Austrian Journal of Statistics
February 2018, Volume 47, 45–76.
http://www.ajs.or.at/
doi:10.17713/ajs.v47i1.155

The Gamma-Weibull-G Family of Distributions with Applications

Broderick O. Oluyede
GSU

Shusen Pu
CWRU

Boikanyo Makubate
BIUST

Yuqi Qiu
UCSD

Abstract

Weibull distribution and its extended families has been widely studied in lifetime applications. Based on the Weibull-G family of distributions and the exponentiated Weibull distribution, we study in detail this new class of distributions, namely, Gamma-Weibull-G (GWG) family of distributions. Some special models in the new class are discussed. Statistical properties of the family of distributions, such as expansion of density function, hazard and reverse hazard functions, quantile function, moments, incomplete moments, generating functions, mean deviations, Bonferroni and Lorenz curves and order statistics are presented. We also present Rényi entropy, estimation of parameters by using method of maximum likelihood, asymptotic confidence intervals and applications using real data sets.

Keywords: gamma distribution, Weibull-G distribution, maximum likelihood estimation.

1. Introduction

Numerous classical distributions have been extensively discussed over the past years for modeling data in various areas such as economics, finance, insurance, actuarial, engineering, demography, biological studies, environmental and medical sciences. However, in many of those applied areas, there is more and more demands for extended forms of these distributions. Hence, several methods for generating new families of distributions have been studied in recent years. Researchers often define new families of probability distributions that extend well-known distributions in order to provide great flexibility in modeling data in practice. Recently, (Alzaatreh and Ghosh 2014) introduced and studied the Weibull-X family of distributions, (Alzaatreh, Famoye, and Lee 2014) developed the properties of the gamma-X family of distributions in general and studied the gamma-normal distribution as a special case, and (Nascimento, Bourguignon, Zea, Santos-Neto, Silva, and Cordeiro 2014) introduced the gamma extended Weibull family of distributions. Moreover, (Nadarajah, Cordeiro, and Ortega 2015) provided a comprehensive analysis of general mathematical and statistical properties of Zografos and Balakrishnan-G distributions. See references therein. Most recently, (Tahir, Zubair, Mansoor, Cordeiro, Alizadeh, and Hamedani 2015) introduced a new generator based on the Weibull random variable, namely, the new Weibull-G family.

We introduce a new extended generator, the gamma-Weibull-G family based on their previous work and combined the gamma-generator with the Weibull-G family of distributions which was defined by (Bourguignon, Silva, and Cordeiro 2014). We hope the new family of distributions yields a better fit in certain practical situations. Additionally, we provide a rigorous and comprehensive account of the mathematical properties of the proposed family of distributions.

The results in this paper are organized in the following manner. The new model called gamma Weibull-G (GWG) distribution and its sub-models are given in section 2. In section 3, statistical properties including expansion of the probability density function, hazard rate, reverse hazard rate and quantile functions, moments, conditional moments, moment generating and characteristics functions, mean deviations, Lorenz and Bonferroni curves, order statistics and Rényi entropy are presented. Maximum likelihood estimates of the model parameters and observed information matrix are given in section 4. The special cases of gamma-Weibull-Uniform (GWU) and gamma-Weibull-Weibull (GWW) distributions are presented in details in sections 5 and 6, respectively. A Monte Carlo simulation study to examine the bias and mean square error of the maximum likelihood estimates are presented in section 7. Section 8 contains applications of the new model to real data sets. A short concluding remark is given in section 9.

2. The model

Considering a continuous cumulative distribution function (cdf) $G(x)$ with probability density function (pdf) $g(x)$ and survival function $\bar{G}(x) = 1 - G(x)$, (Bourguignon *et al.* 2014) defined the Weibull-G family of distribution with cdf:

$$\begin{aligned} F_{WG}(x) &= \int_0^{\frac{G(x;\underline{\theta})}{1-G(x;\underline{\theta})}} \alpha\beta t^{\beta-1} e^{-\alpha t^{\beta}} dt \\ &= 1 - \exp\Big\{ -\alpha\Big[\frac{G(x;\underline{\theta})}{\bar{G}(x;\underline{\theta})}\Big]^{\beta}\Big\}, \quad x, \alpha, \beta > 0, \end{aligned} \tag{1}$$

where $\underline{\theta}$ is a vector of parameters. The pdf of this family of distributions is given by

$$f_{WG}(x) = \alpha\beta g(x;\underline{\theta}) \frac{G(x;\underline{\theta})^{\beta-1}}{\bar{G}(x;\underline{\theta})^{\beta+1}} \exp\Big\{ -\alpha\Big[\frac{G(x;\underline{\theta})}{\bar{G}(x;\underline{\theta})}\Big]^{\beta}\Big\}. \tag{2}$$

Therefore, given a continuous baseline distribution $G(x)$, one can derive the Weibull-G distribution with two extra parameters α and β from pdf (2). In the rest of this section, we consider the combination of Weibull-G (W-G) distribution with distribution proposed by (Zografos and Balakrishnan 2009) with cdf,

$$F(x;\delta) = \frac{1}{\Gamma(\delta)} \gamma\{\delta, -\log[1 - G(x)]\}, \quad x, \delta > 0, \tag{3}$$

where $\gamma(\delta, x) = \int_0^x t^{\delta-1} e^{-t} dt$ is the lower incomplete gamma function and $\Gamma(\cdot)$ is the gamma function. The corresponding pdf is given by

$$f(x;\delta) = \frac{1}{\Gamma(\delta)} \{-\log[1 - G(x)]\}^{\delta-1} g(x). \tag{4}$$

By taking $G(x) = F_{WG}(x)$ in (3), we come up with the gamma-Weibull-G family of distributions (GWG), with cdf:

$$F_{GWG}(x) = \frac{1}{\Gamma(\delta)} \gamma\Big\{\delta, \alpha\Big[\frac{G(x;\underline{\theta})}{\bar{G}(x;\underline{\theta})}\Big]^{\beta}\Big\} \tag{5}$$

and pdf

$$f_{GWG}(x) = \frac{\beta\alpha^{\delta}}{\Gamma(\delta)} g(x;\underline{\theta}) \frac{G(x;\underline{\theta})^{\beta\delta-1}}{\bar{G}(x;\underline{\theta})^{\beta\delta+1}} \exp\Big\{ -\alpha\Big[\frac{G(x;\underline{\theta})}{\bar{G}(x;\underline{\theta})}\Big]^{\beta}\Big\}. \tag{6}$$

Therefore, for each G distribution, we define the Gamma-Weibull-G (GWG) distribution with three extra parameters α, β and δ by the pdf (6). A random variable X with pdf (6) is denoted by $X \sim GWG(\alpha, \beta, \delta, \underline{\theta})$. The additional parameters induced by the Gamma-Weibull (G-W) generator induces a family of more flexible distributions. An interpretation of the GWG family of distributions can be given as follows (Bourguignon *et al.* 2014) in a similar context. Let Z be a lifetime random variable having a certain continuous G distribution. The odds ratio that an individual (or component) following the lifetime Z will die (failure) at time x is $G(x;\underline{\theta})/\bar{G}(x;\underline{\theta})$. Consider that the variability of this odds of death is represented by the random variable X and assume that it follows the Gamma-Weibull model with scale parameter α, and shape parameters δ and β. We can write

$$Pr(Z < x) = Pr\Big(X \leq \alpha\Big[\frac{G(x;\underline{\theta})}{\bar{G}(x;\underline{\theta})}\Big]^{\beta}\Big) = F_{GWG}(x; \alpha, \beta, \delta, \underline{\theta}),$$

which is given by (5). To avoid issues or problems with over parametrization and redundancy, we restrict the parameter vector $\underline{\theta}$ to at most a two component vector.

