}}(x; \mu, \sigma, \delta)} \stackrel{(3.6)}{=} -g(x),$$

is decreasing for each  $x \in \mathcal{D}$ . But this is immediate since  $g(x)$  is increasing on this interval.  $\square$

### 3.6. Moment-generating function

**Theorem 2.** If  $X \sim F_{BG}(\cdot; \mu, \sigma, \delta)$ ,  $t < \min\{0, -m/\sigma\}$  and  $m \in \mathbb{N} \cup \{0\}$ , we have

$$\begin{aligned} \mathbb{E}[X^m \exp(tX)] &= (-1)^{m+2} \exp(t\mu) \frac{\delta^2 \sigma^{m+2} \Gamma^{(m+2)}(1 - \sigma t) + \delta \sigma^{m+1} [2 - \delta \mu(m+2)] \Gamma^{(m+1)}(1 - \sigma t)}{1 + \delta^2 \sigma^2 \frac{\pi^2}{6} + (\delta \mu + \delta \sigma \gamma - 1)^2} \\ &\quad + \frac{\exp(t\mu) \sum_{i=0}^m (-1)^i \sigma^i \mu^{m-i} [2 \binom{m}{i} - 2\delta \mu \binom{m+1}{i} + \delta^2 \mu^2 \binom{m+2}{i}] \Gamma^{(i)}(1 - \sigma t)}{1 + \delta^2 \sigma^2 \frac{\pi^2}{6} + (\delta \mu + \delta \sigma \gamma - 1)^2}, \end{aligned}$$

where  $\Gamma^{(i)}(x) = d^i \Gamma(x) / dx^i$ .

*Proof.* Taking  $g(X) = X^m \exp(tX)$  in Remark 1,

$$\mathbb{E}[X^m \exp(tX)] = \frac{1}{Z_\delta} \{2 \mathbb{E}[Y^m \exp(tY)] - 2\delta \mathbb{E}[Y^{m+1} \exp(tY)] + \delta^2 \mathbb{E}[Y^{m+2} \exp(tY)]\}, \quad (3.11)$$

where  $Y \sim F_G(\cdot; 0, \mu, \sigma)$  and  $Z_\delta$  are as in (1.2) and (2.2), respectively.

Taking the change of variables  $z = \exp[-(y - \mu)/\sigma]$ ,  $dz = -(z/\sigma)dy$ ,  $y = \mu - \sigma \ln(z)$ , and using a binomial expansion we get, for  $k = m, m+1, m+2$ ,

$$\begin{aligned} \mathbb{E}[Y^k \exp(tY)] &= \int_0^{+\infty} [\mu - \sigma \ln(z)]^k z^{(1-\sigma t)-1} \exp(-z) dz, \quad k \in \mathbb{N} \cup \{0\} \\ &= \exp(\mu t) \sum_{i=0}^k \binom{k}{i} (-1)^i \mu^{k-i} \sigma^i \int_0^{+\infty} \ln^i(z) z^{(1-\sigma t)-1} \exp(-z) dz. \quad (3.12) \end{aligned}$$

We claim that

$$\int_0^{+\infty} \ln^i(z) z^{(1-\sigma t)-1} \exp(-z) dz = \Gamma^{(i)}(1 - \sigma t), \quad i = 0, 1, \dots, k; \quad k = m, m+1, m+2. \quad (3.13)$$

Note that, by combining (3.13), (3.12) and (3.11), the proof of theorem follows.

Indeed, since  $\partial^j z^{\alpha-1} / \partial \alpha^j = \ln^j(z) z^{\alpha-1}$ ,  $j = 1, 2, \dots, i$ ,

$$\int_0^{+\infty} \ln^i(z) z^{(1-\sigma t)-1} \exp(-z) dz = \int_0^{+\infty} \frac{\partial^j}{\partial (1 - \sigma t)^j} [\ln^{i-j}(z) z^{(1-\sigma t)-1} \exp(-z)] dz.$$

By Leibniz integral rule we can interchange the derivative with the integral in the above identity. Hence, for  $j = 1, 2, \dots, i$ ,

$$\int_0^{+\infty} \ln^i(z) z^{(1-\sigma t)-1} \exp(-z) dz = \frac{d^j}{d(1 - \sigma t)^j} \int_0^{+\infty} \ln^{i-j}(z) z^{(1-\sigma t)-1} \exp(-z) dz.$$

Letting  $j = i$ , we obtain

$$\begin{aligned} \int_0^{+\infty} \ln^i(z) z^{(1-\sigma t)-1} \exp(-z) dz &= \frac{d^i}{d(1 - \sigma t)^i} \int_0^{+\infty} z^{(1-\sigma t)-1} \exp(-z) dz \\ &= \Gamma^{(i)}(1 - \sigma t), \quad i = 0, 1, \dots, \end{aligned}$$

where in the last line we used the definition of gamma function,  $\Gamma(\alpha) = \int_0^{+\infty} z^{\alpha-1} \exp(-z) dz$ . Then the claimed (3.13) follows.

Thus, this complete the proof of theorem.  $\square$

By taking  $m = 0$  in Theorem 2 we obtain a closed expression for the moment-generating function of BGumbel distribution.



**Corollary 3.6.** *If  $X \sim F_{BG}(\cdot; \mu, \sigma, \delta)$  then, the moment-generating function  $M_X(t) = \mathbb{E}[\exp(tX)]$  with  $t < 0$ , is given by*

$$M_X(t) = \frac{\exp(\mu t)\Gamma(1 - \sigma t)\left[2 - 2\mu\delta + \mu^2\delta^2 + 2\sigma\delta(1 - \mu\delta)\psi(1 - \sigma t) + \frac{\sigma^2\delta^2}{\Gamma(1-\sigma t)}\Gamma^{(2)}(1 - \sigma t)\right]}{1 + \delta^2\sigma^2\frac{\pi^2}{6} + (\delta\mu + \delta\sigma\gamma - 1)^2},$$

where  $\psi(x) = \Gamma'(x)/\Gamma(x)$  is the digamma function and  $\Gamma^{(2)}(1 - \sigma t) = d^2\Gamma(x)/dx^2$ .

**Remark 5.** Letting  $\delta = 0$  in Corollary 3.6 we obtain the known formula  $M_Y(t) = \exp(\mu t)\Gamma(1 - \sigma t)$ , where  $Y \stackrel{d}{=} X \sim F_G(\cdot; 0, \mu, \sigma)$ . Note that, by a by-product of proof of Theorem 2, in this case it is sufficient take  $t < 1/\sigma$  instead  $t < 0$ .

**Remark 6.** The characteristic function of  $X \sim F_{BG}(\cdot; \mu, \sigma, \delta)$ , denoted by  $\phi_X(t)$ , can be obtained from the moment-generating function by the relation  $M_X(t) = \phi_X(-it)$ .

