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A Comparative Study of Goodness-of-Fit Tests for the Laplace Distribution

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Abstract

The Laplace distribution is one of the earliest distributions in probability theory and is a frequently used distribution in many fields. Consequently, various goodness-of-fit tests for the Laplace distribution have been thoroughly derived in the literature. The purpose of this paper is to carry out a comparative study of these tests as well as a new one we develop. Power comparisons of all such tests are performed via Monte Carlo simulations of sample data generated from twenty seven alternatives distributions. Despite the fact that no single test was found to be most powerful in all situations, several useful recommendations however are made.

Keywords: Laplace distribution, Monte Carlo study, goodness-of-fit.

1. Introduction

The Laplace or the double exponential distribution introduced by Laplace (1774) is one of the earliest distribution discussed in probability theory. It is a symmetric distribution which has been used as an alternative to the normal distribution in robustness studies and in modeling phenomena with heavier than normal tails (see Kozubowski and Nadarajah (2010), Cordeiro and Lemonte (2011) and references therein). The areas in which the Laplace distribution has been used are rather wide. A detailed list of these areas along with some references can be found in Kotz, Kozubowski, and Podgorski (2001), Johnson, Kotz, and Balakrishnan (1995), Kozubowski and Nadarajah (2010) and Cordeiro and Lemonte (2011).

Due to its importance in modeling real data, validating the assumption of Laplace distribution has been of great concern. Indeed, numerous studies have been devoted to testing methods for detecting a departure from Laplace. The existing tests can be classified into five classes, namely (i) tests based on the empirical distribution function (c.f. Yen and Moore 1988; Rublík 1997; Puig and Stephens 2000a; Chen 2002, and references cited therein); (ii) tests based on the empirical characteristic function (c.f. Meintanis 2005); (iii) moment based tests, i.e., tests based on sample moments, skewness and kurtosis (González-Estrada and Villaseñor 2016; Gel 2010; Rayner and Best 1989; Langholz and Kronmal 1991; Li and Papadopoulos 2002); (iv) entropy and divergence based tests (Choi and Kim 2006; Rizzo and Haman 2016; Alizadeh Noughabi and Balakrishnan 2016; Alizadeh Noughabi 2019; Alizadeh Noughabi and

Park 2016) (v) other tests (Gulati 2011).

The main goal of this paper is to present a detailed comparison of the various testing procedures for detecting departures from the Laplace distribution. In this context, the rest of the paper is organized as follows. In Section 2 we present some preliminaries on the Laplace distribution. In Section 3 the existing goodness-of fit tests for the Laplace distribution are briefly presented and classified into one of the five classes mentioned above. Moreover, a new modification of the moment structure based test presented in Li and Papadopoulos (2002, p.74) for testing departure from Laplace is presented and studied in detail. Section 4 is devoted to executing Monte Carlo simulations for comparing the goodness-of fit tests for the Laplace distribution described in Section 3. Such a comparison includes 22 tests available in the literature with 27 possible distributions (symmetric or asymmetric) presented as alternatives to the Laplace distribution. To our best knowledge this study is the first attempt made in the literature to compare so many goodness-of-fit tests for the Laplace distribution and alternative distributions (c.f., Best, Rayner, and Thas (2008) and Gel (2010)). Section 5 presents a summary of the results of the paper and some practical relevant conclusions. All proofs related to the new goodness-of fit test for the Laplace distribution introduced in Section 3 appear in an Appendix. `R` codes used to showing and demonstrating the computational and numerical parts of this paper along with their implementation are also presented in the Appendix.

2. Background

In this section we present some preliminaries and notation related to the Laplace distribution. The classical Laplace distribution $\mathcal{C}L(\delta, c)$ has a two-parameter probability density function (p.d.f) and cumulative distribution function (c.d.f) given by

$$f_0\left(x;\delta,c\right) = \frac{1}{2c}e^{-\frac{|x-\delta|}{c}}, \text{ with } \delta \in R, c > 0, x \in R, \tag{1}$$

and

$$F_0\left(x;\delta,c\right) = 0.5 + 0.5sgn(x-\delta)\left(1 - e^{-\frac{|x-\delta|}{c}}\right), \tag{2}$$

where $sgn(x-\delta)$ equals $-1, 0$, or 1, depending on whether $x-\delta$ is negative, zero, or positive, respectively. The case $\mathcal{C}L(0,1)$ with $\delta = 0$ and $c = 1$ is called the classical standard Laplace distribution.

In the sequel we denote, respectively, by $\mu_l = E(X^l) = \int x^l dF(x)$ and $k_l = E[(X-\mu_1)^l] = \int (x-\mu_1)^l dF(x), l \in \mathbb{N}$, the l-th moment and l-th central moment of a r.v. X with c.d.f. F, while $\sqrt{\beta_1} = k_3/k_2^{3/2}$ and $\beta_2 = k_4/k_2^2$ are used to denote the corresponding skewness and kurtosis. For the $\mathcal{C}L(\delta, c)$, the mean, median and mode are all equal to δ, the variance to $2c^2$, while the skewness and kurtosis are 0 and 6, respectively.

Let $(X_1, ..., X_n)$ be a random sample of size n and $X_{(1)} \le X_{(2)} \le ... \le X_{(n)}$ denote its order statistic arrangement. If this sample is taken from a $\mathcal{C}L(\delta, c)$ population then the maximum likelihood estimators (mle) of δ and c are

$$\hat{\delta}_n := \hat{\delta}_n(X_1, ..., X_n) = \text{Median}(X_1, ..., X_n) = \begin{cases} X_{(n+1)/2}, & \text{if } n \text{ is odd} \\ \frac{X_{(n/2)} + X_{(n/2+1)}}{2}, & \text{if } n \text{ is even} \end{cases}, \tag{3}$$

and

$$\hat{c}_n := \hat{c}_n(X_1, ..., X_n) = \frac{1}{n}\sum_{i=1}^{n} |X_i - \hat{\delta}_n|. \tag{4}$$

By using the fact that $E(X) = \delta$ and $Var(X) = 2c^2$ then the corresponding method of moments estimators (mom) of δ and c are

$$\tilde{\delta}_n := \tilde{\delta}_n(X_1, ..., X_n) = \bar{X}_n \tag{5}$$

and

$$\tilde{c}_n := \tilde{c}_n(X_1, ..., X_n) = \sqrt{\frac{S_n^2}{2}}, \tag{6}$$

where

$$S_n^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2. \tag{7}$$

The mle $\hat{\delta}_n$ and the mom estimator $\tilde{\delta}_n$ of δ are affine equivariant, i.e., for any real d and $\beta > 0$

$$\hat{\delta}_n\left(\beta X_1 + d, ..., \beta X_n + d\right) = \beta\hat{\delta}_n\left(X_1, ..., X_n\right) + d,$$

and

$$\tilde{\delta}_n\left(\beta X_1 + d, ..., \beta X_n + d\right) = \beta\tilde{\delta}_n\left(X_1, ..., X_n\right) + d.$$

Moreover, the mle $\hat{c}_n$ and the mom estimator $\tilde{c}_n$ of c are, respectively, location invariant and scale equivariant, i.e., for any real d and $\beta > 0$

$$\hat{c}_n\left(\beta X_1 + d, ..., \beta X_n + d\right) = \beta\hat{c}_n\left(X_1, ..., X_n\right),$$

and

$$\hat{c}_n\left(\beta X_1 + d, ..., \beta X_n + d\right) = \beta\hat{c}_n\left(X_1, ..., X_n\right).$$

In the rest of the paper we shall use the following notation:

$$\begin{aligned} Z_i &= F_0\left(X_i; \hat{\delta}_n, \hat{c}_n\right), Z_{(i)} = F_0\left(X_{(i)}; \hat{\delta}_n, \hat{c}_n\right), Y_i = \frac{X_i - \delta}{c}, \\ \hat{Y}_i &= \frac{X_i - \hat{\delta}_n}{\hat{c}_n}, \tilde{Y}_i = \frac{X_i - \tilde{\delta}_n}{\tilde{c}_n}, \hat{U}_{(i)} = |\hat{Y}_{(i)}|, \text{and } \hat{V}_i = X_i - \hat{\delta}_n \text{ for } i = 1, ..., n. \end{aligned}$$

3. Goodness-of-fit tests for the Laplace distribution

Let $(X_1, ..., X_n)$ be a random sample of size n taken from a c.d.f. $F(x)$ and p.d.f $f(x)$. At a significance level α we consider testing the null hypothesis that the parent distribution is Laplace vs any other alternative, i.e.,

$$H_0 : F(\cdot) \;= F_0\left(\cdot; \delta, c\right), \text{ for some } \delta \in R \text{ and } c > 0,$$

vs.

$$H_1 : F \;\text{ is not } \mathcal{CL}(\delta, c).$$

As if $X \sim \mathcal{CL}(\delta, c)$ then $\beta X + d \sim \mathcal{CL}(\beta\delta + d, \beta c)$ for any $d \in \mathbb{R}$ and $\beta \in R^+$ it follows that the family of Laplace distributions is invariant under affine transformations $X \to \beta X + d$. Thus any test statistic, say $H_n(X_1, ..., X_n)$, used for testing departure from the Laplace distribution should also be affine invariant (see Meintanis (2005), p.927). Accordingly, if $(X_1, ..., X_n)$ is a random sample from $\mathcal{CL}(\delta, c)$ then the following relation should hold

$$H_n\left(\beta X_1 + d, ..., \beta X_n + d\right) = H_n(X_1, ..., X_n). \tag{8}$$

In the sequel, the existing goodness-of fit tests for the Laplace distribution are briefly presented. Moreover, a new modification of the moment structure based test presented in Li and Papadopoulos (2002, p.74) for testing departure from the Laplace distribution is presented and studied in detail. The modification is required as the test proposed by Li and Papadopoulos (2002) is not affine invariant.

As we previously indicated all existing tests are classified into five classes.

3.1. Tests based on the empirical distribution function

The key idea of the empirical distribution function (e.d.f.) tests is to compare the data estimated c.d.f. with the hypothesized c.d.f. Thus e.d.f. tests are based on discrepancy measures between the c.d.f. of the Laplace distribution given in (2) with δ and c being estimated appropriately and the e.d.f. defined by

$$F_n(x) = \frac{\sum_{i=1}^{n} I(X_i \leq x)}{n} = \frac{\#\text{ of observations} \leq x}{n}, -\infty < x < \infty, \tag{9}$$

where $I(\cdot)$ is the indicator function. In this frame, five different e.d.f. tests have been presented and studied in the literature, namely the Cramer-von Mises (W^2), the Watson (U^2), the Anderson-Darling (A^2), the Kolmogorov-Smirnov ($\sqrt{n}D$) and the Kuiper (V) tests (c.f. Yen and Moore 1988; Puig and Stephens 2000a; Chen 2002, and references cited therein). The test statistics W^2, U^2 and A^2 belong to the Cramer von Mises family, while the other two to the Kolmogorov-Smirnov family of tests. These test statistics have the form

$$W^2 = \frac{1}{12n} + \sum_{i=1}^{n} \left(Z_{(i)} - (2i-1)/(2n)\right)^2, \tag{10}$$

$$U^2 = W^2 - n\left(\bar{Z}_n - 0.5\right)^2, \tag{11}$$

$$A^2 = -n - \frac{1}{n}\sum_{i=1}^{n} \left((2i-1)\log(Z_{(i)}) + (2(n-i)+1)\log(1-Z_{(i)})\right), \tag{12}$$

$$D = \max\left\{D^+, D^-\right\} \text{ and } V = D^+ + D^-, \tag{13}$$

where

$$D^+ = \max_{i=1,..,n}\left(\frac{i}{n} - Z_{(i)}\right) \text{ and } D^- = \max_{i=1,..,n}\left(Z_{(i)} - \frac{i-1}{n}\right) \tag{14}$$

with the $Z_{(i)}$ being defined in Section 2.

All of the above tests are right tailed, i.e., the null hypothesis is rejected for large values. Therefore the critical values are the $100(1-\alpha)$-th percentiles of the empirical distribution of the respective test statistic. Asymptotic critical values are also available for the tests belonging to the Cramer von Mises family (see Puig and Stephens (2000a)). A more detailed discussion, asymptotic results on the e.d.f. tests and tables of critical values based on $50,000$ Monte Carlo samples of size n for specific values of n can be found in Puig and Stephens (2000a).

3.2. Tests based on the empirical characteristic function

The characteristic function (c.f.) of $X \sim \mathcal{CL}(\delta, c)$ is

$$\phi_0(t;\delta,c) = \frac{\exp(i\delta t)}{1+c^2t^2}, t \in \mathbb{R}.$$

Meintanis (2005) proposed a class of goodness-of-fit tests for the Laplace distribution based on its c.f. More specifically, the key idea behind his proposal is related to the fact that the c.f. of the Laplace distribution satisfies the relation

$$(1+c^2t^2)\phi(t) - \exp(i\delta t) = 0, t \in \mathbb{R}. \tag{15}$$

Under the null hypothesis of Laplace distribution then for large n the transformed data $\hat{Y}_i$ or $\tilde{Y}_i$ are approximately $\mathcal{CL}(0,1)$. By using these transformed data, Meintanis (2005) proposed a weighted integral of the squared of an empirical counterpart of the equation (15) for $c = 1$ and $\delta = 0$. To be more specific, Meintanis (2005) proposed the following classes of test statistics

$$T_n^j = n\int_{-\infty}^{\infty} |(1+t^2)\phi_n^j(t) - 1|^2 w(t)dt,\ j = ML, MO, \tag{16}$$

where $w(t)$ denotes an appropriate weight function, $\phi_n^{ML}(t)$ and $\phi_n^{MO}(t)$ are the empirical characteristic functions (e.c.f.) of the transformed data $\hat{Y}_i$ and $\tilde{Y}_i$, respectively, defined by

$$\phi_n^{ML}(t) = \frac{\sum_{i=1}^n \exp(it\hat{Y}_i)}{n} \text{ and } \phi_n^{MO}(t) = \frac{\sum_{i=1}^n \exp(it\tilde{Y}_i)}{n}.$$

For computational purposes Meintanis (2005) focused on two parametric classes of weight functions for which both T_n^{ML} and T_n^{MO} take simple forms. In this frame, the test statistics corresponding to $w(t) = \exp(-a|t|)$, $a > 0$ are denoted by $T_{n,a}^{(1,MO)}$ or $T_{n,a}^{(1,ML)}$. Similarly, the test statistics corresponding to $w(t) = \exp(-at^2)$, $a > 0$, are denoted by $T_{n,a}^{(2,MO)}$ or $T_{n,a}^{(2,ML)}$. For simple forms of these statistics and further computational details we refer to p. 928 in Meintanis (2005). Based on the simulation study performed by Meintanis (2005) it is recommended to use $T_{n,2}^{(1,MO)}$, $T_{n,2}^{(1,ML)}$, $T_{n,0.5}^{(1,MO)}$ and $T_{n,0.5}^{(2,ML)}$. Indeed, for this reason we use these statistics in our simulation study.