2.1. Sub-models of the GWG distribution

If $\delta = 1$, the GWG distribution corresponds to the Weibull-generator, which was proposed by (Bourguignon *et al.* 2014). When $\beta = 1$, and $\delta = \beta = 1$, GWG distribution reduces to the gamma-exponential-generator, and exponential-generator (Cordeiro, Ortega, and Cunha 2013b), respectively. Moreover, $\beta = 2$, and $\beta = 2, \delta = 1$ lead to the gamma-Rayleigh-generator, and Rayleigh-generator, respectively. In addition, a lot of sub-models can be obtained when we change the parameter vector $\underline{\theta}$ and these special cases as well as the corresponding parameters are listed in Table 1.

Table 1: Distributions and corresponding $G(x;\underline{\theta})/\bar{G}(x;\underline{\theta})$ functions

Distribution	$G(x;\underline{\theta})/\bar{G}(x;\underline{\theta})$	$\Psi = (\alpha, \beta, \delta, \underline{\theta})$	GWG
Uniform	$x/(\theta - x)$	$(\alpha, \beta, \delta, \theta)$	G-W-Uniform
Exponential	$e^{\lambda x} - 1$	$(\alpha, \beta, \delta, \lambda)$	G-W-Exponential
Weibull	$e^{\lambda x^{\gamma}} - 1$	$(\alpha, \beta, \delta, \lambda, \gamma)$	G-W-Weibull
Fréchet	$(e^{\lambda x^{\gamma}} - 1)^{-1}$	$(\alpha, \beta, \delta, \lambda, \gamma)$	G-W-*Fréchet*
Half-logistic	$(e^{x} - 1)/2$	(α, β, δ)	G-W-Half-logistic
Power function	$[(\theta x)^{-k} - 1]^{-1}$	$(\alpha, \beta, \delta, \theta, k)$	G-W-Power function
Pareto	$(x/\theta)^{k} - 1$	$(\alpha, \beta, \delta, \theta, k)$	G-W-Pareto
Burr XII	$[1 - x^{c}]^{k} - 1$	$(\alpha, \beta, \delta, c, k)$	G-W-Burr XII
Log-logistic	$[1 - x^{c}] - 1$	$(\alpha, \beta, \delta, c)$	G-W-Log-logistic
Lomax	$[1 - x]^{k} - 1$	$(\alpha, \beta, \delta, k)$	G-W-Lomax
Gumbel	$\{\exp[\exp(-(x - \mu)/\sigma)] - 1\}^{-1}$	$(\alpha, \beta, \delta, \mu, \sigma)$	G-W-Gumbel
Normal	$\Phi((x - \mu)/\sigma)/[1 - \Phi((x - \mu)/\sigma)]$	$(\alpha, \beta, \delta, \mu, \sigma)$	G-W-Normal
Kumaraswamy	$(1 - x^{a})^{-b} - 1$	$(\alpha, \beta, \delta, a, b)$	G-W-Kumaraswamy
Modified Exponential	$e^{\theta x e^{\lambda x}} - 1$	$(\alpha, \beta, \delta, \theta, \lambda)$	G-W-Exponential

3. Statistical properties of the GWG distribution

Some statistical properties, such as expansion of density function, hazard and reverse hazard functions, quantile function, moments, incomplete moments, generating functions, mean deviations, Bonferroni and Lorenz curves and distribution of order statistics of the GWG distribution are presented in this section.

3.1. Expansion of the pdf of GWG distribution

Considering the power series for exponential function, we can obtain the following equation:

$$\exp\Big\{-\alpha\Big[\frac{G(x;\underline{\theta})}{\bar{G}(x;\underline{\theta})}\Big]^{\beta}\Big\}=\sum_{i=0}^{\infty}\frac{(-1)^i\alpha^i}{i!}\Big[\frac{G(x;\underline{\theta})}{\bar{G}(x;\underline{\theta})}\Big]^{i\beta}.$$

Inserting this equation into (6), we obtain

$$f_{GWG}(x) = \frac{\beta\alpha^{\delta}}{\Gamma(\delta)}g(x;\underline{\theta})\frac{G(x;\underline{\theta})^{\beta\delta-1}}{\bar{G}(x;\underline{\theta})^{\beta\delta+1}}\sum_{i=0}^{\infty}\frac{(-1)^i\alpha^i}{i!}\Big[\frac{G(x;\underline{\theta})}{\bar{G}(x;\underline{\theta})}\Big]^{i\beta} \tag{7}$$

$$= \frac{\beta\alpha^{\delta}}{\Gamma(\delta)}g(x;\underline{\theta})\sum_{i=0}^{\infty}\frac{(-1)^i\alpha^i}{i!}G(x;\underline{\theta})^{\beta\delta+\beta i-1}\bar{G}(x;\underline{\theta})^{-[\beta\delta+\beta i+1]}. \tag{8}$$

Note that

$$\bar{G}(x;\underline{\theta})^{-[\beta\delta+\beta i+1]}=\sum_{j=0}^{\infty}\frac{\Gamma(\beta(\delta+i)+j+1)}{j!\Gamma(\beta(\delta+i)+1)}G(x;\underline{\theta})^{j}.$$

We obtain

$$f_{GWG}(x) = \frac{\beta\alpha^{\delta}g(x;\underline{\theta})}{\Gamma(\delta)}\sum_{i=0}^{\infty}\frac{(-1)^i\alpha^i}{i!}G(x;\underline{\theta})^{\beta\delta+\beta i-1}\sum_{j=0}^{\infty}\frac{\Gamma(\beta(\delta+i)+j+1)}{j!\Gamma(\beta(\delta+i)+1)}G(x;\underline{\theta})^{j}$$

$$= \sum_{i,j=0}^{\infty}\omega_{i,j}h_{\beta(\delta+i)+j}(x;\underline{\theta}), \tag{9}$$

where

$$\omega_{i,j}=\frac{(-1)^i\beta\alpha^{\delta+i}\Gamma[\beta(\delta+i)+j+1]}{i!j!\Gamma(\delta)\Gamma[\beta(\delta+i)+1][\beta(\delta+i)+j]}, \tag{10}$$

and

$$h_{\beta(\delta+i)+j}(x;\underline{\theta})=[\beta(\delta+i)+j]g(x;\underline{\theta})G(x;\underline{\theta})^{\beta(\delta+i)+j-1}$$

denote the pdf of the exponentiated generalized (EG) (Cordeiro *et al.* 2013b) distribution with parameter $\beta^*=\beta(\delta+i)+j$. The expansion of the pdf of GWG distribution illustrates that the GWG density function is indeed an infinite linear combination of EG density functions. Thus, a lot of the mathematical and statistical properties of the GWG distribution come directly from those of EG distribution.