### 3.7. Moments

**Theorem 3.** *If  $X \sim F_{BG}(\cdot; \mu, \sigma, \delta)$  then*

$$\begin{aligned} \mathbb{E}(X^k) &= \frac{\delta^2\sigma^{k+2}I(k + 2; 0, +\infty) - \delta\sigma^{k+1}[2 - \delta\mu(k + 2)]I(k + 1; 0, +\infty)}{1 + \delta^2\sigma^2\frac{\pi^2}{6} + (\delta\mu + \delta\sigma\gamma - 1)^2} \\ &\quad + \frac{\sum_{i=0}^k \sigma^i\mu^{k-i}[2\binom{k}{i} - 2\delta\mu\binom{k+1}{i} + \delta^2\mu^2\binom{k+2}{i}]I(i; 0, +\infty)}{1 + \delta^2\sigma^2\frac{\pi^2}{6} + (\delta\mu + \delta\sigma\gamma - 1)^2}, \end{aligned}$$

where  $I(i; a, b)$  is defined in (2.4).

*Proof.* From Remark 1 with  $g(X) = X^k$ , we obtain

$$\mathbb{E}(X^k) = \frac{1}{Z_\delta} [2\mathbb{E}(Y^k) - 2\delta\mathbb{E}(Y^{k+1}) + \delta^2\mathbb{E}(Y^{k+2})], \tag{3.14}$$

where  $Y \sim F_G(\cdot; 0, \mu, \sigma)$  is as in (1.2).

By taking  $x \rightarrow +\infty$  in (3.2), Lebesgue dominated convergence theorem gives

$$\mathbb{E}(Y^k) = \lim_{x \rightarrow +\infty} \mathbb{E}(Y^k \mathbf{1}_{\{Y \leq x\}}) = \sum_{i=0}^k \binom{k}{i} \mu^{k-i} \sigma^i I(i; 0, +\infty). \tag{3.15}$$

By combining (3.14) and (3.15), the proof follows. □

**Corollary 3.7.** *If  $X \sim F_{BG}(\cdot; \mu, \sigma, \delta)$  then*

$$\mathbb{E}(X) = \frac{\delta^2\sigma^3\left[2\zeta(3) + \gamma^3 + \frac{\gamma\pi^2}{2}\right] - \delta\sigma^2(2 - 3\delta\mu)\left(\gamma^2 + \frac{\pi^2}{6}\right) + \mu[2 - \delta\mu(2 - \delta\mu)] + \sigma[2 - \delta\mu(4 - 3\delta\mu)]\gamma}{1 + \delta^2\sigma^2\frac{\pi^2}{6} + (\delta\mu + \delta\sigma\gamma - 1)^2}, \tag{3.16}$$

$$\begin{aligned} \mathbb{E}(X^2) &= \frac{\delta^2\sigma^4\left[8\gamma\zeta(3) + \gamma^4 + \gamma^2\pi^2 + \frac{3\pi^4}{20}\right] - 2\delta\sigma^3(1 - 2\delta\mu)\left[2\zeta(3) + \gamma^3 + \frac{\gamma\pi^2}{2}\right] + \mu^2[2 - \delta\mu(2 - \delta\mu)]}{1 + \delta^2\sigma^2\frac{\pi^2}{6} + (\delta\mu + \delta\sigma\gamma - 1)^2} \\ &\quad + \frac{2\sigma\mu[2 - \delta\mu(3 - 2\delta\mu)]\gamma + 2\sigma^2[1 - 3\delta\mu(1 - \delta\mu)]\left(\gamma^2 + \frac{\pi^2}{6}\right)}{1 + \delta^2\sigma^2\frac{\pi^2}{6} + (\delta\mu + \delta\sigma\gamma - 1)^2}, \end{aligned} \tag{3.17}$$

$$\begin{aligned} \mathbb{E}(X^3) &= \frac{\delta^2\sigma^5\left[20\gamma^2\zeta(3) + \frac{10\pi^2\zeta(3)}{3} + 24\zeta(5) + \gamma^5 + \frac{5\gamma^3\pi^2}{3} + \frac{3\gamma\pi^4}{4}\right]}{1 + \delta^2\sigma^2\frac{\pi^2}{6} + (\delta\mu + \delta\sigma\gamma - 1)^2} \\ &\quad + \frac{\mu^3[2 - \delta\mu(2 - \delta\mu)] + \sigma\mu^2[6 - \delta\mu(8 - 5\delta\mu)]\gamma + 2\sigma^2\mu[3 - \delta\mu(6 - 5\delta\mu)]\left(\gamma^2 + \frac{\pi^2}{6}\right)}{1 + \delta^2\sigma^2\frac{\pi^2}{6} + (\delta\mu + \delta\sigma\gamma - 1)^2} \\ &\quad + \frac{2\sigma^3[1 - \delta\mu(4 - 5\delta\mu)]\left[2\zeta(3) + \gamma^3 + \frac{\gamma\pi^2}{2}\right] - \delta\sigma^4(2 - 5\delta\mu)\left[8\gamma\zeta(3) + \gamma^4 + \gamma^2\pi^2 + \frac{3\pi^4}{20}\right]}{1 + \delta^2\sigma^2\frac{\pi^2}{6} + (\delta\mu + \delta\sigma\gamma - 1)^2}, \end{aligned}$$

where  $\gamma$  is the Euler-Mascheroni constant and  $\zeta(s)$  is the Riemann zeta function.

*Proof.* By combining Theorem 3 with the following values for the improper integral  $I(k; 0, +\infty)$  in (2.4), with  $k = 0, 1, 2, 3, 4$  and 5,

$$\begin{aligned}
I(0; 0, +\infty) &= 1, \\
I(1; 0, +\infty) &= \gamma, \\
I(2; 0, +\infty) &= \gamma^2 + \frac{\pi^2}{6}, \\
I(3; 0, +\infty) &= 2\zeta(3) + \gamma^3 + \frac{\gamma\pi^2}{2}, \\
I(4; 0, +\infty) &= 8\gamma\zeta(3) + \gamma^4 + \gamma^2\pi^2 + \frac{3\pi^4}{20}, \\
I(5; 0, +\infty) &= 20\gamma^2\zeta(3) + \frac{10\pi^2\zeta(3)}{3} + 24\zeta(5) + \gamma^5 + \frac{5\gamma^3\pi^2}{3} + \frac{3\gamma\pi^4}{4},
\end{aligned} \tag{3.18}$$

the proof follows. □

For the variance of the BGumbel distribution

$$\text{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2, \tag{3.19}$$

just replace (3.16) and (3.17) in (3.19).