All the above tests are again right tailed. The appropriate critical values which are the $100(1-\alpha)$-th percentiles of the empirical distribution of the respective test statistics are given in Meintanis (2005). A further detailed discussion, asymptotic results on the e.c.f. tests and tables of critical values based on $100,000$ Monte Carlo samples of size $n = 20, 50$ can be found in Meintanis (2005).

3.3. Sample moments based procedures

Moment-based procedures are widely common for testing departures from a hypothesized distribution. In this section some existing tests are briefly presented, while a new modification of the moment structure based test presented in Li and Papadopoulos (2002, p.74) for testing a departure from the Laplace distribution is presented and studied in detail.

Best et al. tests

Smooth tests of goodness of fit, described in Rayner and Best (1989), seek to assess the fit of the data to a given p.d.f. $f(x;\theta)$ within the following class of alternatives of order k given by

$$g_k(x;\theta,\beta) = C(\theta,\beta)\exp\left(\sum_{i=1}^k \beta_i h_i(x;\theta)\right) f(x;\theta).$$

Here, θ is a vector of unknown parameters, $C(\theta,\beta)$ is a normalizing constant and $h_i(x;\theta), i = 1, ..., k$, is a set of functions which are orthonormal on the hypothesized distribution $f(x;\theta)$. If E_0 denotes the expectation when the model that generates the data is $f(x;\theta)$ then orthonormality means that $E_0[h_r(X;\theta)h_s(X;\theta)] = \delta_{rs}$ for $r, s = 0, 1, ...$, where $\delta_{rs} = 1$ if $r = s$ and $\delta_{rs} = 0$ if $r \neq s$. It is obvious that the alternatives are characterized by their order, i.e., the greater the order k the richer is the class of alternatives.

In this frame, based on a sample X_1,...,X_n, if $\widetilde{\theta}_n(X_1, ..., X_n)$ is either the mle or the moment based estimator of the unknown parameters θ of the hypothesized distribution, Rayner and Best (1989) proposed a goodness-of-fit test of a hypothesized distribution using

$$V_r = \sum_{i=1}^n h_r\left(X_i; \widetilde{\theta}_n(X_1, ..., X_n)\right)/\sqrt{n}.$$

Motivated by Rayner and Best (1989) and under the moment estimation method when utilizing the complete orthonormal functions for $r = 3, 4$, Best *et al.* (2008) proposed the following two smooth goodness-of fit tests for the Laplace distribution:

$$V_3 = \sqrt{nb_1/54} \text{ and } V_4 = (b_2 - 6)\sqrt{n/1072.8} \tag{17}$$

where

$$\sqrt{b_1} = \frac{\sum_{i=1}^n \left(X_i - \bar{X}_n\right)^3}{nS_n^3} \text{ and } b_2 = \frac{\sum_{i=1}^n \left(X_i - \bar{X}_n\right)^4}{nS_n^4} \tag{18}$$

and S_n is defined in (7). Here, $\sqrt{b_1}$ and b_2 are estimators of the population skewness and kurtosis, respectively, based on method of moments.

Asymptotically, both statistics follow a normal distribution with mean zero and variance $7/6$ and $165/149$, respectively (see Best *et al.* (2008)). However Best *et al.* (2008) concluded that in practice the use of the asymptotic critical values is not recommended as the the convergence is slow. Taking into account that the null hypothesis of Laplace for both tests is rejected for large absolute values (two-tailed tests), then the critical values are the $100(\alpha/2)$-th and $100(1-\alpha/2)$-th percentiles of the empirical distribution of the test statistics.

Gel tests

Gel (2010) proposed goodness-of-fit procedures for the Laplace distribution based on alternative estimates of population skewness and kurtosis obtained by utilizing the mle of the unknown variance in the denominator of the population skewness and kurtosis. Specifically, the K test by Gel (2010) is defined as

$$K = \frac{n}{C_1}\left(\sqrt{u_1}\right)^2 + \frac{n}{C_2}\left(u_2 - 6\right)^2, C_1, C_2 > 0, \tag{19}$$

where

$$\sqrt{u_1} = \frac{n^{-1}\sum_{i=1}^n (X_i - \bar{X}_n)^3}{(\sqrt{2}\hat{c}_n)^3} \text{ and } u_2 = \frac{n^{-1}\sum_{i=1}^n (X_i - \bar{X}_n)^4}{(\sqrt{2}\hat{c}_n)^4},$$

with $\hat{c}_n$ being the mle of c given in (4) while C_1 and C_2 are the asymptotic variances of $\sqrt{nu_1}$ and $\sqrt{n}u_2$, respectively. Gel (2010) proved that under the Laplace distribution, then K follows asymptotically a chi-square distribution with two degrees of freedom. Here the null hypothesis of Laplace distribution is rejected if $K \geq \chi^2_{1-\alpha,2}$ where $\chi^2_{1-\alpha,2}$ is the upper a percentile of the chi-square distribution with two degrees of freedom. The constants C_1 and C_2 can be obtained using the multivariate Taylor-expansions. However, as these calculations are rather cumbersome, Gel (2010) recommended for small or moderate samples to use $C_1 = 60$ and $C_2 = 1200$ or to approximate them based on functions given in p. 960 by Gel (2010). Also, based on a Monte Carlo study by Gel (2010), the use of asymptotic critical values for small to moderately large samples is not recommended. Instead the use of the empirical critical values is recommended in which case the choice of C_1 and C_2 does not play any role. In this frame, taking into account that the null hypothesis of Laplace distribution is rejected for large values of K - implying a right tailed test - the critical values are taken to be $100(1-\alpha)$-th percentiles of the empirical distribution of the respective test statistic. For more details on the K tests as well as to individual tests based on $\sqrt{u_1}$ and u_2 see Gel (2010).

A ratio gof tests

González-Estrada and Villaseñor (2016) proposed two tests based on ratio of estimators for the scale parameter of the Laplace distribution. To be more specific, the first test statistic denoted by R_n is defined as the ratio of the sample mean absolute deviation around the sample mean to the moment estimator of c, while the second test denoted by $R_n^{'}$ is defined as the ratio of the sample mean absolute deviation around the sample mean to the mle of c, i.e.,

$$R_n = \frac{\sqrt{2}\sum_{i=1}^n |X_i - \bar{X}_n|}{nS_n} \text{ and } R_n^{'} = \frac{\sum_{i=1}^n |X_i - \bar{X}_n|}{\sum_{i=1}^n |X_i - \hat{\delta}_n|}, \tag{20}$$

where S_n^2 is defined in (7) and $\hat{\delta}_n$ is the mle of δ given in (3).

The idea behind the two tests is that under the null hypothesis of Laplace distribution the test statistics are expected to take values close to one. Thus both test are two-tailed tests. González-Estrada and Villaseñor (2016) proved that under the hypothesis of Laplace distribution $\sqrt{4n}(R_n - 1)$ follows asymptotically a standard normal distribution. Thus the null hypothesis is rejected if $|\sqrt{4n}(R_n - 1)| \geq z_{\alpha/2}$, where $z_{\alpha/2} = \Phi^{-1}(1-\alpha/2)$, $\Phi^{-1}(\cdot)$ is the quantile function of the $N(0,1)$ distribution. In the simulation study of the next section we use the $100(\alpha/2)$-th and the $100(1-a/2)$-th percentiles of the empirical distribution of the test statistics. For more details on R_n and R_n' tests see González-Estrada and Villaseñor (2016).

Langholz and Kronmal test

Goodness-of-fit tests for the Laplace distribution can obtained as a special case of the class of tests proposed by Langholz and Kronmal (1991). The key idea behind their method is motivated by the fact that when the hypothesized distribution F is completely specified then $X \sim F$ if and only if $F(X) \sim U(0,1)$. Their approach compares then the estimated Fourier coefficients to those of the $U(0,1)$ density. For the special case of Laplace, their test statistic based on the first Fourier coefficient in the density estimation procedure of Fellner (1974) can be written in the following form

$$K_1 = 2.26n\left(\hat{C}^2 + \hat{S}^2\right) \tag{21}$$

where

$$\hat{C} = n^{-1}\sum_{i=1}^{n}\cos\left(2\pi Z_i'\right) \text{ and } \hat{S} = n^{-1}\sum_{i=1}^{n}\sin\left(2\pi Z_i'\right)$$

are the estimated first trigonometric moments, with $Z_i' = F_0\left(X_i; \tilde{\delta}_n, \tilde{c}_n\right)$. Under the hypothesis of Laplace, K_1 follows asymptotically a chi-square distribution with two degrees of freedom. Since the appropriate test is right-tailed we shall use in our simulation section the empirical critical values being the $100(1-\alpha)$-th percentiles of the empirical distribution. For more details on the K_1 test see Langholz and Kronmal (1991).

A modification of a test based on moment structure

A general and interesting method for testing a departure from a given parametric family of distributions was proposed by Li and Papadopoulos (2002). Their idea is mainly simple as it is based on some moment structure relation holding among the members of the respective parametric family of distributions under the null hypothesis. Then based on such a relation, a test statistic is proposed for testing departures from the relevant family. Li and Papadopoulos (2002) demonstrated their approach for some common parametric families of distributions. Among them is the Laplace family for which a test was proposed but was not studied in details. In the sequel, we present a modification of such a test by requiring that it will be affine invariant and study it in details.

If F is $\mathcal{CL}(\delta, c)$ then it can be simply seen that

$$g(\mu_1, \mu_2, \mu_3) \doteq \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3 = 0, \text{ for any } \delta \text{ and } c. \tag{22}$$

This would imply that if the relation in (22) does not hold then the sample is not taken from $\mathcal{CL}(\delta, c)$. The reverse statement is obviously incorrect as there might exist some other distributions for which (22) holds. Such a test is termed in the literature (c.f. Fang, Zhu, and Bentler 1993; Liang, Fang, and Hickernell 2008; Batsidis and Zografos 2013) as a 'necessary' (but not sufficient). Necessary tests imply that small p-values (say, less than 5%) of the tests indicate evidence for departure from the family of distributions under the null hypothesis whereas larger p-values imply that no sufficient evidence is available for drawing any other statistical conclusion.

Indeed, a necessary test based on (22) for testing H_0 vs. H_1 was proposed by Li and Papadopoulos (2002) and involved with the empirical estimator of $g(\mu_1, \mu_2, \mu_3)$. In the sequel, however, we will modify their test by imposing the property of affine invariance - a desired property for any goodness-of fit test for the Laplace distribution.

More specifically, the proposed test will be based on the transformed data $\hat{Y}_i = (X_i - \hat{\delta}_n)/\hat{c}_n$ and an empirical estimator T_n of $g(\mu_1, \mu_2, \mu_3)$ based on them. The asymptotic distribution of T_n is derived under the null hypothesis of Laplace in the following theorem.

Theorem 3.1. *Let $(X_1, X_2, ..., X_n)$ be a random sample from $\mathcal{CL}(\delta, c)$ and $\hat{Y}_i = (X_i - \hat{\delta}_n)/\hat{c}_n$, $i = 1, ..., n$, be the transformed data. Let*

$$T_n = \frac{1}{n}\sum_{i=1}^{n}\hat{Y}_i^3 - 3\frac{1}{n^2}\sum_{i=1}^{n}\hat{Y}_i\sum_{i=1}^{n}\hat{Y}_i^2 + 2\left(\frac{1}{n}\sum_{i=1}^{n}\hat{Y}_i\right)^3, \tag{23}$$

where

$$\hat{Y}_i = \frac{X_i - \hat{\delta}_n(X_1, ..., X_n)}{\hat{c}_n(X_1, ..., X_n)}, \tag{24}$$

$\hat{\delta}_n = \hat{\delta}_n(X_1, ..., X_n)$ and $\hat{c}_n = \hat{c}_n(X_1, ..., X_n)$ are the mle's of the parameters δ and c, respectively. Then the statistic $\sqrt{n}T_n$ converges in distribution, as $n \longrightarrow \infty$, to the normal distribution $N(0, 504)$.

The proof of the Theorem 3.1 is relegated to Appendix A. The results of Theorem 3.1 can be used to construct a necessary test for testing the hypothesis of Laplace. Indeed by the previous theorem it follows that if the null hypothesis H_0 is true then it is also true that the statistic $Z_n = \sqrt{\frac{n}{504}}T_n$ is asymptotically $N(0, 1)$. Consequently, in view of Theorem 3.1, H_0 should be rejected at a significance level α if $| Z_n | \geq z_{\alpha/2}$, which implies departure from the Laplace distribution. On the contrary, failing to reject the null hypothesis implies that no sufficient information is available for drawing any statistical conclusion on the null hypothesis.