3.2. Hazard, reverse hazard and quantile functions

In this section, we provide the hazard, reverse hazard and quantile functions of the GWG distribution.

Hazard and reverse hazard functions

Note that if X is a continuous random variable with cdf $G(x)$, and pdf $g(x)$, then the hazard rate function (hrf), reverse hazard function (rhf) and mean residual life functions are given by $h_G(x)=g(x)/\overline{G}(x)$, $\tau_G(x)=g(x)/G(x)$, and $\delta_G(x)=\int_x^{\infty}\overline{G}(u)du/\overline{G}(x)$ respectively. The functions $h_G(x)$, $\delta_G(x)$, and $\overline{G}(x)$ are equivalent. See (Shaked and Shanthikumar 1994) for additional details. The hazard rate and reverse hazard rate functions of GWG distribution are given by

$$h_{GWG}(x) = \frac{\beta\alpha^{\delta}g(x;\underline{\theta})\frac{G(x;\underline{\theta})^{\beta\delta-1}}{\overline{G}(x;\underline{\theta})^{\beta\delta+1}}\exp\Big\{-\alpha\Big[\frac{G(x;\underline{\theta})}{\overline{G}(x;\underline{\theta})}\Big]^{\beta}\Big\}}{\Gamma(\delta)-\gamma\Big\{\delta,\alpha\Big[\frac{G(x;\underline{\theta})}{\overline{G}(x;\underline{\theta})}\Big]^{\beta}\Big\}}, \tag{11}$$

and

$$\tau_{GWG}(x) = \frac{\beta\alpha^{\delta} g(x;\underline{\theta}) \frac{G(x;\underline{\theta})^{\beta\delta-1}}{\bar{G}(x;\underline{\theta})^{\beta\delta+1}} \exp\left\{-\alpha\left[\frac{G(x;\underline{\theta})}{\bar{G}(x;\underline{\theta})}\right]^{\beta}\right\}}{\gamma\left\{\delta, \alpha\left[\frac{G(x;\underline{\theta})}{\bar{G}(x;\underline{\theta})}\right]^{\beta}\right\}}, \tag{12}$$

respectively.

Quantile function

The GWG quantile function can be obtained by inverting $F_{GWG}(x) = u$, where $F_{GWG}(x)$ is given by (5). Hence, we obtain the following equation:

$$\left[\frac{G(x;\underline{\theta})}{\bar{G}(x;\underline{\theta})}\right]^{\beta} = \frac{\gamma^{-1}\Big(\delta, u\Gamma(\delta)\Big)}{\alpha}. \tag{13}$$

Thus, the quantile x_u of the GWG$(\alpha, \beta, \delta, \underline{\theta})$ reduces to the real solution of the following equation:

$$G(x;\underline{\theta}) = \frac{\left[\gamma^{-1}\big(\delta, u\Gamma(\delta)\big)\right]^{\frac{1}{\beta}}}{\left[\gamma^{-1}\big(\delta, u\Gamma(\delta)\big)\right]^{\frac{1}{\beta}} + \alpha^{\frac{1}{\beta}}} := q, \tag{14}$$

that is, the quantile x_u of the GWG$(\alpha, \beta, \delta, \underline{\theta})$ can be derived from the quantile x_q of the baseline distribution with cdf $G(x;\underline{\theta})$ and it is given by

$$x_q = G^{-1}(q), \tag{15}$$

where q is defined in equation (14).

3.3. Moments, incomplete moments, moment generating and characteristic functions

Let $X \sim GWG(\alpha, \beta, \delta, \underline{\theta})$, the s^{th} moment of X can be obtained from (9) as

$$E(X^s) = \sum_{i,j=0}^{\infty} \omega_{i,j} E(Z_{i,j}^s), \tag{16}$$

where $E(Z_{i,j}^s)$ denotes the s^{th} moment of $Z_{i,j}$, which follows the EG distribution with the parameter $\beta^* = \beta(\delta + i) + j$ and $\omega_{i,j}$ is given by equation (10). Similarly, the incomplete moments and the moment generating function (mgf) can be obtained as follows:

$$I_X(y) = \int_0^y x^s f_{GWG}(x)dx = \sum_{i,j=0}^{\infty} \omega_{i,j} I_{i,j}(y),$$

where $I_{i,j}(y) = \int_0^y x^s h_{\beta(\delta+i)+j}(x, \underline{\theta})dx$ and

$$M_X(t) = \sum_{i,j=0}^{\infty} \omega_{i,j} E(e^{tZ_{i,j}}),$$

where $E(e^{tZ_{i,j}})$ is the mgf of the EG distribution with parameter $\beta^* = \beta(\delta + i) + j$ and $\omega_{i,j}$ is given by (10). The characteristic function is given by $\phi(t) = E(e^{itX})$, where $i = \sqrt{-1}$, thus one can obtain

$$\phi(t) = \sum_{i,j=0}^{\infty} \omega_{i,j} \phi_{\beta(\delta+i)+j}(t),$$

with $\phi_{\beta(\delta+i)+j}(t)$ denotes the characteristic function of EG distribution, where parameter $\beta^* = \beta(\delta+i)+j$, and $\omega_{i,j}$ is defined as (10).

Since it is always of interest to know conditional expectations for lifetime models, we present it as follow:

$$E(X^t \mid X > x) = \frac{\Gamma(\delta)}{\Gamma(\delta) - \gamma\Big\{\delta, \alpha\Big[\frac{G(x;\underline{\theta})}{\bar{G}(x;\underline{\theta})}\Big]^{\beta}\Big\}} \times \sum_{i,j=0}^{\infty} \omega_{i,j} I_{i,j}(y),$$

where $I_{i,j}(y) = \int_0^y x^t h_{\beta(\delta+i)+j}(x,\underline{\theta})dx$, and $\omega_{i,j}$ is given by (10).