**Remark 7** (Standardized moments). Let  $X \sim F_{\text{BG}}(\cdot; \mu, \sigma, \delta)$  with  $\mathbb{E}(X)$  and  $\sqrt{\text{Var}(X)} > 0$ , both finite. Newton's binomial expansion gives

$$\mathbb{E} \left[ \frac{X - \mathbb{E}(X)}{\sqrt{\text{Var}(X)}} \right]^n = \frac{(-1)^n [\mathbb{E}(X)]^n}{[\text{Var}(X)]^{n/2}} \sum_{k=0}^n \binom{n}{k} (-1)^{-k} [\mathbb{E}(X)]^{-k} \mathbb{E}(X^k). \tag{3.20}$$

By combining (3.20), Theorem 3, identities in (3.18) and the following relation

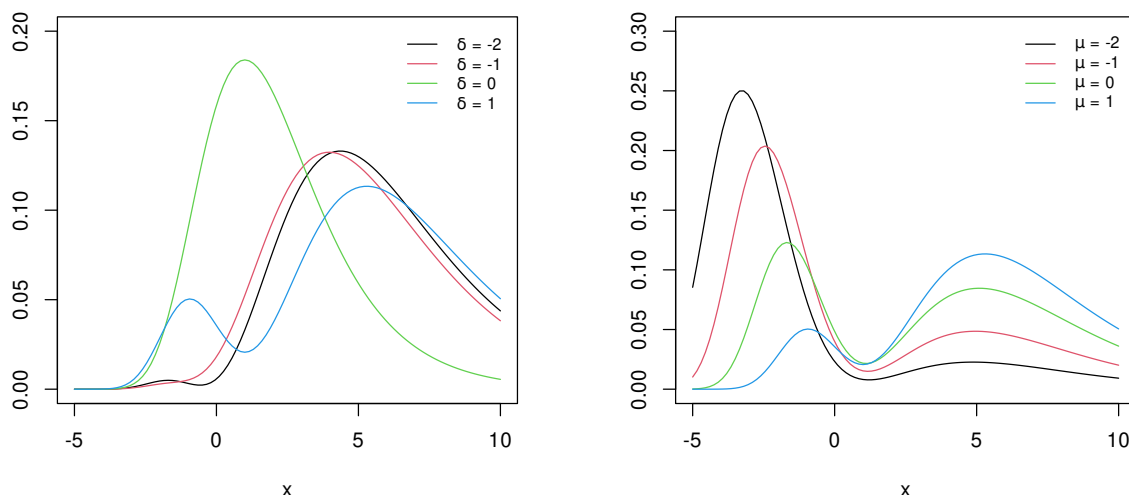
$$I(6; 0, +\infty) = 20\gamma(2\gamma^2 + \pi^2)\zeta(3) + 40[\zeta(3)]^2 + 144\gamma\zeta(5) + \gamma^6 + \frac{5\gamma^4\pi^2}{2} + \frac{9\gamma^2\pi^4}{4} + \frac{61\pi^6}{168},$$

closed formulas for skewness and kurtosis of the random variable  $X \sim F_{\text{BG}}(\cdot; \mu, \sigma, \delta)$  it is possible to obtain.

## 4. Graphical illustrations and simulation results

### 4.1. Graphical illustration

Here, the flexibility of the BG model is shown. Note that the BG model can be unimodal or bimodal. Figures 1a and 1b show how the BG density function is influenced by the shape parameters  $\delta$  and  $\mu$ , since  $\sigma$  is a scale parameter. Figure 1a shows that the bimodality of the BG model is more evident the greater the absolute value of  $\delta$ . The density tends to become unimodal when the values of  $\delta$  and  $\mu$  are far apart. In the bimodal case, the left mode is smaller and the opposite for the right mode as  $\mu$  increases, see Figure 1b. All graphics were generated with our bgumbel package, Brom, Otiniano, Vila, and Pereira (2021).



(a) BG density graphs for  $\mu = 1$ ,  $\sigma = 2$  and  $\delta$  varying as shown in the caption

(b) BG density graphs for  $\delta = 1$ ,  $\sigma = 2$  and  $\mu$  varying as shown in the caption

Figure 1: Some examples of BG densities

## 4.2. Simulation results

In this section, we compare the sample mean and variance with the population mean and variance (3.16) and (3.19). In addition, we compared the cumulative distribution function given in (2.3) with the empirical distribution. For that, we present a Markov chain Monte Carlo (MCMC) simulation study with various sample sizes. In the MCMC simulation method we considered the Metropolis-Hastings method as pseudo-random numbers (PRN) generator Metropolis, Rosenbluth, Rosenbluth, and Teller (1953); Hastings (1970). We chose this method instead of the inverse transform method, because the BG cumulative distribution (3.1) does not admit analytical inversion. To generate PRN was used the MCMCpack package (Martin, Quinn, and Park 2011) available in the R program R-Team (2020). The mean, variance and the corresponding bias of the sample estimates were computed over  $10^5$  iterations of MCMC. These results are shown in Tables 1. The performance of the MCMC algorithm was tested by varying the number of iterations,  $n$ . Graphically, the convergence of the Markov Chain is shown in Figure 2, for the first set of parameters in Table 1. For the other parameters sets, in the Table 1, the performance of the algorithm is illustrated in Appendix A, Figures 5 - 13. The value of  $n$  varies from 1000 to 100000 for each parameters set. In all these figures the transition probability density function (TPDF), defined by the MCMCpack package, is in blue and the estacionary density in red. We can conclude that the algorithm, Brom *et al.* (2021), is efficient for  $n$  greater than 100000.

Table 1: Comparison between the sample moments and population moments ( $n = 10^5$ )

$\mu$	$\sigma$	$\delta$	sample mean	$E(X)$	Bias(mean)	variance sample	$Var(X)$	Bias (variance)
-2	1	-1	-1.0931	-1.0640	-0.0291	4.9268	5.0126	-0.0858
-1	2	-1	3.9504	4.0170	-0.0666	17.126	18.016	-0.8902
-1	2	-2	3.9901	3.9909	-0.0008	21.032	21.575	-0.5435
-2	2	-1	2.0065	1.9512	0.0553	24.104	24.592	-0.4883