However in order to use the asymptotic critical values one should examine the convergence of the percentiles of Z_n to those of the $N(0, 1)$ distribution subject to the assumption that the data are stemming from a Laplace distribution. To achieve this, a total of $l = 100.000$ samples of different sample sizes ($n = 20, 30, 50, 60, 70, 100, 200, 500, 1000, 5000$) were generated from $\mathcal{CL}(0, 1)$. Note that one can confine the study to the case of $\mathcal{CL}(0, 1)$ since the asymptotic distribution of T_n is independent of the parameters δ and c and T_n is affine invariant. For each sample, the value of Z_n was computed. Then, based on all l values of Z_n, Monte Carlo percentiles were computed and compared with the theoretical limiting percentiles. The simulation was carried out by using R (R Core Team 2020). The `R` code used is presented in the Appendix B.

The results are displayed in Table 1. The last row of the table displays the corresponding percentiles of the $N(0, 1)$ distribution. The results indicate a slow convergence of the critical values to their limiting values. This suggests that the limiting critical values may not provide a good approximation. Hence, for small and moderate sample sizes one can use a parametric bootstrap for computing the p-values instead of using the asymptotic distribution. A Monte Carlo study was carried out on the type I error rates in order to examine the performance of the test based on these latter two options. The empirical type I error rates appear in Table 2. They are computed by the relation

$$\text{Empirical Type I error rate} = \frac{\text{Number of rejections}}{\text{Number of replications}}$$

at significance levels $\alpha = 0.05$ and 0.1.

From Table 2 it is clear that for small and moderate sample sizes the use of the parametric bootstrap is recommended. This is not a disadvantage of the test since as Gel (2010) already

Table 1: Simulated lower and upper critical values of Z_n indicated for α and n as shown and 100000 simulated samples from $\mathcal{C}L(0,1)$

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.1$	
n	Z_n^l	Z_n^u	Z_n^l	Z_n^u	Z_n^l	Z_n^u
20	-2.49739	2.40483	-1.35659	1.32971	-0.95007	0.95038
30	-2.75772	2.74174	-1.50211	1.50553	-1.08818	1.09160
50	-2.96832	2.99391	-1.65895	1.69383	-1.22410	1.23730
60	-2.98985	3.07540	-1.71866	1.72124	-1.27531	1.27553
70	-3.07916	3.09744	-1.76075	1.75607	-1.30965	1.29881
100	-3.06851	3.14049	-1.82853	1.83216	-1.37432	1.37556
200	-3.03301	3.03401	-1.89422	1.91534	-1.46851	1.48683
500	-2.91472	2.88816	-1.94998	1.95911	-1.56223	1.57465
1000	-2.79985	2.83282	-1.95724	1.97529	-1.59362	1.60403
5000	-2.64775	2.66607	-1.96253	1.97677	-1.63014	1.63365
$N(0,1)$	-2.57583	2.57583	-1.95996	1.95996	-1.64485	1.64485

Table 2: Simulated type I rate: percentage of samples with p-value smaller that α. $\mathcal{C}L(0,1)$, number of simulated samples: 10,000, p-value: 1) asymptotic and 2) bootstrap p-value: number of bootstrap samples: 1000

	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.1$	
n	asymptotic	bootstrap	asymptotic	bootstrap	asymptotic	bootstrap
20	0.01	0.0116	0.0218	0.0508	0.0325	0.1002
30	0.0125	0.0096	0.0281	0.0533	0.0443	0.0998
50	0.0157	0.0093	0.0307	0.0482	0.0502	0.0944
60	0.0172	0.0095	0.0363	0.0494	0.0555	0.101
70	0.0155	0.0081	0.0346	0.0467	0.0578	0.0947
100	0.0185	0.01	0.0417	0.0521	0.0695	0.1024

pointed out critical values of other available tests detecting departures from the Laplace distribution can be obtained either from special tables or from a Monte Carlo study.

The `R` function for the proposed goodness-of-fit test with options for bootstrap p-value and p-value based on the asymptotic standard normal distribution is displayed in Appendix B. In the simulation study of the next section as well as in practice someone can utilize - taking into account that the test based on Z_n is a two-tailed - the $100(\alpha/2)$-th and $100(1-a/2)$-th percentiles of the empirical distribution of the test statistic.

3.4. Entropy and divergence based tests

Entropy and divergence based tests are widely common for testing departures from a hypothesized distribution. In this section some existing tests are briefly presented.

Maximum entropy test

Choi and Kim (2006) presented three goodness-of-fit tests for the Laplace distribution based on its maximum entropy characterization result. However, based on a Monte Carlo study they recommended the use of only one of these tests. Accordingly we briefly discuss in the sequel only this recommended test.

Let X be a r.v. with density function $f_X(x)$. The Shannon's entropy (see Shannon (1948)) of X is defined by $H(f_X) = E\left(-\log(f_X(x))\right)$. Choi and Kim (2006) proved that under the restriction that $E|X| = c$ the distribution of X maximizing the Shannon's entropy is the $\mathcal{CL}(0, c)$ and its entropy is $H(f_X)$. Utilizing this maximum entropy characterization result, Choi and Kim (2006) proposed a test statistic for testing departures from the Laplace distribution based on an estimation of the entropy difference between the data-generating distribution and the hypothetical distribution. In this frame, a parametric procedure for estimating the entropy of the hypothetical distribution and a nonparametric one for estimating the entropy of the data-generating distribution were used. Following this estimation procedure, the entropy based gof test statistic is given by

$$T^V_{m,n} = \frac{n}{2m\hat{c}_n}\left\{\prod_{i=1}^{n}\left(\hat{V}_{(i+m)} - \hat{V}_{(i-m)}\right)\right\}, \tag{25}$$

where m, denoting the window size, is a positive integer smaller than $n/2$ while $\hat{V}_{(i-m)} = \hat{V}_{(1)}$ for $i \leq m$ and $\hat{V}_{(i+m)} = \hat{V}_{(n)}$ for $i \geq n-m$. According to Choi and Kim (2006), m is selected in advance and its optimal choice corresponding to a given sample size n was studied for various sample sizes up to 100. In Table 4 by Choi and Kim (2006) the results of this study is given for $n \leq 50$, while the rest of the results are available upon request from the authors. In the simulation study the values $m = 3$ for $n = 20$, $m = 6$ for $n = 50$ and $m = 13$ for $n = 100$ were used.

Based on the asymptotic results obtained by Choi and Kim (2006) the null hypothesis of Laplace is rejected when the test statistic is less than the corresponding critical value at a designated significance level α (left-tailed). $T^V_{m,n}$ is approximately normally distributed under the null hypothesis. Its asymptotic variance however is not easy to be derived. Consequently, we use the empirical critical values which are the $100a$-th percentiles of the empirical distribution. For more details on the $T^V_{m,n}$ tests and a table of critical values for selected values of m and n see Choi and Kim (2006).

Energy distance test

The idea behind the class of energy distance goodness-of-fit test is based on the following characterization of equality of distributions: if X_1 and Y_1 are independent random variables such that $E \mid X_1 \mid < \infty$ and $E \mid Y_1 \mid < \infty$, with cumulative distribution functions F_1 and G_1, respectively then

$$D^2(F_1, G_1) = 2E \mid X_1 - Y_1 \mid - \mid X_1 - X_1^{'} \mid - \mid Y_1 - Y_1^{'} \mid \geq 0$$

with equality to zero if and only if X_1 and Y_1 are identically distributed, where $X_1^{'}$ and $Y_1^{'}$ are i.i.d. copies of the random variable X_1 and Y_1, respectively; that is, X_1 and $X_1^{'}$ are i.i.d., and Y_1 and $Y_1^{'}$ are i.i.d. Note that energy distance between the distributions F_1 and G_1 is defined to be the square root of $D^2(F_1, G_1)$.

As pointed out by Rizzo and Szekely (2016), energy distance is a metric that measures the distance between the distributions of random variables or vectors and is zero if and only if the distributions are identical. Thus energy distance characterizes equality of distributions and provides a theoretical foundation for statistical inference and goodness-of-fit tests. For a review and implementation of the energy goodness-of-fit test we refer to Rizzo and Szekely (2016).

In the frame of goodness-of-fit test the distributions to be compared are the hypothesized distribution given in the null hypothesis and the sample distribution. Rizzo and Haman (2016) presented the results related with the expected distance of a random variable X which follows asymmetric Laplace distribution from an arbitrary point and with the expected distance $E \mid X - X^{'} \mid$ when X and $X^{'}$ are independent and identically asymmetric Laplace distributed. Taking into account that symmetric Laplace distribution is a special case of the asymmetric one and the previous mentioned results, the test statistic proposed by Rizzo and Haman (2016) is given by the following relation:

$$E_n = 2\sum_{i=1}^{n} \left(\mid \hat{Y}_i \mid + exp\left(- \mid \hat{Y}_i \mid \right) \right) - 1.5n - \frac{2}{n}\sum_{k=1}^{n}(2k - 1 - n)\hat{Y}_{(k)}, \tag{26}$$

where $\hat{Y}_i$, $i = 1, ..., n$ were defined in Section 2. The null hypothesis of Laplace distribution is rejected for large values of E_n- implying a right tailed test- the critical values are taken to be the $100(1-\alpha)$-th percentiles of the empirical distribution of E_n.

Alizadeh Noughabi and Balakrishnan tests

Divergence measures are indices of similarity or dissimilarity between populations and are used for the development of statistical methods in order to formulate and solve a great variety of statistical problems (see the monograph by Pardo (2006)). One of the widely used and studied divergence measure which includes many others as special cases is the family of divergence measures which is known as ϕ-divergence and were defined simultaneously by Csiszar (1963) and Ali and Silvey (1966). Let P and Q denote two probability measures over a measurable space M such that P is absolutely continuous with respect to Q, then the ϕ-divergence is defined as

$$D_\phi(P, Q) = \int_M \phi\left(\frac{dP}{dQ}\right) dQ \tag{27}$$

where $\phi : [0, +\infty) \to (-\infty, \infty)$ is a convex and continuous function such that $\phi(1) = \phi'(1) = 0$, $0\phi(\frac{0}{0}) = 0$ and $0\phi(\frac{p}{0}) = p \lim_{u\to+\infty} \frac{\phi(u)}{u}$. Well-known divergence measures are constructed by suitable choices of ϕ. For instance the Kullback-Leibler (Kullback and Leibler (1951)) is obtained when $\phi(t) = t\log(t)$. Notice that for all probability measures $D_\phi(P, Q) \geq 0$, while $D_\phi(P, Q) = 0$ if $P = Q$.

As a consequence of this last property, a goodness-of-fit test can be constructed based on an estimate of the ϕ-divergence between the true density of the observations X_1,...,X_n and the hypothesized distribution under the null hypothesis. Based on this idea Alizadeh Noughabi and Balakrishnan (2016) introduced a general goodness-of-fit test based on an estimate through kernel density estimation of this divergence. In their study Alizadeh Noughabi and Balakrishnan (2016) considered five different choices for the function ϕ function and applied them to the goodness-of-fit test for the normal, exponential, uniform and Laplace distributions.

Based on a Monte Carlo study Alizadeh Noughabi and Balakrishnan (2016) concluded that for the Laplace distribution, the test based on Kullback-Leibler divergence performs quite well as compared to the EDF tests and the other four tests proposed by them by using different member of the ϕ-divergence. For this reason, we only consider this test in our simulation study which is defined as follows:

$$T_{KL} = \frac{1}{n}\sum_{i=1}^{n} \log\left(\frac{\hat{f}(X_i)}{f_0(X_i;\hat{\delta}_n,\hat{c}_n)}\right) \tag{28}$$

where $f_0(x;\delta,c)$ is the p.d.f. of the classical Laplace distribution $\mathcal{CL}(\delta,c)$ given in (1), while $\hat{f}$ is a kernel density estimator of the unknown true density. Alizadeh Noughabi and Balakrishnan (2016) proposed (see p. 414) to use the following kernel density estimator

$$\hat{f}(X_i) = \frac{1}{ns1.06n^{-1/5}}\sum_{j=1}^{n} f\left(\frac{X_i - X_j}{1.06S_n n^{-1/5}}\right), \tag{29}$$

where S_n denotes the sample standard deviation which is obtained from relation (7), while f is the p.d.f. of the standard normal distribution. The null hypothesis of Laplace distribution is rejected for large values of T_{KL}- implying a right tailed test- the critical values are taken to be the $100(1-\alpha)$-th percentiles of the empirical distribution of T_{KL}. For more details about this class of tests we refer to Alizadeh Noughabi and Balakrishnan (2016), while for more details about ϕ-divergence measures see Pardo (2006).

Alizadeh Noughabi test

Let $X_1,...,X_n$ be a sample of size n from a population with unknown true density say $g(x)$ and suppose that we interest to test the null hypothesis $H_0 : g(x) = f(x;\theta)$, for some $\theta \in \Theta$ with $f(x;\theta)$ the p.d.f. of a parametric family of distributions. Then, as mentioned in the previous subsection, a goodness-of-fit test can be constructed based on an estimator of a divergence measure between $g(x)$ and $f(x;\theta)$. Motivated by this idea, Alizadeh Noughabi (2019) proposed recently a general statistic for the goodness-of-fit test of statistical distribution which is constructed based on an estimate of Kullback-Leibler divergence. Recall that Kullback-Leibler is a special case of the ϕ divergence given in (27) with $\phi(t) = tlogt$. Thus the Kullback-Leibler between $g(x)$ and $f(x;\theta)$ is given by:

$$\begin{aligned} D_{KL}(g,f) =& \int g(x)\log\left(\frac{g(x)}{f(x;\theta)}\right)dx \\ =& E_g\left(\log g(X)\right) - E_g\left(\log f(X;\theta)\right) \\ =& -H(g_X) - E_g\left(\log f(X;\theta)\right) \end{aligned} \tag{30}$$

where $H(g_X)$ is Shannon's entropy. In this frame, Alizadeh Noughabi (2019) proposed to estimate the entropy term by Vasicek's estimate and the term $E_g(\log f(X;\theta))$ by a semi-parametric estimate. In this frame the proposed test statistic for testing departures from Laplace distribution is the following:

$$DA = -\frac{1}{n}\sum_{i=1}^{n} \log\left(\frac{n}{2m}\left(Z_{(i+m)} - Z_{(i-m)}\right)\right) \tag{31}$$

where m denoting the window size is a positive integer smaller than $n/2$, while $Z_{(i-m)} = Z_{(1)}$ for $i \le m$ and $Z_{(i+m)} = Z_{(n)}$ for $i \ge n-m$. Alizadeh Noughabi (2019) mentioned that the optimal choice of m equals 4 for $n = 10$, $m = 7$ for $n = 20$, $m = 15$ for $n = 100$ and that it increases with n while the ration m/n tends to zero. In our simulation study the value $m = 10$ for $n = 50$ was also used. The null hypothesis of Laplace distribution is rejected for large values of DA- implying a right tailed test- the critical values are taken to be the

$100(1-\alpha)$-th percentiles of the empirical distribution of DA. For more details about this test we refer to Alizadeh Noughabi (2019).