3.4. Mean deviation, Bonferroni and Lorenz curves

Let $X \sim GWG(\alpha,\beta,\delta,\underline{\theta})$, the mean deviation about the mean and the mean deviation about the median are defined by $\delta_1(X) = \int_0^\infty |x-\mu| f_{GWG}(x)dx$, and $\delta_2(X) = \int_0^\infty |x-M| f_{GWG}(x)dx$, respectively, where $\mu = E(X)$ and $M = Median(X)$ denotes the median. They can be expressed as

$$\delta_1(X) = 2\mu F_{GWG}(\mu) - 2\int_0^{\mu} x f_{GWG}(x)dx \quad \text{and} \quad \delta_2(X) = \mu - 2\int_0^{M} x f_{GWG}(x)dx,$$

where $m(z) = \int_0^z x f_{GWG}(x)dx$ is the first incomplete moment. These quantities have been applied to a wide variety of fields, such as studying of income and property in economics, reliability, demography, insurance and medicine. Bonferroni and Lorenz curves are defined by

$$B(p) = \frac{1}{p\mu}\int_0^q x f(x)dx \quad and \quad L(p) = \frac{1}{\mu}\int_0^q x f(x)dx,$$

respectively, where $\mu = E(X)$ and $q = F^{-1}_{GWG}(p)$ is obtained from equation (14) and (15). Using similar methods in deriving the moments, we can show that

$$B(p) = \frac{1}{p\mu}\int_0^q x f(x)dx = \frac{1}{p\mu}\sum_{i,j=0}^{\infty} \omega_{i,j} I_{\beta(\delta+i)+j}(i,j),$$

and

$$L(p) = \frac{1}{\mu}\int_0^q x f(x)dx = \frac{1}{\mu}\sum_{i,j=0}^{\infty} \omega_{i,j} I_{\beta(\delta+i)+j}(i,j),$$

where $I_{\beta(\delta+i)+j}(i,j) = \int_0^q x f_{\beta(\delta+i)+j}(x,\underline{\theta})dx$, is the first incomplete moment of the EG distribution with parameter $\beta^* = \beta(\delta+i)+j$ and $\omega_{i,j}$ is given by (10).

3.5. Order statistics

In this section, distribution of the i^{th} order statistic for the GWG distribution is presented. Let $X_1,\ldots,X_n$ be independent and identically distributed (i.i.d) random variables distributed according to (6). The pdf of the i^{th} order statistic, for example $X_{i:n}$, is given by

$$f_{i:n}(x) = \frac{n! f_{GWG}(x)}{(i-1)!(n-i)!}[F_{GWG}(x)]^{i-1}[1-F_{GWG}(x)]^{n-i}.$$

Using the binomial theorem, the pdf of i^{th} order statistic can be written as

$$f_{i:n}(x) = \frac{n! f_{GWG}(x)}{(i-1)!(n-i)!}\sum_{k=0}^{n-i}\frac{(-1)^k}{[\Gamma(\delta)]^{i+k-1}}\binom{n-i}{k}\left\{\gamma\Big(\delta, \alpha\Big[\frac{G(x;\underline{\theta})}{\bar{G}(x;\underline{\theta})}\Big]^{\beta}\Big)\right\}^{i+k-1}.$$

Note that for the incomplete gamma function $\gamma(x,\delta)$, we apply the following power series (see (Gradshteyn and Ryzhik 2000) for additional details)

$$\gamma(x,\delta) = \sum_{m=0}^{\infty}\frac{(-1)^m x^{m+\delta}}{(m+\delta)m!},$$

to obtain

$$
\begin{aligned}
f_{i:n}(x) &= \frac{n! f_{GWG}(x)}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} \binom{n-i}{k} \frac{(-1)^k}{[\Gamma(\delta)]^{i+k-1}} \left\{ \alpha \left[\frac{G(x;\underline{\theta})}{\bar{G}(x;\underline{\theta})} \right]^{\beta} \right\}^{\delta(i+k-1)} \\
&\times \left\{ \sum_{m=0}^{\infty} \frac{(-1)^m \left(\alpha \left[\frac{G(x;\underline{\theta})}{\bar{G}(x;\underline{\theta})} \right]^{\beta} \right)^m}{(m+\delta) m!} \right\}^{k+i-1}.
\end{aligned}
$$

Let $c_m = \frac{(-1)^m}{m!(m+\delta)}$ and using the result on a power series raised to a positive integer, we can write

$$
\left\{ \sum_{m=0}^{\infty} c_m \left(\alpha \left[\frac{G(x;\underline{\theta})}{\bar{G}(x;\underline{\theta})} \right]^{\beta} \right)^m \right\}^{k+i-1} = \sum_{m=0}^{\infty} d_{m,k+i-1} \left(\alpha \left[\frac{G(x;\underline{\theta})}{\bar{G}(x;\underline{\theta})} \right]^{\beta} \right)^m,
$$

where $d_{0,k+i-1} = c_0^{(k+i-1)}$ and $d_{m,k+i-1} = (mc_0)^{-1} \sum_{l=1}^{m} [(k+i-1)l - m + l] c_l d_{m-l,k+i-1}$. Applying the above equation and replacing $f_{GWG}(x)$ by the right hand side of (6), we obtain

$$
\begin{aligned}
f_{i:n}(x) &= \frac{n! f_{GWG}(x)}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} \sum_{m=0}^{\infty} \binom{n-i}{k} \frac{(-1)^k d_{m,k+i-1}}{[\Gamma(\delta)]^{i+k-1}} \left\{ \alpha \left[\frac{G(x;\underline{\theta})}{\bar{G}(x;\underline{\theta})} \right]^{\beta} \right\}^{\delta(i+k-1)+m} \\
&= \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} \sum_{m=0}^{\infty} \binom{n-i}{k} \frac{(-1)^k d_{m,k+i-1} \Gamma(\delta i + \delta k + m)}{[\Gamma(\delta)]^{i+k}} \\
&\times \; f_{GWG}(x;\alpha,\beta,\delta_*).
\end{aligned}
$$

In other words, the pdf of the i^{th} order statistic is exactly a linear combination of GWG densities with parameters $(\alpha, \beta, \delta_*)$, where $\delta_* = \delta i + \delta k + m$. This is a very useful result since the properties of the order statistics of the GWG distribution can be obtained from those of EG distribution. For instance, we can obtain the moments of the i^{th} order statistic from GWG distribution as:

$$
\begin{aligned}
E(X_{i:n}^t) &= \sum_{l,j=0}^{\infty} \sum_{k=0}^{n-i} \sum_{m=0}^{\infty} \frac{n! \omega_{i,j}}{(i-1)!(n-i)!} \binom{n-i}{k} \frac{(-1)^k d_{m,k+i-1} \Gamma(\delta i + \delta k + m)}{[\Gamma(\delta)]^{i+k}} \\
&\times E\big(Z_{l,j}(x;\alpha,\beta,\delta_*)^t\big),
\end{aligned}
$$

where $E(Z_{l,j}(x;\alpha,\beta,\delta_*)^t)$ is the t^{th} moment of EG distribution with parameter $\beta^{**} = \beta(\delta_* + i) + j$, where $\delta_* = \delta i + \delta k + m$ and $\omega_{i,j}$ is given by (10).