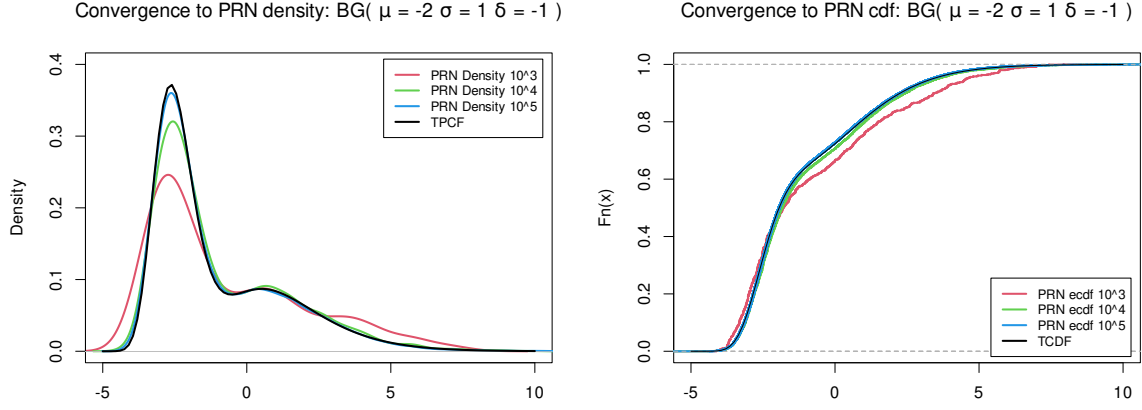


Figure 2: Transition probability density function (TPDF)(left) and transition cumulative distribution function (TCDF)(right), varying  $n$  from  $10^3$  to  $10^5$

## 5. Maximum likelihood estimation

In this section, we determine the maximum likelihood estimates (MLEs) of the parameters of the BG distribution. Let  $X \sim F_{BG}(\cdot; \mu, \sigma, \delta)$  be a random variable with PDF  $f_{BG}(x; \Theta)$  defined in (2.1), where  $\Theta = (\mu, \sigma, \delta)$ . Let  $X_1, \dots, X_n$  be a random sample of  $X$  and  $x = (x_1, \dots, x_n)$  the corresponding observed values of the random sample  $X_1, \dots, X_n$ . The log-likelihood function for  $\Theta = (\mu, \sigma, \delta)$  is given by

$$\ell(\Theta; x) = -n \ln(\sigma Z_\delta) + \sum_{i=1}^n \left\{ \ln \left[ (1 - \delta x_i)^2 + 1 \right] - \left( \frac{x_i - \mu}{\sigma} \right) - \exp \left[ - \left( \frac{x_i - \mu}{\sigma} \right) \right] \right\}. \quad (5.1)$$

The first-order partial derivatives of  $Z_\delta$  are given by

$$\begin{aligned} \frac{\partial Z_\delta}{\partial \mu} &= 2\delta [\delta(\mu + \sigma\gamma) - 1]; \\ \frac{\partial Z_\delta}{\partial \sigma} &= \frac{\sigma\delta\pi^2}{3} + 2\delta\gamma [\delta(\mu + \sigma\gamma) - 1]; \\ \frac{\partial Z_\delta}{\partial \delta} &= \frac{\sigma\delta\pi^2}{3} + 2[\delta(\mu + \sigma\gamma) - 1](\mu + \sigma\gamma). \end{aligned}$$

Then, the mle of  $\mu, \sigma, \delta$  are the solutions of the following system of equations

$$\begin{aligned} \frac{\partial \ell(\Theta; x)}{\partial \mu} &= -\frac{n}{Z_\delta} \frac{\partial Z_\delta}{\partial \mu} + \frac{n}{\sigma} - \frac{1}{\sigma} \sum_{i=1}^n \exp \left[ - \left( \frac{x_i - \mu}{\sigma} \right) \right] = 0; \\ \frac{\partial \ell(\Theta; x)}{\partial \sigma} &= -\frac{n}{Z_\delta} \frac{\partial Z_\delta}{\partial \sigma} - \frac{n}{\sigma} - \frac{1}{\sigma^2} \sum_{i=1}^n (x_i - \mu) \left\{ 1 - \exp \left[ - \left( \frac{x_i - \mu}{\sigma} \right) \right] \right\} = 0; \\ \frac{\partial \ell(\Theta; x)}{\partial \delta} &= -\frac{n}{Z_\delta} \frac{\partial Z_\delta}{\partial \delta} - 2 \sum_{i=1}^n \frac{x_i(1 - \delta x_i)}{(1 - \delta x_i)^2 + 1} = 0. \end{aligned}$$

The MLEs  $\hat{\mu}, \hat{\sigma}$  and  $\hat{\delta}$  of  $\mu, \sigma$  and  $\delta$  are defined as the values of  $\hat{\mu}, \hat{\sigma}$  and  $\hat{\delta}$  that maximize the log-likelihood function in (5.1). There will be, in general, no closed form for the MLE and their obtention will need, in practice, numerical methods.

Since the second order partial derivatives of  $Z_\delta$  are

$$\begin{aligned} \frac{\partial^2 Z_\delta}{\partial \mu^2} &= 2\delta^2; & \frac{\partial^2 Z_\delta}{\partial \sigma^2} &= \frac{\delta\pi^2}{3} + 2\delta^2\gamma^2; \\ \frac{\partial^2 Z_\delta}{\partial \delta^2} &= \frac{\sigma\pi^2}{3} + 2(\mu + \sigma\gamma)^2; & \frac{\partial^2 Z_\delta}{\partial \mu \partial \sigma} &= \frac{\partial^2 Z_\delta}{\partial \sigma \partial \mu} = 2\delta^2\gamma; \\ \frac{\partial^2 Z_\delta}{\partial \mu \partial \delta} &= \frac{\partial^2 Z_\delta}{\partial \delta \partial \mu} = 4\delta(\mu + \sigma\gamma); & \frac{\partial^2 Z_\delta}{\partial \sigma \partial \delta} &= \frac{\partial^2 Z_\delta}{\partial \delta \partial \sigma} = \frac{\sigma\pi^2}{3} + 4\delta\gamma(\mu + \sigma\gamma); \end{aligned}$$

the second-order partial derivatives of the log-likelihood function  $\ell(\Theta; x)$  are given by

$$\begin{aligned}\frac{\partial^2 \ell(\Theta; x)}{\partial \mu^2} &= -D_{\mu, \mu}(\Theta; n) - \frac{1}{\sigma^2} \sum_{i=1}^n \exp \left[ - \left( \frac{x_i - \mu}{\sigma} \right) \right]; \\ \frac{\partial^2 \ell(\Theta; x)}{\partial \sigma^2} &= -D_{\sigma, \sigma}(\Theta; n) + \frac{n}{\sigma^2} + \frac{1}{\sigma^3} \sum_{i=1}^n (x_i - \mu) \left\{ 2 - \left[ 2 - \left( \frac{x_i - \mu}{\sigma} \right) \right] \exp \left[ - \left( \frac{x_i - \mu}{\sigma} \right) \right] \right\}; \\ \frac{\partial^2 \ell(\Theta; x)}{\partial \delta^2} &= -D_{\delta, \delta}(\Theta; n) - 2 \sum_{i=1}^n \frac{x_i^2 [(1 - \delta x_i)^2 - 1]}{[(1 - \delta x_i)^2 + 1]^2}.\end{aligned}$$