Alizadeh Noughabi and Park tests

Alizadeh Noughabi and Park (2016) proposed several test statistics for testing departures from the Laplace distribution. Their idea is also motivated by the Kullback-Leibler divergence between the unknown true density $g(x)$ and the hypothesized density under the null hypothesis, i.e. the density of the Laplace distribution. From relation (30) it is easily obtained that in the case of Laplace distribution

$$D_{KL}(g, f_0) = -H(g_X) + \log(2c)E_g\left(\mid X-\delta \mid\right). \tag{32}$$

In this frame, Alizadeh Noughabi and Park (2016) proposed to use the minimum discriminant information loss estimator for the unknown parameters instead of the maximum likelihood (see Alizadeh Noughabi and Park (2016) for further details). On the other hand since the problem of estimation of $H(g_X)$ has been considered by several authors, Alizadeh Noughabi and Park (2016) obtained five different test statistics based on five different estimators of Shannon's entropy and the moments of nonparametric distribution functions of the aforementioned estimators. Based on a Monte Carlo study Alizadeh Noughabi and Park (2016) concluded that the statistic which uses the Vasicek entropy estimator and the minimum discriminant information loss estimator has a good performance against symmetric alternatives and better performance against asymmetric alternatives. According we only consider this last test in our simulation study which is defined as follows:

$$TV_{mn} = \log(2\hat{c}_u) + 1 - HV_{mn}, \tag{33}$$

where

$$HV_{mn} = \frac{1}{n}\sum_{i=1}^{n}\log\left(\frac{n}{2m}\left(X_{(i+m)} - X_{(i-m)}\right)\right), \tag{34}$$

where m denoting the window size is a positive integer smaller than $n/2$, while $\hat{c}_u$ is defined (see Lemma 3 in Alizadeh Noughabi and Park (2016))

$$\hat{c}_u = \begin{cases} -\frac{1}{n}\sum_{i=1}^{n/2}\frac{\xi_i+\xi_{i+1}}{2} + \frac{1}{n}\sum_{i=n/2+1}^{n}\frac{\xi_i+\xi_{i+1}}{2} & \text{if } n \text{ is even} \\ -\frac{1}{n}\sum_{i=1}^{(n-1)/2}\frac{\xi_i+\xi_{i+1}}{2} + \frac{\xi_{(n+1)/2+1}-\xi_{(n+1)/2}}{4n} + & \\ \frac{1}{n}\sum_{i=(n+1)/2+1}^{n}\frac{\xi_i+\xi_{i+1}}{2} & \text{if } n \text{ is odd} \end{cases}, \tag{35}$$

where

$$\xi_i = \frac{X_{(i-m)} + \ldots + X_{(i+m-1)}}{2m} \tag{36}$$

with $X_{(i-m)} = X_{(1)}$ for $i \leq m$ and $X_{(i+m)} = X_{(n)}$ for $i \geq n-m$. The proposed values of m for different values of sample size n are given in Table 6 by Alizadeh Noughabi and Park (2016). In our simulation study the values $m = 4$ for $n = 20$, $m = 6$ for $n = 50$ and $m = 8$ for $n = 100$ were used. The null hypothesis of Laplace distribution is rejected for large values of TV_{mn}- implying a right tailed test- the critical values are taken to be the $100(1-\alpha)$-th percentiles of the empirical distribution of TV_{mn}. For more details about this class of tests we refer to Alizadeh Noughabi and Park (2016).

3.5. Other tests

In this section a goodness-of fit test for the Laplace distribution which cannot be classified in either one of the previous classes is briefly presented.

Gulati test

It is well known that if $X \sim \mathcal{C}L(\delta, c)$ then $Y = |X - \delta| \sim Exp(c)$ where $Exp(c)$ stands for the exponential distribution with mean c. Based on this property Gulati (2011) proposed a goodness-of fit test for the Laplace distribution based on the regression test of Brain and Shapiro (1983) for exponentiality. In this frame, let

$$W_i = (n - i + 1)\left(\hat{U}_{(i)} - \hat{U}_{(i-1)}\right), i = 1, ..., n, \tag{37}$$

with $\hat{U}_{(0)} = 0$ and $\hat{U}_{(i)}$, $i = 1, ..., n$ as defined in Section 2. Also let $l_i = \frac{\sum_{j=1}^{i} W_j}{\sum_{j=1}^{n} W_j}$, for $i = 1, ..., n-1$ and $\bar{l} = \frac{\sum_{i=1}^{n-1} l_i}{n-1}$. Then the test statistic proposed by Gulati (2011) is defined by

$$Z = Z_1^2 + Z_2^2 \tag{38}$$

where

$$Z_1 = \sqrt{12(n-1)}(\bar{l} - 0.5) \text{ and } Z_2 = \sqrt{\frac{5(n-1)}{(n+1)(n-2)}}\left(n - 2 + 6n\bar{l} - 12\sum_{i=1}^{n-1}\frac{il_i}{n-1}\right).$$

Based on Gulati (2011) under the null hypothesis this test statistic is asymptotically a chi-square variate with 2 degrees of freedom so that the null hypothesis is rejected if $Z \geq \chi^2_{1-\alpha,2}$. Based on a Monte Carlo study Gulati (2011) concluded that the empirical percentiles were fairly close to the theoretical percentiles of the chi-square distribution. Since the test is right-tailed we use in our simulation section the empirical critical values which are the $100(1-\alpha)$-th percentiles of the empirical distribution. For more details on Z test see Gulati (2011).

4. Monte Carlo study

Our purpose is to present a detailed comparison of the existing procedures for departure from the Laplace distribution. To assess the performance of the above tests we apply them to alternatives distributions (symmetric and asymmetric) which were previously considered in other studies of testing departures from the Laplace distribution (c.f. Puig and Stephens 2000b; Best *et al.* 2008; González-Estrada and Villaseñor 2016). In this context, the performance of the power of all the tests is investigated using Monte Carlo simulations by generating samples for the following alternatives:

- Symmetric alternatives: normal $N(0,1)$, Logistic $L(0, 0.551)$, Cauchy, Uniform $(-1.732, 1.732)$, Beta$(2,2)$, t with $10, 6$ and 3 degrees of freedom (d.f.'s), Tukey, contaminated normal $CN_{3.2,0.2}$ and $CN_{3.5,0.1}$, two special cases of the normal inverse Gaussian (NIG) distribution denoted by NIG_1 and NIG_2 by Gel (2010), i.e., these are NIG distributions with common skewness and location parameters 0 and 1, respectively, but with a combination of shape and scale parameters $(0.4, 0.6)$ and $(0.7, 0.2)$, respectively.
- Asymmetric alternatives: Exp(1), Gamma$(2,1)$, standard Gumbel, skew-normal with slant parameter 3, skew-t with slant parameter 3 and 10 d.f., Log-Normal, Weibull $(2,1)$, Weibull $(3,1)$, χ^2 with 2 degrees of freedom (d.f.'s), Extreme Value $(0,1)$, Inverse Gaussian IG(4), two special cases of Normal Inverse Gaussian distribution denoted by NIG_3 and NIG_4 by Gel (2010), i.e., NIG distributions with common location parameter 1, but with a combination of shape, skewness and scale parameters $(1, 0.5, 0.43)$ and $(0.5, 0.2, 0.5)$, respectively.

Only the exact simulated critical values are utilized for the power analysis in order to ensure that the correct size of the test is preserved. For this purpose we initially use the `many.crit` function of the PoweR package (see Lafaye de Micheaux and Tran (2016)) for the determination of the critical values based on $l = 100,000$ samples from $\mathcal{CL}(0,1)$. Then by employing the function powcomp.fast of the PoweR package (see Lafaye de Micheaux and Tran (2016)), the empirical power of the tests are obtained by using the above set of critical values. Note that for implementing the existing gof tests for the Laplace distribution we have used the functions of the PoweR package with the exception for the K test by Gel (2010), the two tests R_n and $R_n^{'}$ proposed by González-Estrada and Villaseñor (2016), the modified test based on moment structure studied in this paper in details and the tests presented in the subsection 3.4, i.e. the entropy and divergence based tests, denoted by $T_{m,n}^V$, E_n, T_{KL}, DA and TV_{mn}, proposed by Choi and Kim (2006) Rizzo and Haman (2016), Alizadeh Noughabi and Balakrishnan (2016), Alizadeh Noughabi (2019) and Alizadeh Noughabi and Park (2016), respectively. The empirical power is then obtained by calculating the proportion of times in $l = 100,000$ Monte Carlo simulations for which the false null hypothesis is rejected. For all of the latter we take into account the specified significance level for which $l = 100,000$ samples of size n ($n = 20, 50, 100$) are simulated from the previous distributions. A relevant R code is presented in the Appendix.

Based on the results given in Tables 3 and 4 we conclude the following:

a) For symmetric alternatives it is observed that R_n and $R_n^{'}$ proposed by González-Estrada and Villaseñor (2016) produce similar results, while R_n outperforms $R_n^{'}$ in the majority of the asymmetric alternatives. Accordingly, the rest of the conclusions is restricted to R_n only. Despite the fact that the R_n test is very simple and is based on a ratio of two estimators for the scale parameter of the Laplace distribution, it is found to be rather competitive for the symmetric alternatives.

b) Between the five e.d.f. gof tests, i.e., W^2, U^2, A^2, $\sqrt{n}D$ and V, the Watson (U^2) is found to be the best versus symmetric alternatives. For non-symmetric alternatives the Anderson-Darling (A^2) is found to be the best (in 10 out 14 alternatives), though the U^2 is rather good and the best for 4 out 14 non-symmetric alternatives. These conclusions coincide with that reached by Puig and Stephens (2000a) and with that by Choi and Kim (2006) based on simulation studies with lesser alternatives distributions and simulation runs. Consequently, for the rest of our conclusions, the U^2 and A^2 are reported, respectively, for symmetric and asymmetric alternatives.

c) When comparing the e.c.f. tests proposed by Meintanis (2005) it is concluded that for the symmetric alternatives considered with population kurtosis less or equal to 6, i.e., for the alternatives $U(0,1)$, Tukey (0.5), Beta$(2,2)$, Normal, t with 10 d.f., Logistic, t with 6 d.f., $T_{n,0.5}^{(2,ML)}$ has the best power. On the other hand for the symmetric alternatives considered with population kurtosis greater than 6 (see $CN_{3.2,0.2}$, $CN_{3.5,0.1}$, NIG_1 and NIG_2) or with undefined or non finite population kurtosis (see Cauchy and t with 3 d.f.), it seems that $T_{n,2}^{(1,MO)}$ is a good choice among them. Finally, the MO-based tests are slightly more powerful for all asymmetric alternatives considered with the exception of the Weibull. The performance of $T_{n,2}^{(1,MO)}$ and $T_{n,0.5}^{(2,MO)}$ is almost similar. Consequently, we restrict the rest of the analysis to $T_{n,0.5}^{(2,ML)}$ for symmetric alternatives with population kurtosis less or equal to 6 and to $T_{n,2}^{(1,MO)}$ for any other alternatives.

d) When comparing for symmetric alternatives the performance of e.c.f. based test previously recommended with the Watson e.d.f. test, we conclude that e.c.f. based tests outperforms U^2 (with the exception when testing against the Cauchy and the contaminated normal distributions). For asymmetric alternatives, the performance of $T_{n,2}^{(1,MO)}$ is better in almost all cases than the Anderson Darling test. Thus in a manner similar to that in Meintanis (2005) we conclude that for the majority of the cases considered the e.c.f. based tests either outperform or remain competitive with the best e.d.f. test.

Table 3: Empirical power multiplied by 100 of goodness of fit tests based on empirical critical values, when $l = 100.000$ samples of size n ($n = 20, 50, 100$) were simulated from several symmetric distributions ($\alpha = 0.05$). In bold are marked the highest powers per distribution and per sample size.