3.6. Rényi entropy

In information theory, Rényi entropy generalizes Shannon entropy, Hartley entropy, min-entropy, and collision entropy. Entropies quantify the diversity, uncertainty, or randomness of a system. Rényi entropy is named after Alfréd Rényi (Renyi 1961) and is very important in ecology and statistics as indices of diversity. Rényi entropy is also important in quantum information, where it can be used as a measure of entanglement. As an extension of Shannon entropy, Rényi entropy is a popular measure of entropy, which is defined by

$$
I_R(\nu) = (1-\nu)^{-1} \log \left[\int_{-\infty}^{\infty} g^{\nu}(x) dx \right],
$$

where $\nu > 0$ and $\nu \neq 1$. We obtain Rényi entropy for the GWG distribution as follows:

$$
\begin{aligned}
I_R(\nu) &= (1-\nu)^{-1} \log \left[\int_{-\infty}^{\infty} g^{\nu}(x) dx \right] \\
&= (1-\nu)^{-1} \Big\{ \nu \log(\beta) + \delta\nu \log(\alpha) - \nu \log(\Gamma(\delta)) \\
&+ \log \left[\int_{-\infty}^{\infty} g^{\nu}(x;\underline{\theta}) \frac{G(x;\underline{\theta})^{(\beta\delta-1)\nu}}{\bar{G}(x;\underline{\theta})^{(\beta\delta+1)\nu}} \exp\left\{ -\alpha\nu \left[\frac{G(x;\underline{\theta})}{\bar{G}(x;\underline{\theta})} \right]^{\beta} \right\} dx \right] \Big\}. \qquad (17)
\end{aligned}
$$

Using similar expansion for the pdf, that is,

$$\exp\Big\{-\alpha\nu\Big[\frac{G(x;\underline{\theta})}{\bar{G}(x;\underline{\theta})}\Big]^{\beta}\Big\} = \sum_{i=0}^{\infty}\frac{(-1)^i(\alpha\nu)^i}{i!}\Big[\frac{G(x;\underline{\theta})}{\bar{G}(x;\underline{\theta})}\Big]^{i\beta}, \tag{18}$$

we obtain

$$\begin{aligned} I_R(\nu) &= (1-\nu)^{-1}\Big\{\nu\log(\beta)+\delta\nu\log(\alpha)-\nu\log(\Gamma(\delta)) \\ &\quad +\log\Big[\sum_{i=0}^{\infty}\frac{(-1)^i(\alpha\nu)^i}{i!}\int_{-\infty}^{\infty}g^{\nu}(x;\underline{\theta})\frac{\big[G(x;\underline{\theta})\big]^{(\beta\delta-1)\nu+\beta i}}{\big[\bar{G}(x;\underline{\theta})\big]^{(\beta\delta+1)\nu+\beta i}}dx\Big]\Big\}. \end{aligned} \tag{19}$$

Note that

$$\frac{1}{\bar{G}(x;\underline{\theta})} = \frac{1}{1-G(x;\underline{\theta})} = \sum_{s=0}^{\infty}\big[G(x;\underline{\theta})\big]^s \quad and \quad \Big(\sum_{s=0}^{\infty}G(x;\underline{\theta})^s\Big)^m = \sum_{s=0}^{\infty}b_{s,m}\big[G(x;\underline{\theta})\big]^s,$$

where

$$b_{s,m} = s^{-1}\sum_{l=1}^{s}\Big[m(l+1)-s\Big]b_{s-l,m}, \tag{20}$$

and $b_{0,m}=1, m=(\beta\delta+1)\nu+\beta i$. Now, we can write Rényi entropy as

$$\begin{aligned} I_R(\nu) &= (1-\nu)^{-1}\Big\{\nu\log(\beta)+\delta\nu\log(\alpha)-\nu\log(\Gamma(\delta))+\log\Big[\sum_{i=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^i(\alpha\nu)^i}{i!} \\ &\times\ b_{s,(\beta\delta+1)\nu+\beta i}\int_{-\infty}^{\infty}g^{\nu}(x;\underline{\theta})\big(G(x;\underline{\theta})\big)^{(\beta\delta-1)\nu+\beta i+s}dx\Big]\Big\} \\ &= (1-\nu)^{-1}\Big\{\nu\log(\beta)+\delta\nu\log(\alpha)-\nu\log(\Gamma(\delta)) \\ &+\ \log\Big[\sum_{i=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^i(\alpha\nu)^i}{i!}b_{s,(\beta\delta+1)\nu+\beta i}\frac{\nu^{\nu}}{[(\beta\delta-1)\nu+\beta i+s+\nu]^{\nu}} \\ &\times\ \int_{-\infty}^{\infty}\Big[\frac{(\beta\delta-1)\nu+\beta i+s+\nu}{\nu}g(x;\underline{\theta})\big(G(x;\underline{\theta})\big)^{\frac{(\beta\delta-1)\nu+\beta i+s}{\nu}}\Big]^{\nu}dx\Big]\Big\} \\ &= (1-\nu)^{-1}\Big\{\nu\log(\beta)+\delta\nu\log(\alpha)-\nu\log(\Gamma(\delta))+\log\Big[\sum_{i=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^i(\alpha\nu)^i}{i!} \\ &\times\ b_{s,(\beta\delta+1)\nu+\beta i}\frac{\nu^{\nu}}{[(\beta\delta-1)\nu+\beta i+s+\nu]^{\nu}}\times e^{(1-\nu)I_{REG}}\Big]\Big\}, \end{aligned} \tag{21}$$

where I_{REG} denotes the Rényi entropy for the exponentiated generalized distribution (Cordeiro *et al.* 2013b) with parameter $\beta^* = \frac{(\beta\delta-1)\nu+\beta i+s+\nu}{\nu}$ and $b_{s,m} = s^{-1}\sum_{l=1}^{s}\Big[m(l+1)-s\Big]b_{s-l,m}$ with $b_{0,m}=1, m=(\beta\delta+1)\nu+\beta i$. Thus, we can obtain Rényi entropy of the GWG distribution from Rényi entropy of the exponentiated generalized distribution by equation (21).