Already, the second-order mixed derivatives of  $\ell(\Theta; x)$  can be written as

$$\begin{aligned}\frac{\partial^2 \ell(\Theta; x)}{\partial \mu \partial \sigma} &= \frac{\partial^2 \ell(\Theta; x)}{\partial \sigma \partial \mu} = -D_{\mu, \sigma}(\Theta; n) - \frac{n}{\sigma^2} + \frac{1}{\sigma^2} \sum_{i=1}^n \left[ 1 - \left( \frac{x_i - \mu}{\sigma} \right) \right] \exp \left[ - \left( \frac{x_i - \mu}{\sigma} \right) \right]; \\ \frac{\partial^2 \ell(\Theta; x)}{\partial \mu \partial \delta} &= \frac{\partial^2 \ell(\Theta; x)}{\partial \delta \partial \mu} = -D_{\mu, \delta}(\Theta; n); \\ \frac{\partial^2 \ell(\Theta; x)}{\partial \sigma \partial \delta} &= \frac{\partial^2 \ell(\Theta; x)}{\partial \delta \partial \sigma} = -D_{\sigma, \delta}(\Theta; n),\end{aligned}$$

where

$$D_{u, v}(\Theta; n) = \frac{\partial}{\partial u} \left( \frac{n}{Z_\delta} \frac{\partial Z_\delta}{\partial v} \right) = \frac{n}{Z_\delta} \left( \frac{\partial^2 Z_\delta}{\partial u \partial v} - \frac{1}{Z_\delta} \frac{\partial Z_\delta}{\partial u} \frac{\partial Z_\delta}{\partial v} \right), \quad u, v \in \{\mu, \sigma, \delta\}. \quad (5.2)$$

Thus, the elements of the Fisher information matrix  $I_X(\Theta)$  are defined by

$$[I_X(\Theta)]_{jk} = \mathbb{E} \left[ \frac{\partial \ln f_{\text{BG}}(X; \Theta)}{\partial \theta_j} \frac{\partial \ln f_{\text{BG}}(X; \Theta)}{\partial \theta_k} \right],$$

for  $\theta_j, \theta_k \in \{\mu, \sigma, \delta\}$  and  $j, k = 1, 2, 3$ . Under known regularity conditions, the elements  $[I_X(\Theta)]_{jk}$  are written as  $\mathbb{E} \left[ - \frac{\partial^2 \ln f_{\text{BG}}(X; \Theta)}{\partial \theta_j \partial \theta_k} \right]$ . Hence, by considering the following notations:

$$\begin{aligned}F_1(X) &= \exp \left[ - \left( \frac{X - \mu}{\sigma} \right) \right], \\ F_2(X) &= \left[ 1 - \left( \frac{X - \mu}{\sigma} \right) \right] \exp \left[ - \left( \frac{X - \mu}{\sigma} \right) \right], \\ F_3(X) &= (X - \mu) \left\{ 2 - \left[ 2 - \left( \frac{X - \mu}{\sigma} \right) \right] \exp \left[ - \left( \frac{X - \mu}{\sigma} \right) \right] \right\}, \\ F_4(X) &= X^2 [(1 - \delta X)^2 - 1] / [(1 - \delta X)^2 + 1]^2,\end{aligned}$$

we get

$$I_X(\Theta) = \begin{bmatrix} D_{\mu, \mu}(\Theta; 1) + \frac{1}{\sigma^2} \mathbb{E}[F_1(X)] & D_{\mu, \sigma}(\Theta; 1) + \frac{1}{\sigma^2} - \frac{1}{\sigma^2} \mathbb{E}[F_2(X)] & D_{\mu, \delta}(\Theta; 1) \\ D_{\mu, \sigma}(\Theta; 1) + \frac{1}{\sigma^2} - \frac{1}{\sigma^2} \mathbb{E}[F_2(X)] & D_{\sigma, \sigma}(\Theta; 1) - \frac{1}{\sigma^2} - \frac{1}{\sigma^3} \mathbb{E}[F_3(X)] & D_{\sigma, \delta}(\Theta; 1) \\ D_{\mu, \delta}(\Theta; 1) & D_{\sigma, \delta}(\Theta; 1) & D_{\delta, \delta}(\Theta; 1) + 2\mathbb{E}[F_4(X)] \end{bmatrix},$$

where  $D_{u, v}(\Theta; n)$  is as in (5.2). We emphasize that, by using Theorem 2 and Corollary 3.6, closed expressions for the expectations  $\mathbb{E}[F_1(X)]$ ,  $\mathbb{E}[F_2(X)]$  and  $\mathbb{E}[F_3(X)]$  can be found. Furthermore, by using Theorem 3, a simple argument shows that  $F_4(X)$  is an integrable random variable.

## 6. Real-world data analysis

In this section, we used the BG model to fit one data set coming from daily pressure observations of the state of Mato Grosso do Sul-Brazil, during the period of the year 2015

to 2020. The data series employed were taken from Corumbá automatic station of INMET (Instituto Nacional de Meteorologia) at <https://portal.inmet.gov.br/dadoshistoricos>. The required numerical evaluations are implemented using the R software R-Team (2020); Hastings (1970); Martin *et al.* (2011). The maximum likelihood estimates of the parameters were obtained as in Section 5, using the BGumbel package Brom *et al.* (2021).

For a random sample of the pressure  $\{X_i\}_{i=1}^T$ ,  $T = 1774$ , we fit their maximum values by Gumbel and BG models. For this, we used the maximum block technique. That is, we provide  $\tau = 29$  non-overlapping subsamples of length  $N = 60$ ;  $\{m_i\}_{i=1}^\tau$ , where  $m_i = \max\{x_{(i-1)N+1} \dots x_{iN+1}\}$ ,  $\forall i = 1, \dots, \tau = \lceil T/N \rceil$  ( $\lceil \cdot \rceil$  denotes the integer part). We applied Ljung – Box’s test (Trapletti 2016) for verifying the null hypothesis of serial independence of the new sample of maxima. The test statistics did not reject the null hypothesis at the significance level of 1,7% for the 28 blocks of size  $N = 60$  and the last of size 34.

The descriptive statistics of sample  $\{m_i\}_{i=1}^\tau$  and the sample centered by the mean are shown in Table 2.

Table 2: Descriptive statistics

Distribution	Mean	Median	Maximum	Minimum	Std Dev
Extreme values	1009.2	1010.0	1017.9	1001.1	4.9301
Centralized extreme values	0.0000	0.7893	8.7393	−8.0607	4.9301

For these data, we fit the BG and Gumbel models, and compare it with the bimodal normal model defined in Gómez-Déniz *et al.* (2021). The parameter estimates and their corresponding standard error estimates (SE) from the BG and Gumbel models are shown in Table 3. The results of goodness-of-fit tests based on the Kolmogorov–Smirnov (KS) test, Akaike information criterion (AIC) and Bayesian information criterion (BIC) indicate that the BG model is a better fit, judging on the basis of  $p$ -values and the lowest value of the AIC and BIC, see Table 3. The Figures 3 and 4 also indicates that the adjustment of the data by the BG model is better than the classic Gumbel model and bimodal normal.