		Goodness-of-fit tests																					
Alternative	n	W^2	U^2	A^2	$\sqrt{n}D$	V	$T_{n,2}^{(1,MO)}$	$T_{n,2}^{(1,ML)}$	$T_{n,0.5}^{(2,MO)}$	$T_{n,0.5}^{(2,ML)}$	$T_{m,n}^{V}$	E_n	T_{KL}	DA	TV_{mn}	Z	K_1	V_3	V_4	K	R_n	R_n'	Z_n
CL(0,1)	20	4.976	5.011	4.917	4.848	4.944	4.788	4.857	4.8	4.961	4.927	4.895	4.817	4.87	4.861	4.954	4.885	5.006	4.87	5.04	4.898	4.961	5.044
	50	4.89	5.024	5.002	4.811	5.13	4.946	5.066	4.896	5.077	4.955	4.975	4.891	4.863	4.799	4.862	4.979	5.155	5.04	5.177	5.046	5.125	5.181
	100	5.082	5.132	5.053	5.024	5.043	5.013	5.236	5.012	5.176	5.046	5.004	5.036	5.219	5.065	4.994	4.879	4.874	5.083	4.987	5.019	5.054	4.874
N(0,1)	20	7.622	10.829	6.682	8.628	10.415	10.918	13.87	12.057	15.261	18.041	6.539	**19.358**	15.585	10.221	7.904	8.941	0.119	15.171	0.075	15.566	16.358	0.08
	50	17.018	33.621	14.819	19.132	29.396	30.678	35.753	32.269	38.315	45.567	13.777	40.175	38.409	52.305	30.586	29.48	0.013	42.943	0.001	51.017	**52.345**	0.004
	100	40.285	69.48	35.738	37.62	60.398	65.759	68.368	66.632	70.805	73.781	33.149	70.426	62.982	86.669	72.73	67.348	0.001	76.054	0.001	88.12	**88.776**	0.001
Logistic(0,0.551)	20	6.009	7.089	5.35	6.392	6.98	7.025	8.9	7.591	9.296	11.052	5.229	**12.238**	9.859	7.49	5.444	5.639	1.45	8.455	1.09	8.726	9.178	1.117
	50	9.91	17.177	8.542	11.566	15.632	14.499	18.311	15.322	19.074	20.101	8.063	20.901	16.654	**26.339**	13.597	12.509	1.123	15.997	0.645	21.968	22.709	0.737
	100	18.59	36.379	14.934	20.274	31.521	30.008	35.592	30.923	36.388	29.245	13.409	35.619	22.929	**47.666**	32.704	27.238	0.909	26.104	0.399	43.249	44.304	0.501
Cauchy	20	53.788	62.97	55.673	50.929	60.794	60.549	55.822	58.896	55.312	36.159	55.161	2.106	46.148	45.448	67.574	63.12	58.383	53.224	**68.132**	60.692	66.52	64.835
	50	86.019	91.989	87.063	82.893	90.416	92.206	88.893	91.881	88.787	59.685	86.673	8.971	87.726	64.905	**94.659**	93.077	78.248	84.232	93.61	93.566	94.543	87.434
	100	98.62	99.482	98.727	97.631	99.233	99.551	99.1	99.536	99.087	58.873	98.655	30.469	99.373	89.356	**99.772**	99.632	87.303	97.366	99.536	99.712	99.758	95.538
U(-1.732,1.732)	20	25.139	44.426	24.62	24.135	36.752	48.866	58.004	52.033	63.118	**76.555**	24.405	70.458	71.655	49.544	31.454	40.708	0	71.056	0.001	60.448	58.877	0.001
	50	76.331	94.504	81.511	62.012	87.966	96.289	97.808	96.406	98.407	**99.988**	81.801	99.462	99.924	99.777	94.702	93.521	0	99.892	0	99.36	99.225	0
	100	99.483	99.99	99.889	93.87	99.849	99.996	99.998	99.995	99.999	**100**	99.92	**100**	**100**	**100**	99.991	99.978	0	**100**	0	**100**	**100**	0
Beta(2,2)	20	13.074	23.292	12.097	14.766	20.498	24.6	31.033	27.209	35.444	**43.288**	11.899	43.156	38.559	23.038	16.977	21.175	0	42.333	0	37.079	37.557	0
	50	42.297	71.624	42.464	38.527	61.9	72.535	77.686	73.967	81.559	96.229	41.24	87.773	92.763	93.842	77.178	71.755	0	**96.624**	0	92.751	92.947	0
	100	86.473	98.217	89.446	72.287	94.348	98.636	98.859	98.51	99.169	**99.999**	89.27	99.83	99.971	99.99	99.7	98.527	0	**99.999**	0	99.976	99.98	0
t(df=3)	20	6.539	7.922	6.909	6.723	7.681	8.095	6.706	8	6.641	7.054	6.968	7.871	8.712	9.15	10.614	8.462	**12.907**	12.229	12.801	11.506	12.12	12.66
	50	7.745	10.608	8.546	7.932	9.947	11.76	8.935	11.484	8.236	6.136	8.66	18.508	13.418	10.246	16.204	12.378	21.506	19.723	**22.024**	18.849	19.132	21.316
	100	9.147	14.31	10.169	9.426	13.103	15.9	11.789	15.128	10.229	3.395	10.341	**33.382**	21.44	10.215	25.515	15.935	29.226	28.551	32.451	25.994	26.222	29.603
t(df=6)	20	5.892	6.915	5.33	6.282	6.72	6.823	8.381	7.23	8.687	10.351	5.211	**11.574**	9.418	7.578	5.789	5.646	3.137	8.801	2.593	8.755	9.257	2.657
	50	9.088	15.355	7.965	10.499	14.142	12.85	16.607	13.423	16.979	16.808	7.522	20.579	14.931	**23.394**	12.285	10.85	4.086	14.572	3.013	18.722	19.495	3.236
	100	16.367	31.683	13.072	17.926	27.382	24.896	31.548	25.588	32.01	21.348	11.819	35.08	19.388	**39.925**	27.664	21.151	4.765	20.608	3.384	33.105	34.068	3.653
t(df=10)	20	6.415	8.164	5.697	6.983	7.972	8.029	10.137	8.69	10.882	12.79	5.571	**14.041**	11.29	8.088	6.154	6.4	1.113	10.131	0.795	10.535	11.227	0.817
	50	11.926	21.615	10.15	13.827	19.417	18.757	23.282	19.814	24.479	25.823	9.499	26.338	21.619	**33.506**	17.623	16.61	0.987	21.694	0.528	29.23	30.173	0.61
	100	24.21	46.713	19.805	25.206	40.311	40.155	45.992	41.279	47.687	38.954	17.971	46.382	32.228	**60.292**	43.876	37.83	0.919	36.654	0.437	56.597	57.756	0.525
Tukey(0.5)	20	14.42	25.981	13.524	16.075	22.705	27.735	34.698	30.581	39.429	**48.397**	13.292	47.496	43.355	26.241	18.714	23.844	0	47.123	0	40.798	41.203	0
	50	47.155	76.559	48.179	42.077	66.668	78.007	82.786	79.129	86.075	97.932	46.995	91.452	95.67	95.962	81.953	76.536	0	**98.043**	0	94.724	94.879	0
	100	90.427	99.023	93.381	76.787	96.176	99.343	99.474	99.269	99.622	**100**	93.414	99.95	99.994	99.998	99.855	99.174	0	99.998	0	99.986	99.986	0
$CN_{3.5,0.1}$	20	6.537	8.376	6.585	6.982	7.981	8.192	8.424	8.398	8.403	9.991	6.627	13.517	11.056	10.889	10.763	7.916	14.635	**15.28**	13.574	13.39	14.173	13.473
	50	8.955	14.165	9.112	9.597	12.964	12.085	15.235	12.012	14.365	9.866	9.048	**33.715**	15.457	18.49	18.379	10.154	22.507	21.314	22.312	19.597	19.963	20.773
	100	13.296	24.545	13.866	13.48	21.182	17.29	27.929	16.449	25.707	4.748	13.829	**58.884**	20.353	21.669	32.97	11.144	26.223	25.269	30.301	21.631	21.898	25.164
$CN_{3.2,0.2}$	20	5.937	7.007	6.134	6.138	6.794	6.654	6.412	6.601	6.201	6.712	6.159	8.701	7.759	8.982	8.749	6.617	**11.993**	9.594	11.521	9.198	9.649	11.328
	50	7.147	9.674	7.684	7.579	9.067	8.323	9.566	8.083	8.217	4.844	7.734	**18.728**	7.42	8.563	13.674	7.325	14.094	9.102	14.893	11.082	11.235	13.784
	100	8.88	14.58	9.938	9.396	13.315	10.859	15.55	10.013	12.393	1.765	10.098	**33.388**	9.463	7.109	25.652	7.669	13.763	8.775	17.424	12.808	12.91	14.362
NIG_1	20	8.164	11.459	9.354	8.402	10.721	11.185	8.135	10.569	8.158	5.128	9.427	4.768	7.754	9.476	15.578	13.051	17.807	13.22	**19.716**	14.395	15.784	19.089
	50	10.273	17.085	12.234	10.708	15.456	18.859	12.311	18.312	12.467	2.761	12.413	11.531	13.334	3.424	26.772	21.614	25.981	21.115	**32.988**	27.418	28.323	29.517
	100	14.993	27.048	17.709	15.464	23.686	30.904	20.125	30.372	20.489	0.697	17.973	23.786	29.68	1.489	43.677	34.185	31.369	31.128	**47.131**	44.386	44.993	37.288
NIG_2	20	12.684	19.265	14.42	12.398	17.816	18.221	13.517	17.109	13.505	6.11	14.477	4.04	10.718	12.574	24.874	21.107	25.263	19.059	**29.18**	22.213	24.572	27.836
	50	21.098	34.879	24.121	20.313	31.31	36.557	26.835	35.67	26.988	3.989	23.967	13.647	25.971	4.296	46.963	40.367	36.907	32.912	**50.52**	46.43	47.926	43.663
	100	36.941	56.849	40.635	33.973	51.252	60.697	47.256	60.222	47.757	0.676	40.472	35.899	57.011	3.309	72.374	64.187	44.499	49.991	70.561	72.213	**73.115**	54.716

Table 4: Empirical power multiplied by 100 of goodness of fit tests based on empirical critical values, when $l = 100.000$ samples of size n ($n = 20, 50, 100$) were simulated from several asymmetric distributions ($\alpha = 0.05$). In bold are marked the highest powers per distribution and per sample size.