4. Estimation and observed information matrix

Estimation of the parameters of the GWG distributions can be obtained by the method of maximum likelihood. Let $X \sim GEMW(\alpha,\beta,\delta,\underline{\theta})$ and $\mathbf{\Delta} = (\alpha,\beta,\delta,\underline{\theta})^T$ be the parameter vector. The log-likelihood for $\mathbf{\Delta}$ can be written as

$$\begin{aligned} \ell=\ell(\mathbf{\Delta}) &= n\log(\beta)+n\delta\log(\alpha)-n\log[\Gamma(\delta)]-\alpha\sum_{i=1}^{n}\Big[\frac{G(x_i;\underline{\theta})}{1-G(x_i;\underline{\theta})}\Big]^{\beta}+\sum_{i=1}^{n}\log[g(x_i;\underline{\theta})] \\ &-\ (\beta\delta+1)\sum_{i=1}^{n}\log[1-G(x_i;\underline{\theta})]+(\beta\delta-1)\sum_{i=1}^{n}\log[G(x_i;\underline{\theta})]. \end{aligned} \tag{22}$$

The first derivative of the log-likelihood function with respect to the parameters $\mathbf{\Delta} = (\alpha, \beta, \delta, \underline{\theta})^T$ are given by

$$\frac{\partial \ell}{\partial \alpha} = \frac{n\delta}{\alpha} - \sum_{i=1}^{n} \left[\frac{G(x_i;\underline{\theta})}{1-G(x_i;\underline{\theta})}\right]^{\beta}, \tag{23}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \beta} &= \frac{n}{\beta} - \alpha \sum_{i=1}^{n} \left[\frac{G(x_i;\underline{\theta})}{1-G(x_i;\underline{\theta})}\right]^{\beta} \log\left[\frac{G(x_i;\underline{\theta})}{1-G(x_i;\underline{\theta})}\right] \\ &\quad -\delta \sum_{i=1}^{n} \log[1-G(x_i;\underline{\theta})] + \delta \sum_{i=1}^{n} \log[G(x_i;\underline{\theta})],\end{aligned} \tag{24}$$

$$\frac{\partial \ell}{\partial \delta} = n\log(\alpha) - \frac{n\Gamma'(\delta)}{\Gamma(\delta)} - \beta \sum_{i=1}^{n} \log[1-G(x_i;\underline{\theta})] + \beta \sum_{i=1}^{n} \log[G(x_i;\underline{\theta})], \tag{25}$$

and

$$\begin{aligned}\frac{\partial \ell}{\partial \underline{\theta}_k} &= \sum_{i=1}^{n} \frac{1}{g(x_i;\underline{\theta})} \frac{\partial g(x_i;\underline{\theta})}{\partial \underline{\theta}_k} + (\beta\delta+1) \sum_{i=1}^{n} \frac{1}{1-G(x_i;\underline{\theta})} \frac{\partial G(x_i;\underline{\theta})}{\partial \underline{\theta}_k} \\ &\quad -\alpha\beta \sum_{i=1}^{n} \frac{G(x_i;\underline{\theta})^{\beta-1}}{\left[1-G(x_i;\underline{\theta})\right]^{\beta+1}} \frac{\partial G(x_i;\underline{\theta})}{\partial \underline{\theta}_k} \\ &\quad +(\beta\delta-1) \sum_{i=1}^{n} \frac{1}{G(x_i;\underline{\theta})} \frac{\partial G(x_i;\underline{\theta})}{\partial \underline{\theta}_k},\end{aligned} \tag{26}$$

where $\underline{\theta}_k$ is the k^{th} element of the vector of parameters $\underline{\theta}$.
The partitioned observed information matrix $\hat{\mathbf{\Delta}} = (\hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\underline{\theta}})$ for the GWG distribution is

$$\mathbf{J}(\hat{\mathbf{\Delta}}) = -\begin{bmatrix} \frac{\partial^2 \ell}{\partial \alpha^2} & \frac{\partial^2 \ell}{\partial \alpha \partial \beta} & \frac{\partial^2 \ell}{\partial \alpha \partial \delta} & \frac{\partial^2 \ell}{\partial \alpha \partial \underline{\theta}} \\ \frac{\partial^2 \ell}{\partial \beta \partial \alpha} & \frac{\partial^2 \ell}{\partial \beta^2} & \frac{\partial^2 \ell}{\partial \beta \partial \delta} & \frac{\partial^2 \ell}{\partial \beta \partial \underline{\theta}} \\ \frac{\partial^2 \ell}{\partial \delta \partial \alpha} & \frac{\partial^2 \ell}{\partial \delta \partial \beta} & \frac{\partial^2 \ell}{\partial \delta^2} & \frac{\partial^2 \ell}{\partial \delta \partial \underline{\theta}} \\ \frac{\partial^2 \ell}{\partial \underline{\theta} \partial \alpha} & \frac{\partial^2 \ell}{\partial \underline{\theta} \partial \beta} & \frac{\partial^2 \ell}{\partial \underline{\theta} \partial \delta} & \frac{\partial^2 \ell}{\partial \underline{\theta}^2} \end{bmatrix},$$

which is symmetric matrix. We provide these elements as follows:

$$\frac{\partial^2 \ell}{\partial \alpha^2} = \frac{-n\delta}{\alpha^2}, \quad \frac{\partial^2 \ell}{\partial \alpha \partial \delta} = \frac{n}{\alpha},$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \beta} = -\sum_{i=1}^{n} \left[\frac{G(x_i;\underline{\theta})}{1-G(x_i;\underline{\theta})}\right]^{\beta} \log\left[\frac{G(x_i;\underline{\theta})}{1-G(x_i;\underline{\theta})}\right],$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \underline{\theta}_k} = -\beta \sum_{i=1}^{n} \frac{G(x_i;\underline{\theta})^{\beta-1}}{\left[1-G(x_i;\underline{\theta})\right]^{\beta+1}} \frac{\partial G(x_i;\underline{\theta})}{\partial \underline{\theta}_k},$$

$$\frac{\partial^2 \ell}{\partial \beta^2} = \frac{n}{\beta} - \alpha \sum_{i=1}^{n} \left[\frac{G(x_i;\underline{\theta})}{1-G(x_i;\underline{\theta})}\right]^{\beta} \left(\log\left[\frac{G(x_i;\underline{\theta})}{1-G(x_i;\underline{\theta})}\right]\right)^2,$$

$$\frac{\partial^2 \ell}{\partial \beta \partial \delta} = -\sum_{i=1}^{n} \log[1-G(x_i;\underline{\theta})] + \sum_{i=1}^{n} \log[G(x_i;\underline{\theta})],$$

$$\frac{\partial^2\ell}{\partial\beta\partial\underline{\theta}_k} = \delta\sum_{i=1}^{n}\frac{1}{1-G(x_i;\underline{\theta})}\frac{\partial G(x_i;\underline{\theta}_k)}{\partial\underline{\theta}} + \delta\sum_{i=1}^{n}\frac{1}{G(x_i;\underline{\theta})}\frac{\partial G(x_i;\underline{\theta})}{\partial\underline{\theta}_k} -\alpha\sum_{i=1}^{n}\frac{G(x_i;\underline{\theta})^{\beta-1}}{\left[1-G(x_i;\underline{\theta})\right]^{\beta+1}}\left(1+\beta\log\left[\frac{G(x_i;\underline{\theta})}{1-G(x_i;\underline{\theta})}\right]\right)\frac{\partial G(x_i;\underline{\theta})}{\partial\underline{\theta}_k},$$