Table 3: Parameters estimates of Gumbel, bimodal normal and BG models

Model	Parameters	Estimate	SE	ks $p$ -value	AIC	BIC
BG	$\mu$	−3.6972	0.4613	0.8239	169.8228	173.8194
	$\sigma$	2.4661	0.2222			
	$\delta$	−0.4128	0.1220			
Gumbel	$\mu$	−2.4080	0.8667	0.6576	173.2287	175.8932
	$\beta$	4.3326	0.6405			
Bimodal Normal	$\theta$	4.2988	0.1890	0.2265	232.8265	235.4909
	$\lambda$	−0.0165	0.0441			

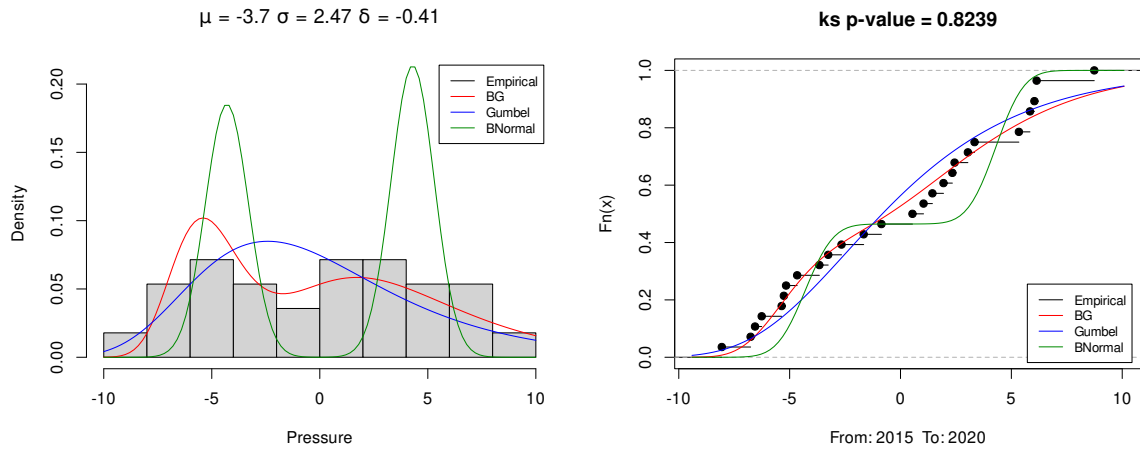


Figure 3: On the left: centralized extreme values histogram versus fitted Gumbel, bimodal normal and BG models. On the right: empirical cdf versus theoretical cdf of Gumbel, bimodal normal and BG models.

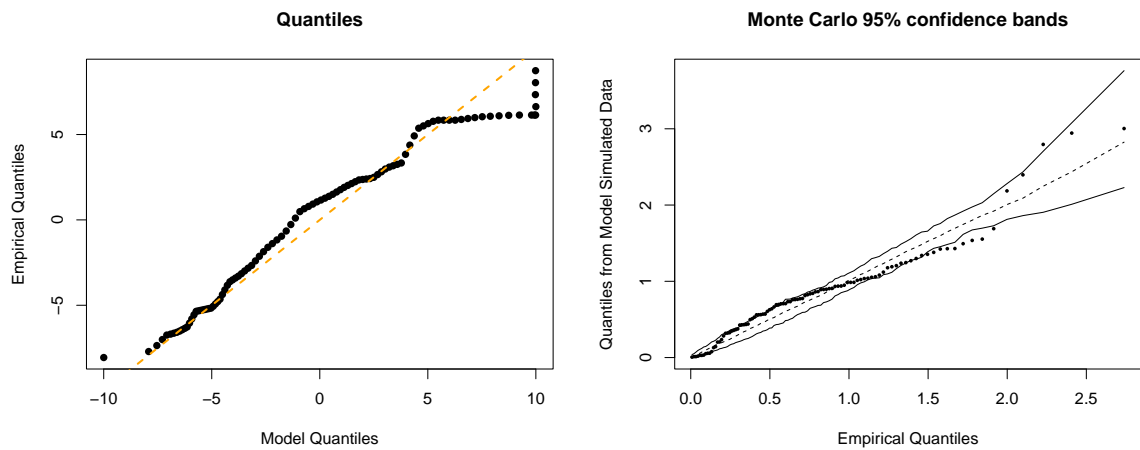


Figure 4: QQ-plot empirical versus theoretical of BG model

### 7. Conclusions

In this paper, we proposed an extension to the Gumbel distribution by using a quadratic transformation technique used to generate bimodal functions produced due to using the quadratic expression, with an additional bimodality parameter which modifies the mode of the distribution, composing as an alternative models for single maxima events. In this generalization, the Gumbel distribution appears as a particular case. The advantage of proposed model is the fact that it can be used to model extreme data with one or two modes. We provide a mathematical treatment of the new model including the mode, bimodality, moment generating function and moments. We performed the modelling under a frequentist approach and the estimation of the parameters was proposed using the maximum likelihood estimation. Finally, we have performed a statistical modeling with real data by using the new proposed model in the article. The application demonstrated the practical relevance of the new model, which also showed the advantage of Gumbel and bimodal normal models. We hope this new model attracts wider applications for bimodal extreme analysis. As part of future research, we plan to explore other estimation methods for the model, such as the Bayesian approach.