		Goodness-of-fit tests																					
Alternative	n	W^2	U^2	A^2	$\sqrt{n}D$	V	$T_{n,2}^{(1,MO)}$	$T_{n,2}^{(1,ML)}$	$T_{n,0.5}^{(2,MO)}$	$T_{n,0.5}^{(2,ML)}$	$T_{m,n}^V$	E_n	T_{KL}	DA	TV_{mn}	Z	K_1	V_3	V_4	K	R_n	R_n'	Z_n
Exp(1)	20	43.742	45.972	53.515	47.831	47.055	63.252	50.109	62.153	42.571	82.594	55.992	51.65	**86.792**	84.91	19.638	47.253	22.202	14.493	15.566	14.614	14.139	17.572
	50	88.593	93.224	97.166	95.787	95.801	99.103	97.429	98.956	96.618	99.976	97.887	99.805	**99.998**	99.993	39.917	92.723	50.593	13.994	26.224	21.413	15.657	35.504
	100	99.908	99.973	**100**	**100**	99.998	**100**	99.999	**100**	99.999	**100**	**100**	**100**	**100**	**100**	72.414	99.918	86.53	15.363	43.578	27.705	16.58	64.251
Gamma(2,1)	20	21.011	21.329	26.3	18.343	19.165	33.594	26.849	34.412	25.698	47.548	27.736	37.046	**51.532**	50.426	10.181	26.594	9.566	14.06	5.945	15.276	13.344	6.872
	50	54.095	61.682	71.959	50.286	56.644	85.356	75.373	85.648	75.403	96.421	74.86	93.87	**98.55**	98.364	21.203	72.197	21.024	15.214	7.892	31.528	20.794	11.953
	100	91.58	94.856	98.975	92.798	93.46	99.769	98.734	99.755	98.749	99.929	99.353	99.988	**99.999**	**99.999**	44.094	97.632	46.18	14.936	10.964	50.031	29.286	22.334
Gamma(6,1)	20	11.076	13.319	11.853	10.27	12.174	16.766	17.347	18.05	18.399	**25.205**	12.133	25.042	24.191	20.209	8.069	14.008	2.373	15.034	1.313	15.836	15.329	1.527
	50	27.059	40.074	32.589	22.598	33.493	50.434	46.97	51.919	49.375	68.303	33.403	64.565	68.327	**76.683**	23.965	45.309	3.391	28.349	0.976	43.254	37.91	1.558
	100	60.602	77.277	72.594	45.323	66.247	88.729	83	88.975	84.644	92.89	73.998	95.33	95.766	**98.851**	53.401	84.336	6.084	39.466	0.726	73.821	65.173	1.901
Gumbel	20	12.104	12.586	13.936	10.634	11.546	17.913	15.772	18.766	16.036	24.379	14.593	23.671	**25.074**	23.922	7.813	15.286	5.826	12.886	3.917	13.15	12.602	4.349
	50	28.605	35.579	38.27	22.532	29.807	52.632	43.107	53.462	44.234	64.342	40.208	66.135	69.285	**75.93**	17.404	46.208	11.443	18.757	4.653	30.257	24.431	6.667
	100	61.052	71.51	78.29	46.792	61.073	90.088	79.524	89.908	80.534	88.133	80.464	95.987	95.766	**98.414**	37.2	84.124	23.313	21.48	5.664	50.809	38.775	11.041
SkN(3)	20	10.863	12.771	11.409	10.272	11.723	16.289	16.715	17.441	17.867	24.046	11.68	**24.097**	22.697	18.201	7.637	13.724	1.228	14.865	0.705	16.075	15.347	0.803
	50	27.542	39.889	31.703	23.715	33.01	49.278	45.716	50.627	47.877	62.943	32.2	59.711	60.567	**71.008**	24.298	46.077	1.153	32.13	0.233	47.442	41.916	0.422
	100	60.986	77.358	70.284	46.483	65.969	87.689	81.4	87.863	82.798	88.928	71.119	91.616	89.227	**97.04**	55.468	85.839	1.598	50.429	0.084	81.392	73.537	0.311
Skt(3,10)	20	39.916	41.769	49.995	43.668	42.519	58.308	43.339	57.211	37.493	74.434	52.512	41.806	**80.866**	80.41	27.207	46.332	33.579	22.974	27.233	18.385	22.641	29.252
	50	81.682	87.813	94.602	91.813	91.458	97.982	93.8	97.716	92.275	99.866	95.95	98.255	**99.984**	99.957	53.187	89.838	68.91	33.923	50.495	28.077	34.734	58.023
	100	99.445	99.822	99.995	99.986	99.959	**100**	99.982	**100**	99.973	**100**	99.998	99.453	**100**	**100**	82.959	99.751	94.912	49.309	75.35	36.532	48.83	85.795
LN(0,0.5)	20	18.82	17.96	23.825	16.128	16.214	29.235	21.248	29.793	20.599	37.187	25.347	30.71	42.053	**43.432**	10.938	24.299	13.474	14.436	9.52	13.634	13.025	10.511
	50	45.556	49.616	64.173	40.435	43.69	77.349	61.183	77.546	60.734	87.061	67.554	88.219	93.766	**94.07**	20.177	66.034	30.287	16.212	15.319	23.766	18.058	20.502
	100	83.791	87.353	96.9	81.595	82.646	99.019	94.633	98.948	94.719	98.141	97.779	99.887	99.975	**99.986**	39.849	95.334	59.531	17.144	24.264	33.448	21.628	38.226
Weibull(3,1)	20	8.556	13.257	7.649	9.777	12.43	13.516	17.134	14.937	19.259	22.945	7.53	**24.273**	19.937	12.531	9.528	11.284	0.064	19.88	0.031	19.896	20.842	0.039
	50	21.693	42.716	19.754	23.261	36.899	40.736	45.883	42.472	49.379	63.035	18.698	53.759	55.013	**67.326**	41.275	40.275	0.006	59.74	0	64.848	65.676	0
	100	53.065	81.211	50.915	46.391	71.852	79.981	81.203	80.259	83.533	93.193	48.652	86.921	86.002	**96.571**	86.14	81.781	0	90.663	0	95.994	96.145	0
Weibull(2,1)	20	11.384	15.526	11.607	11.163	13.892	18.361	20.857	19.984	22.554	**30.517**	11.731	29.551	28.344	21.135	9.629	15.146	0.918	19.96	0.44	20.385	19.821	0.532
	50	30.695	49.928	34.562	26.602	41.583	57.125	57.889	59.031	61.305	81.657	34.873	72.973	80.067	**85.74**	35.075	52.061	0.703	43.038	0.091	59.562	54.839	0.196
	100	69.124	87.372	78.381	52.866	77.988	93.503	91.666	93.551	93.014	98.963	79.071	98.136	99.287	**99.847**	72.997	90.535	0.87	63.615	0.03	90.762	86.518	0.119
χ_2^2	20	43.956	45.928	53.653	48.171	47.02	63.372	50.038	62.233	42.377	82.535	56.085	51.537	**86.687**	84.871	19.426	47.323	22.049	14.535	15.588	14.766	14.098	17.589
	50	88.47	93.215	97.202	95.709	95.884	99.172	97.477	99.014	96.623	99.982	97.96	99.815	**99.999**	99.995	39.7	92.711	50.522	13.967	26.306	21.686	15.618	35.438
	100	99.905	99.973	99.999	99.999	99.996	**100**	99.996	**100**	99.997	**100**	**100**	**100**	**100**	**100**	72.195	99.926	86.431	15.395	43.62	27.834	16.834	64.027
EV(0,1)	20	12.186	12.452	14.061	10.652	11.429	17.853	15.713	18.675	16.061	24.504	14.598	23.367	**25.115**	23.948	7.707	15.044	5.734	12.776	3.76	13.004	12.402	4.217
	50	28.369	35.365	37.866	22.64	29.554	52.19	43.03	52.995	44.235	64.403	39.767	66.003	69.205	**75.852**	17.16	45.746	11.544	18.545	4.528	30.144	24.308	6.572
	100	61.082	71.716	78.174	46.738	60.962	90	79.686	89.859	80.679	88.097	80.465	96.012	95.78	**98.451**	37.394	83.984	23.135	21.356	5.572	50.989	38.744	11.064
IG(1,4)	20	92.635	94.458	95.795	95.799	95.276	97.291	93.344	96.565	87.576	99.289	96.351	27.078	**99.65**	99.591	79.827	91.49	73.732	42.616	71.12	30.177	61.823	73.871
	50	99.99	99.999	**100**	**100**	**100**	**100**	**100**	**100**	99.998	**100**	**100**	79.955	**100**	**100**	98.965	99.982	98.644	68.476	96.127	57.551	91.338	97.848
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	95.847	**100**	**100**	99.998	**100**	99.999	88.414	99.923	81.776	99.45	99.989
NIG_3	20	12.77	13.307	16.236	11.848	12.303	18.297	10.207	17.542	10.062	11.227	17.003	8.397	15.888	19.364	16.368	19.738	**21.726**	14.513	20.96	12.156	15.714	21.584
	50	22.807	22.806	33.12	20.585	20.227	40.329	18.809	39.041	17.933	16.519	35.314	29.128	33.601	20.197	28.066	**42.263**	41.958	23.005	38.207	20.551	27.166	41.138
	100	40.521	39.824	58.069	37.457	34.562	68.25	34.684	66.633	33.043	14.159	61.251	58.361	59.35	29.421	45.708	**69.309**	64.69	34.573	58.07	31.781	41.924	63.789
NIG_4	20	12.481	15.452	15.236	11.953	14.389	17.852	11.089	16.965	11	8.55	15.709	5.733	13.16	16.176	20.165	20.053	23.755	16.542	**24.983**	16.17	19.481	24.782
	50	21.339	27.444	28.435	20.119	24.626	37.417	21.164	36.305	20.959	9.903	29.603	20.675	28.577	10.958	37.097	40.424	40.964	28.05	**44.667**	31.957	37.078	43.664
	100	36.485	45.73	48.01	34.691	40.353	61.996	37.479	60.931	37.077	5.404	49.981	45.737	55.136	12.373	59.136	64.722	58.218	42.74	**64.729**	51.838	58.384	62.189

e) For all the alternatives considered E_n has less power than the respective of the e.d.f. and e.c.f. based tests previously recommended. Thus this test cannot be recommended. Accordingly the rest of the conclusions is restricted to the rest of the tests belonging to the class of entropy and divergence based tests.

f) When comparing the rest of the entropy and divergence based tests with the e.d.f. and e.c.f. tests previously recommended it is concluded that for symmetric alternatives $T^V_{m,n}$ performs better than them under symmetric alternatives with population kurtosis less than or equal to 6. For this type of alternatives T_{KL}, DA and TV_{mn} behave also similar in almost all cases in comparison with the recommended e.d.f and e.c.f. tests. It seems that for such alternatives TV_{mn} and $T^V_{m,n}$ stand out as the best ones, for moderate and large sample sizes, i.e. for $n = 50, 100$, while for $n = 20$ $T^V_{m,n}$ and T_{KL} are good choices among the entropy and divergence based tests. For the rest of the symmetric alternatives we have that the entropy and divergence based tests have less power than the respective of the e.d.f. and e.c.f. previously recommended for the NIG_1, NIG_2 and Cauchy distributions, with DA being the best among them. On the other hand T_{KL} outperforms the e.d.f. and e.c.f. tests previously recommended for contaminated normal and t with 3 d.f. Moreover, for asymmetric alternatives the entropy and divergence based tests have in almost all cases better performance than the recommended e.c.f. and e.d.f. tests with the exception of the NIG distributions. Note that for such alternatives these tests, namely $T^V_{m,n}$, T_{KL}, TV_{mn} and DA, have the better performance among the tests considered. Specifically, for asymmetric alternatives with the exception of NIG distributions, TV_{mn} is recommended for moderate and large sample sizes for, while DA is recommended for $n = 20$.

However at this point we have to note that there are two points that make the use of the four previously mentioned entropy and divergence based tests difficult or even impossible in some cases in practical applications. The first point is related to the fact that $T^V_{m,n}$, DA and TV_{mn} depend on the optimal choice of window sizes m. These choices are available for the TV_{mn} for several sample sizes in Table 6 by Alizadeh Noughabi and Park (2016), while for the other two tests are available for specific choices of n. Nonetheless, this problem can be tackled by noticing, based on the available information, that for $T^V_{m,n}$ it seems that the optimal choice of m is about $13\% - 15\%$ of the size of n. On the other hand, DA was constructed based on specific choice of kernel density estimator. The second point is that for instance $T^V_{m,n}$ is based on the geometric mean of a suitable transformation of the data which involves differences related to their median, while the rest of the tests involve the logarithm of some differences. As a result, whenever we tackle with data points having identical numerical values with that of the median or with differences which are equal to zero, the tests cannot be executed and are not recommended. Note that such a situation is not rare in real data sets due to the rounding up to a certain number of decimal points.

g) As previously noted, the main idea behind the Z_n test is that for the Laplace distribution the third central moment equals zero which implies that the population skewness is also zero. This means that its difference with the test based on sample skewness proposed by Rayner and Best (1989), denoted by V_3, is that only the numerator is used and this does not estimate the standard deviation as V_3 does. The Z_n test has similar or superior performance against some symmetrical distributions (Cauchy, NIG_i, $i = 1, 2$ distributions) since estimating the standard deviation in such cases add an extra layer of doubt in the estimation. On the other hand the Z_n test has inferior or similar performance for non symmetric alternatives since the presence of an estimator of the standard deviation allows the existence of a skewed distribution to be more easily spotted.

h) The K test by Gel (2010) which is based on a combination of the sample skewness and kurtosis; the smooth V_3 test by Rayner and Best (1989) which is related to sample skewness; and the modified moment based gof Z_n test do not perform well and are not competitive under symmetric alternatives with population kurtosis less than or equal to 6. On the other hand these tests are competitive for symmetric alternatives such as the Cauchy, t with 3 d.f.'s, NIG_1, NIG_2 and contaminated normal distributions, where in some cases they attain the maximum power. Note also that these tests outperforms the popular edf tests for t with 3 d.f.'s, contaminated normal and normal inverse Gaussian distributions. Furthermore, they do not perform well and are not competitive in comparison with the popular edf tests under the asymmetric alternative distributions that are considered (with the exception of the asymmetric normal inverse Gaussian distribution). For the latter alternative distribution their performance is found to be the best among all tests considered and thus is recommended for possible applications. Note that for the asymmetric alternatives considered, V_3 is better than Z_n which is better than K with the exception of the NIG_4 distribution.

i) Based on the results given in Table 3, we note that for symmetric alternatives, with the exception of uniform distribution, the Z test by Gulati (2011) outperforms the K_1 test by Langholz and Kronmal (1991). However, based on the results of Table 4, the situation is reversed in favor of the K_1 test. Moreover, the smooth test V_4 by Rayner and Best (1989) which is related to sample kurtosis is better than the Z and the K_1 tests for symmetric alternatives with the exception of the Cauchy and the two normal inverse Gaussian distributions.

Based on the previous results and comments it is concluded that for the majority of the considered asymmetric alternatives there is a power advantage in using the DA and TV_{mn} tests, while for symmetric alternatives a single recommendation is not that clear, since, as expected, there is no test that can detect all types of symmetric alternatives. There are several tests including TV_{mn}, T_{KL}, TV_{mn}, Z, V_4 and K that present the highest power under different alternatives. Although, it is for note, that TV_{mn} present a robust, acceptable behavior under the symmetric alternatives used with tails not heavier than that of the standard Laplace distribution. On the other hand, despite the fact that in general some tests, for instance the Z_n or V_3, are not useful for detecting several alternatives, they are quite competitive when alternatives with tails heavier than that of the standard Laplace distribution as the Cauchy, t with 3 d.f.'s, contaminated normals and normal inverse Gaussian distributions are considered. These tests may not present the highest power but have a competitive behavior to that of the tests with the highest power under such alternatives. The same holds for these two tests and for asymmetric alternatives with heavy tailness of large degree. Finally, for symmetric alternatives the simple R_n test could be also recommended.

5. Conclusions

The main contribution of the paper is linked to the fact that twenty two different goodness-of-fit tests for the Laplace distribution, included a new one, have been compared with 27 possible alternative distributions (symmetric and asymmetric). As has been initially expected there is no test which outperforms the others in all cases. However if a particular alternative is suspected like the ones mentioned in the simulation study of the previous section, one can utilize the recommended test based on the conclusions of the simulation study. Moreover, extremely practical and important `R` codes are provided in the appendix for implementing the comparison study of the various tests for detecting a departure from the Laplace distribution while taking into account a numerous alternative distributions.

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Appendices

Appendix A

Proof of Theorem 3.1. Consider T_n as a function of $\hat{\theta}_n$ and expand it into a Taylor series around the true value of $\theta = (\delta, c)^T$, where T denotes the transpose of a vector or matrix. We will obtain that

$$\sqrt{n}T_n = \sqrt{n}\mathcal{T}_n + \sqrt{n}\left(\frac{\partial \mathcal{T}_n}{\partial \delta}, \frac{\partial \mathcal{T}_n}{\partial c}\right)\Bigg|_{\theta=\theta_n^\star,} \left(\hat{\theta}_n - \theta\right), \tag{39}$$

where

$$\mathcal{T}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i^3 - 3\frac{1}{n^2}\sum_{i=1}^{n} Y_i \sum_{i=1}^{n} Y_i^2 + 2\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right)^3, \tag{40}$$

$Y_i = \frac{X_i - \delta}{c}$ and $\theta_n^\star = a\hat{\theta}_n + (1-a)\theta$, for some $a \in (0,1)$ (note that $\theta_n^\star \xrightarrow{\mathrm{P}} \theta$).