$$\frac{\partial^2\ell}{\partial\delta^2} = \frac{n\left(\left[\Gamma'(\delta)\right]^2-\Gamma(\delta)\Gamma''(\delta)\right)}{\Gamma^2(\delta)},$$

$$\frac{\partial^2\ell}{\partial\delta\partial\underline{\theta}_k} = \beta\sum_{i=1}^{n}\frac{1}{1-G(x_i;\underline{\theta})}\frac{\partial G(x_i;\underline{\theta})}{\partial\underline{\theta}_k} + \beta\sum_{i=1}^{n}\frac{1}{G(x_i;\underline{\theta})}\frac{\partial G(x_i;\underline{\theta})}{\partial\underline{\theta}_k},$$

and

$$\begin{aligned}\frac{\partial^2\ell}{\partial\underline{\theta}_k\underline{\theta}_j} &= \sum_{i=1}^{n}\left(\frac{-1}{g^2(x_i;\underline{\theta})}\frac{\partial g(x_i;\underline{\theta})}{\partial\underline{\theta}_k}\frac{\partial g(x_i;\underline{\theta})}{\partial\underline{\theta}_j} + \frac{1}{g(x_i;\underline{\theta})}\frac{\partial^2 g(x_i;\underline{\theta})}{\partial\underline{\theta}_k\underline{\theta}_j}\right) + (\beta\delta+1)\\ &\times\sum_{i=1}^{n}\left(\frac{1}{[1-G(x_i;\underline{\theta})]^2}\frac{\partial G(x_i;\underline{\theta})}{\partial\underline{\theta}_k}\frac{\partial G(x_i;\underline{\theta})}{\partial\underline{\theta}_j} + \frac{1}{1-G(x_i;\underline{\theta})}\frac{\partial^2 G(x_i;\underline{\theta})}{\partial\underline{\theta}_k\underline{\theta}_j}\right)\\ &-\alpha\beta\sum_{i=1}^{n}\left(\ell_1\frac{\partial G(x_i;\underline{\theta})}{\partial\underline{\theta}_k}\frac{\partial G(x_i;\underline{\theta})}{\partial\underline{\theta}_j} + \frac{G(x_i;\underline{\theta})^{\beta-1}}{\left[1-G(x_i;\underline{\theta})\right]^{\beta+1}}\frac{\partial^2 G(x_i;\underline{\theta})}{\partial\underline{\theta}_k\underline{\theta}_j}\right)\\ &+(\beta\delta-1)\sum_{i=1}^{n}\left(\frac{1}{G^2(x_i;\underline{\theta})}\frac{\partial G(x_i;\underline{\theta})}{\partial\underline{\theta}_k}\frac{\partial G(x_i;\underline{\theta})}{\partial\underline{\theta}_j} + \frac{1}{G(x_i;\underline{\theta})}\frac{\partial^2 G(x_i;\underline{\theta})}{\partial\underline{\theta}_k\underline{\theta}_j}\right),\end{aligned}$$

where

$$\ell_1 = \frac{(\beta-1)G(x_i;\underline{\theta})^{\beta-2}\left[1-G(x_i;\underline{\theta})\right]^{\beta+1} + (\beta+1)G(x_i;\underline{\theta})^{\beta-1}\left[1-G(x_i;\underline{\theta})\right]^{\beta}}{\left[1-G(x_i;\underline{\theta})\right]^{2(\beta+1)}},$$

and $\underline{\theta}_k$ is the k^{th} element of the vector of parameters $\underline{\theta}$.

The asymptotic confidence intervals for the parameters of the GWG distribution are as follows. The expectations in the Fisher Information Matrix (FIM) can be obtained numerically. Let $\hat{\Delta} = (\hat{\alpha},\hat{\beta},\hat{\delta},\hat{\underline{\theta}})$ be the maximum likelihood estimate of $\mathbf{\Delta} = (\alpha,\beta,\delta,\underline{\theta})$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}(\hat{\mathbf{\Delta}}-\mathbf{\Delta}) \xrightarrow{d} N_p(\underline{0}, I^{-1}(\mathbf{\Delta}))$, where $I(\mathbf{\Delta})$ is the expected Fisher information matrix. The asymptotic behavior is still valid if $I(\mathbf{\Delta})$ is replaced by the observed information matrix evaluated at $\hat{\mathbf{\Delta}}$, that is $J(\hat{\mathbf{\Delta}})$. The multivariate normal distribution $N_p(\underline{0}, J(\hat{\mathbf{\Delta}})^{-1})$, where the mean vector $\underline{\mathbf{0}} = (0,0,0,\underline{0})^T$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions. That is, the approximate $100(1-\eta)\%$ two-sided confidence intervals for α, β, δ, $\underline{\theta}_k$ are given by:

$$\widehat{\alpha} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}_{\alpha\alpha}^{-1}(\widehat{\mathbf{\Delta}})},\quad \widehat{\beta} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}_{\beta\beta}^{-1}(\widehat{\mathbf{\Delta}})},\quad \widehat{\delta} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}_{\delta\delta}^{-1}(\widehat{\mathbf{\Delta}})},\quad \textit{and}\quad \widehat{\underline{\theta}_k} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}_{\underline{\theta}_k\underline{\theta}_k}^{-1}(\widehat{\mathbf{\Delta}})},$$

respectively, where $\mathbf{I}_{\alpha\alpha}^{-1}(\widehat{\mathbf{\Delta}}), \mathbf{I}_{\beta\beta}^{-1}(\widehat{\mathbf{\Delta}}), \mathbf{I}_{\delta\delta}^{-1}(\widehat{\mathbf{\Delta}})$, and $\mathbf{I}_{\underline{\theta}_k\underline{\theta}_k}^{-1}(\widehat{\mathbf{\Delta}})$, are the diagonal elements of $\mathbf{I}_n^{-1}(\widehat{\mathbf{\Delta}}) = (n\mathbf{I}(\widehat{\mathbf{\Delta}}))^{-1}$, and $Z_{\frac{\eta}{2}}$ is the upper $\frac{\eta}{2}^{th}$ percentile of a standard normal distribution.

5. The gamma-Weibull-uniform distribution

Suppose that the baseline distribution is a uniform distribution on the interval $(0,\theta)$, with $\theta > 0$. Then, $g(x;\theta) = 1/\theta$ and $G(x;\theta) = x/\theta$, where $0 < x < \theta$. The cdf and pdf of the gamma-Weibull-uniform (GWU) distribution are given by

$$F_{GWU}(x) = \frac{1}{\Gamma(\delta)}\gamma\Big[\delta, \alpha\Big(\frac{x}{\theta - x}\Big)^{\beta}\Big], \tag{27}$$

and

$$f_{GWU}(x) \quad = \quad \frac{\beta\theta\alpha^{\delta}}{\Gamma(\delta)}\frac{x^{\beta\delta-1}}{(\theta - x)^{\beta\delta+1}}\exp\Big[-\alpha\Big(\frac{x}{\theta - x}\Big)^{\beta}\Big], \tag{28}$$

respectively, where $0 < x < \theta, \alpha, \beta, \delta > 0$.