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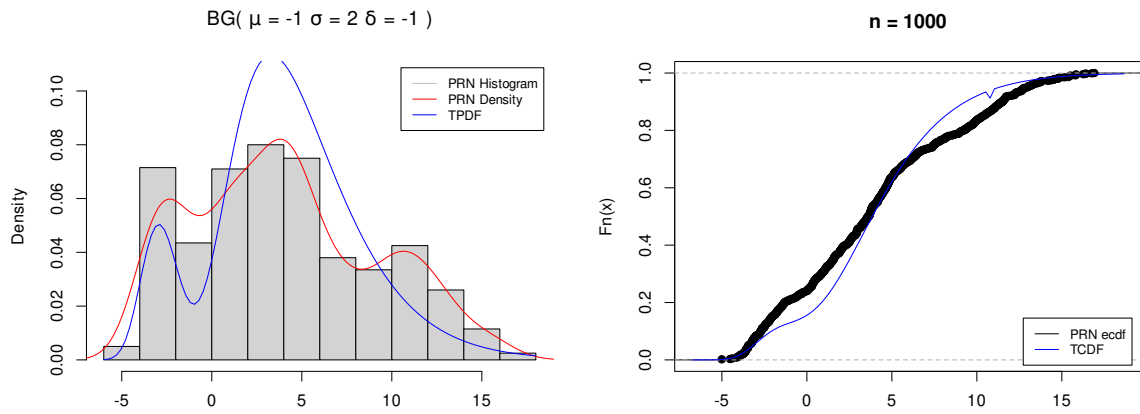
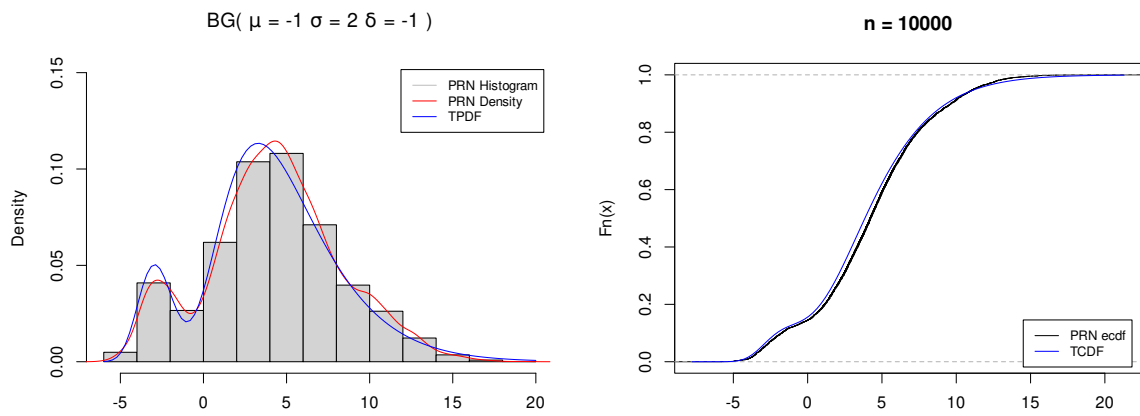
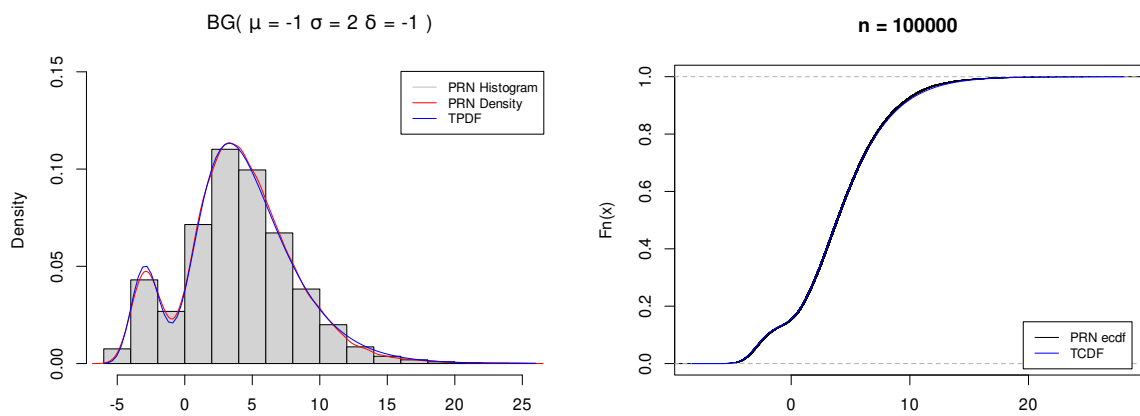
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## Appendix

Figure 5: TPDF (left) and TCDF (right),  $n = 1000$ Figure 6: TPDF (left) and TCDF (right),  $n = 10000$ Figure 7: TPDF (left) and TCDF (right),  $n = 100000$