We first determine the asymptotic joint distribution of $\frac{1}{n}\sum_{i=1}^{n} Y_i$, $\frac{1}{n}\sum_{i=1}^{n} Y_i^2$ and $\frac{1}{n}\sum_{i=1}^{n} Y_i^3$, under the hypothesis that $X_i \sim \mathcal{C}L(\delta, c)$, $i = 1, ..., n$, or equivalently, that $Y_i \sim \mathcal{C}L(0,1)$, $i = 1, ..., n$. From the multivariate Central Limit Theorem we easily obtain that under the null hypothesis of Laplace distribution

$$\sqrt{n}\begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n} Y_i - 0 \\ \frac{1}{n}\sum_{i=1}^{n} Y_i^2 - 2 \\ \frac{1}{n}\sum_{i=1}^{n} Y_i^3 - 0 \end{pmatrix} \xrightarrow{\mathrm{L}} N_3\left(0_3, \Sigma\right), \tag{41}$$

where 0_3 is a 3-dimensional column vector with zero elements and $\Sigma = (\sigma_{lk})_{3\times 3}$ with $\sigma_{lk} = EY^{k+l} - EY^l EY^k$. As under the null hypothesis $EY^r = 0$ if r is odd and $EY^r = r!$ if r is even, we obtain that

$$\Sigma = \begin{pmatrix} 2 & 0 & 24 \\ 0 & 20 & 0 \\ 24 & 0 & 720 \end{pmatrix}.$$

In order to find the asymptotic distribution of $\mathcal{T}_n$ we note that

$$\mathcal{T}_n = g\left(\frac{1}{n}\sum_{i=1}^{n} Y_i, \frac{1}{n}\sum_{i=1}^{n} Y_i^2, \frac{1}{n}\sum_{i=1}^{n} Y_i^3\right),$$

where $g(x,y,z) = z - 3xy + 2x^3$ and $\dot{g}(x,y,z) = \left(\frac{\partial}{\partial x}g, \frac{\partial}{\partial y}g, \frac{\partial}{\partial z}g\right) = \left(-3y + 6x^2, -3x, 1\right)$. Then by the Cramér's Theorem (see Theorem 7, p. 45, in Ferguson, 1996), we find that under H_0

$$\sqrt{n}\left(\mathcal{T}_n - g(0,2,0)\right) \xrightarrow{\mathrm{L}} N\left(0, \sigma^2\right),$$

where

$$\sigma^2 = \dot{g}(0,2,0)\,\Sigma \dot{g}^T(0,2,0).$$

As $g(0,2,0) = 0$ and $\dot{g}(0,2,0) = (-6,0,1)$ it follows that $\sigma^2 = 504$. Therefore, under the null hypothesis, the first term $\sqrt{n}\mathcal{T}_n$ in the right side of (39) converges in distribution to $N(0, 504)$.

For the second term we first note that the quantity $\sqrt{n}\left(\hat{\theta}_n - \theta\right)$ is bounded in probability. Then by evaluating the derivative of $\mathcal{T}_n$ given in (40) with respect to θ at the point $\theta = \theta_n^\star$ it follows that

$$\frac{\partial \mathcal{T}_n}{\partial \delta} = 0$$

and

$$\frac{\partial \mathcal{T}_n}{\partial c} = -\frac{3}{nc^3}\sum_{i=1}^{n} Y_i + \frac{9}{n^2 c}\sum_{i=1}^{n} Y_i \sum_{i=1}^{n} Y_i^2 - \frac{3}{n^3 c}\sum_{i=1}^{n} Y_i^3.$$

Since under the null hypothesis $X_i \sim \mathcal{CL}(\delta, c)$ one obtains that

$$\left.\frac{\partial \mathcal{T}_n(Y)}{\partial \theta}\right|_{\theta=\theta_n^\star} \xrightarrow{\mathrm{P}} 0 \text{ as } n \to \infty,$$

and thus the second term on the right side of (39) converges in probability to zero. This completes the proof. □

Appendix B

Next the `R` code used to obtain the results appearing in the Tables is presented. Note that the following versions of R packages were loaded during the preparation of the revised version of the current paper:

- `PoweR`, version 1.0.7 published on August 28, 2018
- `univariateML`, version 1.1.0 published on August 5, 2020
- `MuMIn`, version 1.43.17 published on April 15, 2020
- `psych`, version 1.0.7 published on December 16, 2020
- `skewt`, version 0.1 published on October 10, 2012
- `statmod`, version 1.4.35 published on October 19, 2020
- `teachingApps`, version 1.0.8 published on May 13, 2020
- `rmutil`, version 1.1.5 published on June 9, 2020

```
#------------------Main function for---------------------#
#----------------the proposed gof test------------------#

# load the following packages

library(PoweR)
library(univariateML)
library(MuMIn)


gofMomentsLaplace <- function (data, levels, boot){
  # Arguments
  # data:    a list of data points
  # levels:  vector of significance levels for the test.
  #          set levels=NA if only the value of the test
  #          statistic is required
  # boot:    number of bootstrap samples to generate
  #          set boot=0 if only the asymptotic
  #          distribution is to be used

  # Value - output
  # statistic:       the value of the test statistic
  # pvalue:          the asymptotic pvalue
  # pvalueboot:      the bootstrap pvalue (NA if boot=0)
  # levels:          vector of significance levels for the test.
  # decision_asymp:  the vector of decisions based on the
  #                  asymptotic distribution same length as levels
  #                  1 if we reject the null, 0 otherwise
```

```
  # decision_boot:  the vector of decisions based on bootstrap
  #                 same length as levels
  #                 1 if we reject the null, 0 otherwise
  #                 NA if boot=0
  # stat.pars:      A vector of the MLE of the Laplace parameters.
  # boot_crit:      A vector of bootstrap critical values at .
  #                 sort(c(levels,levels/2,1-levels/2,1-levels))
  #                 NA if boot=0

  n <- length(data)

  mledata=mllaplace(data)
  hatdelta=mledata[1]
  hatc=mledata[2]
  datatransf=stdize(data, center = hatdelta, scale = hatc)

  mtr=mean(datatransf)
  tn=mean(datatransf^3)-3* mtr*mean(datatransf^2)+2*(mtr)^3
  zn=sqrt(n/504)*tn

  pvalue=2*(1-pnorm(abs(zn),0,1))

  decision_asymp <- as.numeric(pvalue<levels)
  stat.pars = mledata

  if(boot>0){
    znboot=rep(0,boot)
    countmc<-0
    for (kk in 1:boot){
      datamc=gensample(law.index=1,n,law.pars=c(hatdelta,hatc))$sample
      mledatamc=mllaplace(datamc)
      hatdeltamc=mledatamc[1]
      hatcmc=mledatamc[2]
      datatransfmc=stdize(datamc, center = hatdeltamc, scale = hatcmc)
      mtrmc=mean(datatransfmc)
      tnmc=mean(datatransfmc^3)-3*mtrmc*mean(datatransfmc^2)+2*(mtrmc)^3
      znboot[kk]=sqrt(n/504)*tnmc
      if(abs(sqrt(n/504)*tnmc)>abs(zn)){countmc<-countmc+1}

    }
    pvalueboot=countmc/boot

    decision_boot <- as.numeric(pvalueboot<levels)
    }else{
    pvalueboot=NA
    decision_boot=NA
  }

  if(boot>0){
    quant=sort(c(levels,levels/2,1-levels/2,1-levels))
    boot_crit=quantile(znboot,probs =quant)
  }else{
    boot_crit=NA
  }

  return(list(statistic = zn, pvalue = pvalue, pvalueboot=pvalueboot,levels=levels,
              decision_asymp = decision_asymp,decision_boot=decision_boot, stat.pars = stat.pars[1:2],boot_crit=boot_crit
  ))
}
```

```
#-----------------------code for Table 1-----------------------#

# standard Laplace
law.index<-1
```

```
mu<- 0
b<- 1

# simulation setup
nsim<-100000
nall<- c(20,30,50,60,70,100,200,500,1000,5000)
quant=c(c(0.005,0.025,0.05),sort(1-c(0.005,0.025,0.05)))
typeI_crit<-matrix( rep( NA, (length(quant)+1)*length(nall)), nrow = length(nall))
nsimtotal=nsim*length(nall)

# setting progress bar
pb <- winProgressBar(title = "progress bar", min = 1, max = nsimtotal, width = 500)
simrunning=0
for(jj in 1:length(nall)){
  crit <- rep(NA, length = nsim)

  # set seed for reproducibility purposes
  set.seed(0)

  for (ii in 1:nsim){
    n=nall[jj]
    data=gensample(law.index=1,n,law.pars=c(mu,b))$sample
    LapM<-gofMomentsLaplace(data,levels=NA,boot=0) #compute only the test statistic
    crit[ii]=LapM$statistic
    simrunning=simrunning+1
    setWinProgressBar(pb, simrunning, title=paste( round(simrunning/nsimtotal*100, 4),"% done ", n))
    }
  typeI_crit[jj,]=c(n,quantile(crit,probs =quant))

}
# calculating asymptotic critical values
typeI_crit<-rbind(typeI_crit,c(NA,qnorm(quant)))

# Formating results to obtain Table 1
typeI_crit_table<-cbind(typeI_crit[,1],typeI_crit[,2],typeI_crit[,7],typeI_crit[,3],typeI_crit[,6],typeI_crit[,4],typeI_crit[,5])
close(pb)

#-------------------code end for Table 1-------------------#
```

```
#---------------------code for Table 2---------------------#
law.index<-1
mu<- 0
b<- 1

nsim<-10000
nboot=1000
nall<- c(20,30,50,60,70,100)
alpha=c(0.01,0.05,0.1)


typeI_boot<-matrix( rep( NA, (2*length(alpha)+1)*length(nall)), nrow = length(nall))
nsimtotal=nsim*length(nall)

pb <- winProgressBar(title = "progress bar", min = 1, max = nsimtotal, width = 500)
simrunning=0
for(jj in 1:length(nall)){
  crit <- matrix(rep(NA, length = 2*nsim), nrow = nsim)

  set.seed(0)
```

```

  for (ii in 1:nsim){
    n=nall[jj]
    data=gensample(law.index=1,n,law.pars=c(mu,b))$sample
    LapM<-gofMomentsLaplace(data,levels=NA,boot=nboot,return_boot_critical=0)
    crit[ii,]<-c(LapM$pvalue,LapM$pvalueboot)
    simrunning=simrunning+1
    setWinProgressBar(pb, simrunning, title=paste( round(simrunning/nsimtotal*
        100, 4),"% done  ", n))
  }
  typeI_boot[jj,]=c(n*nsim,sum(as.numeric(crit[,1]<alpha[1])),sum(as.numeric(
      crit[,2]<alpha[1])),sum(as.numeric(crit[,1]<alpha[2])),sum(as.numeric(crit
      [,2]<alpha[2])),sum(as.numeric(crit[,1]<alpha[3])),sum(as.numeric(crit
      [,2]<alpha[3])))/nsim

}
close(pb)
#------------------code end for Table 2------------------#
```

```
#------------------code for Tables 3 and 4------------------#

# load the following packages
library("psych")
library(skewt)
library(statmod)
library(teachingApps)
library(PoweR)
library(univariateML)
library(rmutil)


# properly defined function for the proposed test
# in order to be used in many.crit and powcomp.fast
# commands of PowerR package
gofMomentsLaplace_PowerR <- function (data, levels,usecrit=0,critvalL=0,
    critvalR=0){

  n <- length(data)
  mledata=mllaplace(data)
  hatdelta=mledata[1]
  hatc=mledata[2]
  datatransf=(data-hatdelta)/hatc

  mtr=mean(datatransf)
  tn=mean(datatransf^3)-3* mtr*mean(datatransf^2)+2*(mtr)^3
  zn=sqrt(n/504)*tn

  pvalue=2*(1-pnorm(abs(zn),0,1))

  decisions <- rep(0, length(levels))
  for (i in 1:length(levels)) {
    if (usecrit == 0) {
      decisions[i] <- if (pvalue < levels[i]) 1 else 0
    } else {
      decisions[i] <- if (zn < critvalL[i] | zn > critvalR[i]) 1 else 0
    }
  }
   stat.pars = NULL
  return(list(statistic = zn, pvalue = pvalue,decision= decisions, alter=0,stat
      .pars = NULL, pvalcomp=1L,nbparstat=0))
}


# gof51Laplace is the test $T_{m,n}^{V}$ proposed by Choi and Kim (2006)
gof51Laplace <- function (data, levels,usecrit=0,critvalL=0,critvalR=0){