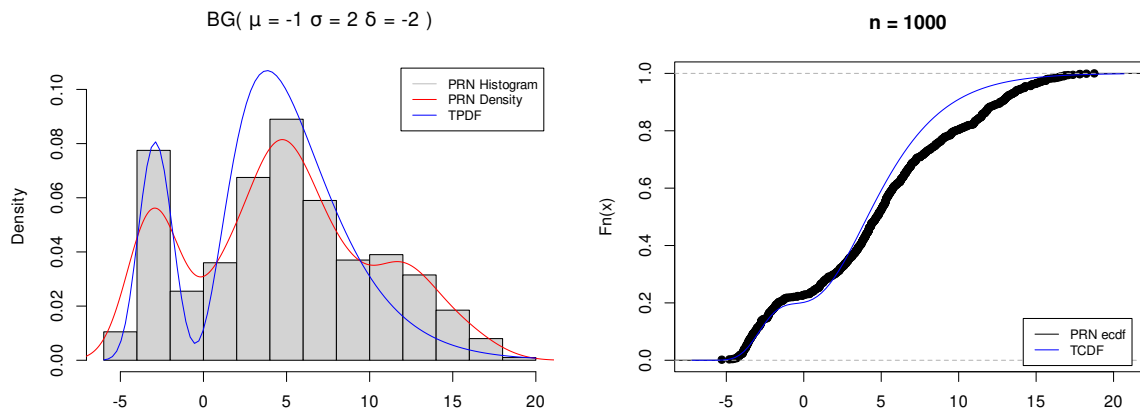


Figure 8: TPDF (left) and TCDF (right),  $n = 1000$

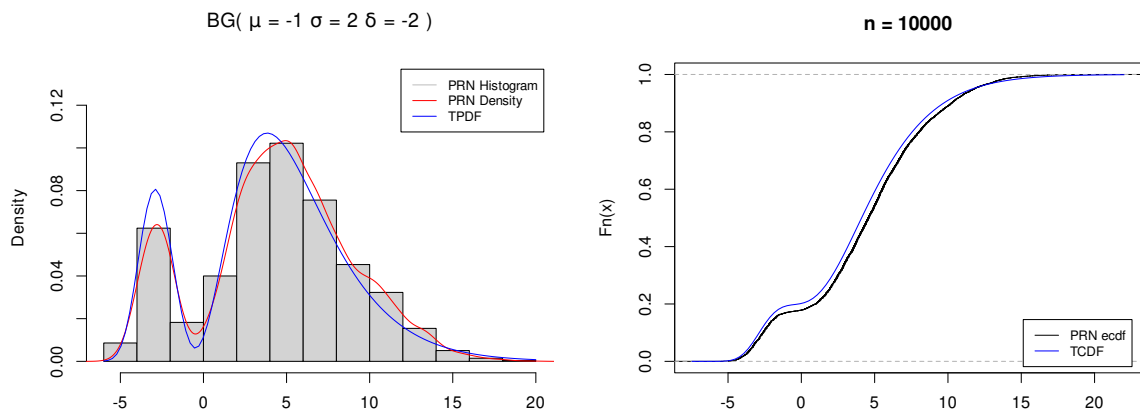


Figure 9: TPDF (left) and TCDF (right),  $n = 10000$

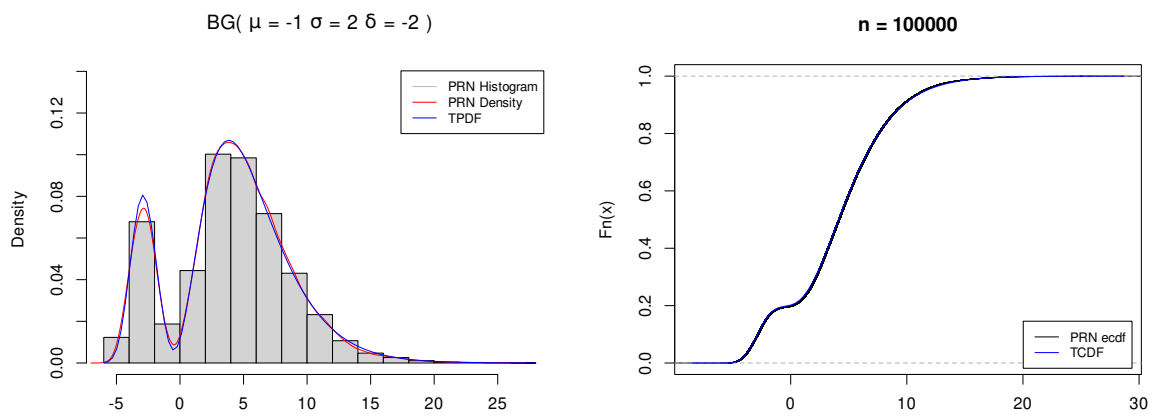
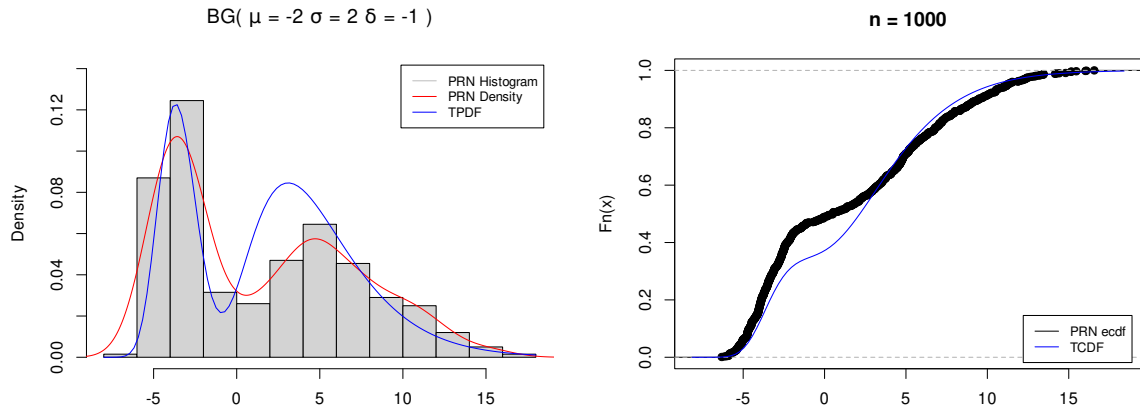
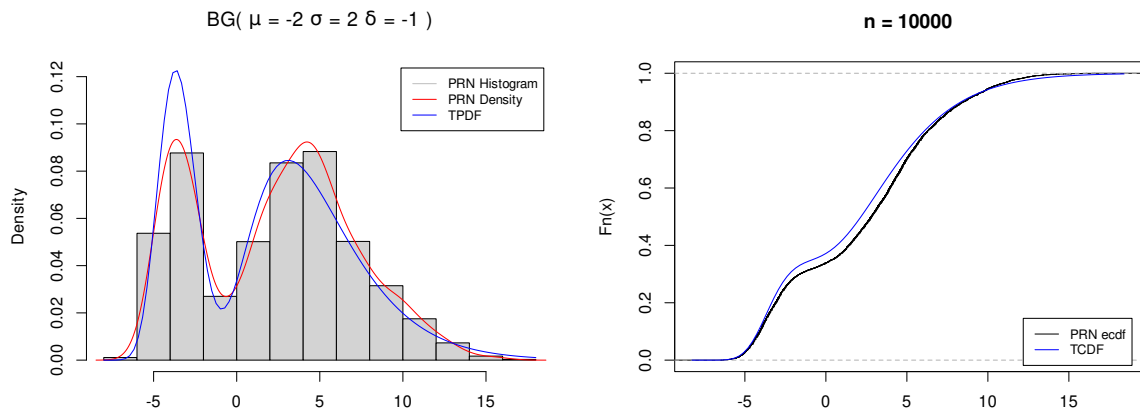
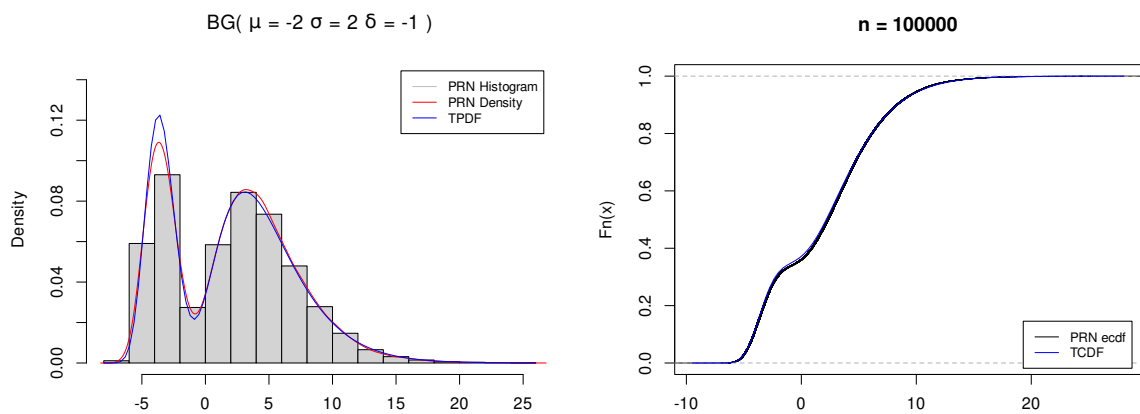


Figure 10: TPDF (left) and TCDF (right),  $n = 100000$

Figure 11: TPDF (left) and TCDF (right),  $n = 1000$ Figure 12: TPDF (left) and TCDF (right),  $n = 10000$ Figure 13: TPDF (left) and TCDF (right),  $n = 100000$

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