```

```
  n <- length(data)
  muhat <- median(data)
  tildaY_sorted <- sort(data-muhat)
  thetahat <- mean(abs(data-muhat))

  if(n==20){
    m<-3
  }else if(n==50){
    m<-6
  }else if(n==100){
    m<-13
  }else{
    m<-round(0.13*n) # set by noticing that the best m is close to the 0.13*n
  }

  newd<-rep(NA,n)
  for (i in 1:n){
    if(i<=m){
      x<-tildaY_sorted[1]
    }else{
      x<-tildaY_sorted[i-m]
    }
    if(i>=n-m){
      y<-tildaY_sorted[n]
    }else{
      y<-tildaY_sorted[i+m]
    }
    newd[i]<-y-x
  }
 GM <- geometric.mean(newd)
  rnstar=n/(2*m*thetahat)*GM
  pvalue <-0
  stat.pars = NULL
  decisions <- rep(0, length(levels))
  for (i in 1:length(levels)) {
    if (usecrit == 0) {
      decisions[i] <- 0
    } else {
      decisions[i] <- if ( rnstar < critvalL[i]) 1 else 0
    }
  }
  return(list(statistic = rnstar, pvalue = pvalue,decision= decisions, alter=4,
       stat.pars = NULL, pvalcomp=0L,nbparstat=0))
}

# gof60Laplace is the test proposed by Gel (2010).
gof60Laplace <- function (data, levels,usecrit=0,critvalL=0,critvalR=0){

  n <- length(data)
  mledata=mllaplace(data)
  hatdelta=mledata[1]
  hatc=mledata[2]
  datanew3=(data-mean(data))^3
  datanew4=(data-mean(data))^4
  u1sqrt=mean(datanew3)/((sqrt(2)*hatc)^3)
  u2=mean(datanew4)/((sqrt(2)*hatc)^4)

  c1=60
  c2=1200

  k=n/c1*(u1sqrt)^2+n/c2*(u2-6)^2
  pvalue=1-pchisq(k,2)

  decisions <- rep(0, length(levels))
  for (i in 1:length(levels)) {
    if (usecrit == 0) {
```

```
      decisions[i] <- if (pvalue < levels[i]) 1 else 0
    } else {
      decisions[i] <- if (k > critvalR[i]) 1 else 0
    }
  }
  stat.pars = NULL
  return(list(statistic = k, pvalue = pvalue,decision= decisions, alter=3,stat.pars = NULL, pvalcomp=1L,nbparstat=0))
}


# gof91Laplace is the 1st ratio test proposed by Gonzalez-Estrada and Villasenor (2016)
gof91Laplace <- function (data, levels,usecrit=0,critvalL=0,critvalR=0){

  n <- length(data)
  mledata=mllaplace(data)
  hatdelta=mledata[1]
  hatc=mledata[2]

  samplevariance=(n-1)/n*var(data)
  datanew=abs((data-mean(data)))

  bnrelation3=mean(datanew)
  bnrelationunder=sqrt(samplevariance/2)
  rnstar=sqrt(4*n)*(bnrelationunder/bnrelation3 -1)

  pvalue=2*(1-pnorm(abs(rnstar),0,1))

  decisions <- rep(0, length(levels))
  for (i in 1:length(levels)) {
    if (usecrit == 0) {
      decisions[i] <- if (pvalue < levels[i]) 1 else 0
    } else {
      decisions[i] <- if (rnstar < critvalL[i] | rnstar > critvalR[i]) 1 else 0
    }
  }

  stat.pars = NULL

  return(list(statistic = rnstar, pvalue = pvalue,decision= decisions, alter=0,stat.pars = NULL, pvalcomp=1L,nbparstat=0))
}


# gof92Laplace is the 2nd ratio test proposed by Gonzalez-Estrada and Villasenor (2016)
gof92Laplace <- function (data, levels,usecrit=0,critvalL=0,critvalR=0){

  n <- length(data)

  mledata=mllaplace(data)
  hatdelta=mledata[1]
  hatc=mledata[2]

  samplevariance=(n-1)/n*var(data)
  bnrelationunder=sqrt(samplevariance/2)
  rnstar=sqrt(4*n)*(bnrelationunder/ hatc-1)

  pvalue=2*(1-pnorm(abs(rnstar),0,1))

  decisions <- rep(0, length(levels))
  for (i in 1:length(levels)) {
    if (usecrit == 0) {
      decisions[i] <- if (pvalue < levels[i]) 1 else 0
    } else {
```

```
      decisions[i] <- if (rnstar < critvalL[i] | rnstar > critvalR[i]) 1 else 0
    }
  }
  stat.pars = NULL
  return(list(statistic = rnstar, pvalue = pvalue,decision= decisions, alter=0,
      stat.pars = NULL, pvalcomp=1L,nbparstat=0))
}

# gof97Laplace is the test proposed by Rizzo and Haman (2016)
gof97Laplace <- function (data, levels,usecrit=0,critvalL=0,critvalR=0){

  n <- length(data)

  mledata=mllaplace(data)
  hatdelta=mledata[1]
  hatc=mledata[2]

  datatransf=(data-hatdelta)/hatc
  y <- sort(datatransf)

  voit <- rep(0,n)
  for (k in 1:n){
    voit[k]<-(2*k-1-n)*y[k]
  }

  En=2*sum(abs(y)+exp(-abs(y)))-1.5*n-2/n *sum(voit)

  pvalue<-0

  decisions <- rep(0, length(levels))
  for (i in 1:length(levels)) {
    if (usecrit == 0) {
      decisions[i] <- if (pvalue < levels[i]) 1 else 0
    } else {
      decisions[i] <- if (En > critvalR[i]) 1 else 0
    }
  }
  stat.pars = NULL
  return(list(statistic = En, pvalue = pvalue,decision= decisions, alter=3,stat.pars = NULL, pvalcomp=1L,nbparstat=0))
}


# goftest1Laplace is the test proposed by Hadi Alizadeh Noughabi &
    Narayanaswamy Balakrishnan (2016) Tests
# DOI: 10.1080/02664763.2015.1063116
goftest1Laplace <- function (data, levels,usecrit=0,critvalL=0,critvalR=0){

  n <- length(data)
  mledata=mllaplace(data)
  hatdelta=mledata[1]
  hatc=mledata[2]
  w<-1/(2*hatc)*(exp(-abs(data-hatdelta)/hatc))

  s<-sd(data)*sqrt((n-1)/n)
  h<-1.06*s*n^(-1/5)

  fhat <- rep(0, n)

  for (i in 1:n){
    fhat[i]<-sum(dnorm((data[i]-data)/h,mean = 0, sd = 1, log = FALSE))/(n*h)
  }

  TkL<-sum(log(fhat/w))/n

  pvalue<-0
```

```
  decisions <- rep(0, length(levels))
  for (i in 1:length(levels)) {
    if (usecrit == 0) {
      decisions[i] <- if (pvalue < levels[i]) 1 else 0
    } else {
      decisions[i] <- if (TkL > critvalR[i]) 1 else 0
    }
  }
  stat.pars = NULL
  return(list(statistic = TkL, pvalue = pvalue,decision= decisions, alter=3,
      stat.pars = NULL, pvalcomp=1L,nbparstat=0))
}








# goftes2Laplace is the test proposed by Hadi Alizadeh Noughabi & Sangun Park
    (2016)
# DOI: 10.1080/00949655.2015.1104685
goftest2Laplace <- function (data, levels,usecrit=0,critvalL=0,critvalR=0){

  n <- length(data)
  mledata=mllaplace(data)
  hatdelta=mledata[1]
  hatc=mledata[2]


  if(n==20){
    m<-4
  }else if(n==50){
    m<-6
  }else if(n==100){
    m<-8
  }else{
    m<-round(3.22449 +0.04898*n) # by fitting the linear model lm(c(4,6,8)~c
        (20,50,100))
  }
  data_sorted <- sort(data)


  xii <- rep(0, n+1)
  for(ii in 1:(n+1)){
    if(ii-m<1){
      rdused<-length(data_sorted[1:(ii+m-1)])
      augdata<-2*m-rdused
      xii[ii]=mean(c(rep(data_sorted[1],augdata),data_sorted[1:(ii+m-1)]))
    }else if(ii-m>=1 & ii+m-1<=n){
      xii[ii]=mean(data_sorted[(ii-m):(ii+m-1)])
    }else{
      rdused<-length(data_sorted[(ii-m):n])
      augdata<-2*m-rdused
      xii[ii]=mean(c(data_sorted[(ii-m):n],rep(data_sorted[n],augdata)))
    }
  }

  if((n %% 2) == 0){
    cuhat<- -1/n * sum( (xii[1:(n/2)]+xii[(1+1):(n/2+1)])/2)+
              1/n*sum((xii[(n/2+1):n]+xii[(n/2+1+1):(n+1)])/2)
  }else{
    cuhat<- -1/n*sum((xii[1:((n-1)/2)]+xii[(1+1):((n-1)/2+1)])/2)+
      1/(4*n)*(xii[(n+1)/2+1]-xii[(n+1)/2])+
```

```
      1/n*sum((xii[((n+1)/2+1):n]+xii[((n+1)/2+1+1):(n+1)])/2)
    }

  difxm<-rep(0, n)
  for(ii in 1:n){
    if(ii-m<=1){
      difxm[ii]=data_sorted[ii+m]-data_sorted[1]
    }else if(ii-m>1 & ii+m<=n){
      difxm[ii]=data_sorted[ii+m]-data_sorted[ii-m]
    }else{
      difxm[ii]=data_sorted[n]-data_sorted[ii-m]
    }
  }
  HVmn<-sum(log(difxm*n/(2*m)))/n

  TVmn<-log(2*cuhat)+1-HVmn


  pvalue<-0

  decisions <- rep(0, length(levels))
  for (i in 1:length(levels)) {
    if (usecrit == 0) {
      decisions[i] <- if (pvalue < levels[i]) 1 else 0
    } else {
      decisions[i] <- if (TVmn > critvalR[i]) 1 else 0
    }
  }
  stat.pars = NULL
  return(list(statistic = TVmn, pvalue = pvalue,decision= decisions, alter=3,
      stat.pars = NULL, pvalcomp=1L,nbparstat=0))
}




# goftest3Laplace is the test proposed by Hadi Alizadeh Noughabi (2019)
# DOI: 10.1080/00949655.2019.1602870
goftest3Laplace <- function (data, levels,usecrit=0,critvalL=0,critvalR=0){

  n <- length(data)
  mledata=mllaplace(data)
  hatdelta=mledata[1]
  hatc=mledata[2]


  if(n==10){
    m<-4
  }else if(n==20){
    m<-7
  }else if(n==33){
    m<-7
  }else if(n==45){
    m<-10
  }else if(n==100){
    m<-15
  }else{
    m<-round(3.8021+0.1153*n ) # by fitting the linear model lm(c(4,7,7,15)~c
        (10, 20,33,100))
  }
  data_sorted <- sort(data)

  z=plaplace(data_sorted, m=hatdelta, s=hatc)

  difzm<-rep(0, n)
  for(ii in 1:n){
```

```
    if(ii-m<=1){
      difzm[ii]=z[ii+m]-z[1]
    }else if(ii-m>1 & ii+m-1<n){
      difzm[ii]=z[ii+m]-z[ii-m]
    }else{
      difzm[ii]=z[n]-z[ii-m]
    }
  }
  DA<- -sum(log(difzm*n/(2*m)))/n



  pvalue<-0

  decisions <- rep(0, length(levels))
  for (i in 1:length(levels)) {
    if (usecrit == 0) {
      decisions[i] <- if (pvalue < levels[i]) 1 else 0
    } else {
      decisions[i] <- if (DA > critvalR[i]) 1 else 0
    }
  }
  stat.pars = NULL
  return(list(statistic = DA, pvalue = pvalue,decision= decisions, alter=3,stat
      .pars = NULL, pvalcomp=1L,nbparstat=0))
}














# simulation setup
law.index<-1
M<-10^5
vectn<- c(20,50,100)
levels<- 0.05


stind<-c(43,44,42,45,46,47,48,49,50, 0,0,0,0,0,59,57,55,56,0,0,0,0)

alter <- list(stat43 = 3,stat44 = 3,stat42 = 3,stat45 = 3,stat46 = 3,stat47 =
    3,stat48 = 3,stat49 = 3,stat50 = 3,stat0=4,stat0=3,stat0=3, stat0=3,stat0=3,
    stat59 = 3,stat57 = 3,stat55 = 0,stat56 = 0,stat0=3,stat0=0,stat0=0,stat0=0)

set.seed(0)

critval<-many.crit(law.index,stat.indices=stind,M,vectn,levels,alter,law.pars=c
    (0,1),Rstats=list(NULL,NULL,NULL,NULL,NULL,NULL,NULL,NULL,NULL,gof51Laplace,
    gof97Laplace,goftest1Laplace,goftest2Laplace,goftest3Laplace, NULL,NULL,NULL
    ,NULL,gof60Laplace,gof91Laplace,gof92Laplace,gofMomentsLaplace_PowerR)
)

print(critval)

# law indices for Table 3
law.indices<-c(1,2,4,3,7,6,8,8,8,18,31,31,37,37)
parlaws=list(law1=c(0,1),law2 = c(0, 1),law4=c(0,sqrt(3)/pi),law3=c(0,1),law7=c
    (-sqrt(12)/2,sqrt(12)/2),law6=c(2,2),law8=3,law8=6,law8=10,law18=0.5,law31=c
```

```
(0.1,0,3.5),law31=c(0.2,0,3.2),law37=c(0.4,0,0.6,1),law37=c(0.7,0,0.2,1))

table3<-powcomp.fast(law.indices,stind,vectn,M,levels,critval=critval,alter,
    Rlaws=list(NULL,NULL,NULL,NULL,NULL,NULL,NULL,NULL,NULL,NULL,NULL,NULL,NULL,
    NULL)
,parlaws=parlaws,Rstats=list(NULL,NULL,NULL,NULL,NULL,NULL,NULL,NULL,NULL,
    gof51Laplace,gof97Laplace,goftest1Laplace,goftest2Laplace,goftest3Laplace,
    NULL,NULL,NULL,NULL,gof60Laplace,gof91Laplace,gof92Laplace,gofMomentsLaplace
    _PowerR)
)


print(table3)

# law indices for Table 4
law.indices<-c(35,5,5,26,21,0,10,11,11,9,0,0,37,37)
parlaws=list(law35=1,law5=c(2,1),law5=c(6,1),law26=c(0,1),law21=c(0,1,3),law0=c
    (3,10),law10=c(0,0.5),law11=c(3,1), law11=c(2,1),law9=c(2),law0=c(0,1),law0=
    c(1,4),law37=c(1,0.5,0.43,1),law37=c(0.5,0.2,0.5,1))

table4<-powcomp.fast(law.indices,stind,vectn,M,levels,critval=critval,alter,
    Rlaws=list(NULL,NULL,NULL,NULL,NULL,rskt,NULL,NULL,NULL,NULL,rsev,rinvgauss,
    NULL,NULL),parlaws=parlaws,Rstats=list(NULL,NULL,NULL,NULL,NULL,NULL,NULL,
    NULL,NULL,gof51Laplace,gof97Laplace,goftest1Laplace,goftest2Laplace,
    goftest3Laplace, NULL,NULL,NULL,NULL,gof60Laplace,gof91Laplace,gof92Laplace,
    gofMomentsLaplace_PowerR)
)



print(table4)
#-----------------code end for Tables 3 and 4-----------------#
```

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