5.1. Some sub-models of the GWU distribution

When $\alpha = \beta = \delta = 1$, the GWU distribution reduces to the exponential uniform (EU(θ), EU1) distribution with pdf as follows:

$$f(x) \quad = \quad \frac{\theta}{(\theta - x)^2}\exp\Big(-\frac{x}{\theta - x}\Big).$$

By letting $\alpha = \delta = 1$, we obtain the Weibull uniform (WU(θ, β)) distribution as a special case of the GWU distribution, whose pdf is given by

$$f(x) \quad = \quad \frac{\beta\theta x^{\beta-1}}{(\theta - x)^{\beta+1}}\exp\Big[-\Big(\frac{x}{\theta - x}\Big)^{\beta}\Big].$$

We can also obtain the gamma exponential uniform (GEU(θ, δ), GEU1) distribution by letting $\alpha = \beta = 1$ with pdf given by

$$f(x) \quad = \quad \frac{\theta x^{\delta-1}}{\Gamma(\delta)(\theta - x)^{\delta+1}}\exp\Big(-\frac{x}{\theta - x}\Big).$$

When $\beta = \delta = 1$, the GWU distribution reduces to the exponential uniform (EU(θ, α), EU2) distribution with pdf given by

$$f(x) \quad = \quad \frac{\theta\alpha}{(\theta - x)^2}\exp\Big[-\alpha\Big(\frac{x}{\theta - x}\Big)\Big].$$

By setting $\delta = 1$, we obtain the Weibull uniform distribution, which is also known as the Phani distribution (Phani 1987). The pdf of Phani distribution is given by

$$f(x) \quad = \quad \frac{\beta\theta\alpha x^{\beta-1}}{(\theta - x)^{\beta+1}}\exp\Big[-\alpha\Big(\frac{x}{\theta - x}\Big)^{\beta}\Big].$$

Moreover, when $\alpha = 1$, we can obtain the gamma Weibull uniform (GWU(θ, β, δ), GWU1) distribution as a special case of the GWU distribution, whose pdf is given by

$$f(x) \quad = \quad \frac{\beta\theta}{\Gamma(\delta)}\frac{x^{\beta\delta-1}}{(\theta - x)^{\beta\delta+1}}\exp\Big[-\Big(\frac{x}{\theta - x}\Big)^{\beta}\Big].$$

We can obtain the gamma exponential uniform (GWU(θ, α, δ), GU (Torabi and Montazeri 2012)) distribution by setting $\beta = 1$ and the corresponding pdf is given by

$$f(x) \quad = \quad \frac{\theta\alpha^{\delta}}{\Gamma(\delta)}\frac{x^{\delta-1}}{(\theta - x)^{\delta+1}}\exp\Big[-\alpha\Big(\frac{x}{\theta - x}\Big)\Big].$$

When $\beta = 2$ and $\beta = 2, \delta = 1$, we can obtain gamma Rayleigh uniform (GRU) distribution and Rayleigh uniform (RU) distribution, and the corresponding pdf are given by

$$f(x) \quad = \quad \frac{2\theta\alpha^{\delta}}{\Gamma(\delta)}\frac{x^{2\delta-1}}{(\theta - x)^{2\delta+1}}\exp\Big[-\alpha\Big(\frac{x}{\theta - x}\Big)^{2}\Big],$$

and

$$f(x) \quad = \quad \frac{2\theta\alpha x}{(\theta-x)^3}\exp\Big[-\alpha\Big(\frac{x}{\theta-x}\Big)^2\Big],$$

respectively.

5.2. Expansion of the GWU density

From equation (9), we can obtain

$$f_{GWG}(x) \quad = \quad \sum_{i,j=0}^{\infty}\omega_{i,j}h_{\beta(\delta+i)+j}(x;\theta), \tag{29}$$

where

$$\omega_{i,j} = \frac{(-1)^i\beta\alpha^{\delta+i}\Gamma[\beta(\delta+i)+j+1]}{i!j!\Gamma(\delta)\Gamma[\beta(\delta+i)+1][\beta(\delta+i)+j]}, \tag{30}$$

and

$$h_{\beta(\delta+i)+j}(x;\theta) = \big(\beta(\delta+i)+j\big)\frac{1}{\theta}\Big(\frac{x}{\theta}\Big)^{\beta(\delta+i)+j-1}$$

denotes pdf of the exponentiated uniform (EU) distribution with power $\beta(\delta+i)+j$. Thus, we can obtain statistical properties of the GWU distribution from those of EU distribution.

5.3. Hazard, reverse hazard and quantile functions

The hazard and reverse hazard functions of the GWU distribution are given by

$$h_{GWU}(x) \quad = \quad \frac{\beta\theta\alpha^\delta\frac{x^{\beta\delta-1}}{(\theta-x)^{\beta\delta+1}}\exp\big[-\alpha\big(\frac{x}{\theta-x}\big)^\beta\big]}{\Gamma(\delta)-\gamma\big[\delta,\alpha\big(\frac{x}{\theta-x}\big)^\beta\big]}, \tag{31}$$

and

$$\tau_{GWU}(x) \quad = \quad \frac{\beta\theta\alpha^\delta\frac{x^{\beta\delta-1}}{(\theta-x)^{\beta\delta+1}}\exp\big[-\alpha\big(\frac{x}{\theta-x}\big)^\beta\big]}{\gamma\big[\delta,\alpha\big(\frac{x}{\theta-x}\big)^\beta\big]}, \tag{32}$$

respectively. In addition, we can also obtain the quantile function of GWU distribution from equation (14) and (15) as

$$x_u = q\times\theta = \frac{\theta\Big[\gamma^{-1}\big(\delta,u\Gamma(\delta)\big)\Big]^{\frac{1}{\beta}}}{\Big[\gamma^{-1}\big(\delta,u\Gamma(\delta)\big)\Big]^{\frac{1}{\beta}}+\alpha^{\frac{1}{\beta}}}. \tag{33}$$

Several plots of pdf and hrf of the GWU distribution for selected parameter values are given by Figure 1 and 2. The graphs of the pdf take various types, including unimodal, decreasing, left and right skewed. As we can see from Figure 2, the plot for hrf of the GWU distribution contains decreasing, increasing, bathtub, unimodal and upside down bathtub shapes for the selected values of the GWU parameters.

5.4. Moments, incomplete moments, moment generating and characteristic functions

Let $X\sim GWU(\alpha,\beta,\delta,\theta), 0<x<\theta,\alpha,\beta,\delta>0$, the s^{th} moment of X can be obtained from (29) as

$$E(X^s) = \sum_{i,j=0}^{\infty}\omega_{i,j}E(Z_{i,j}^s),$$

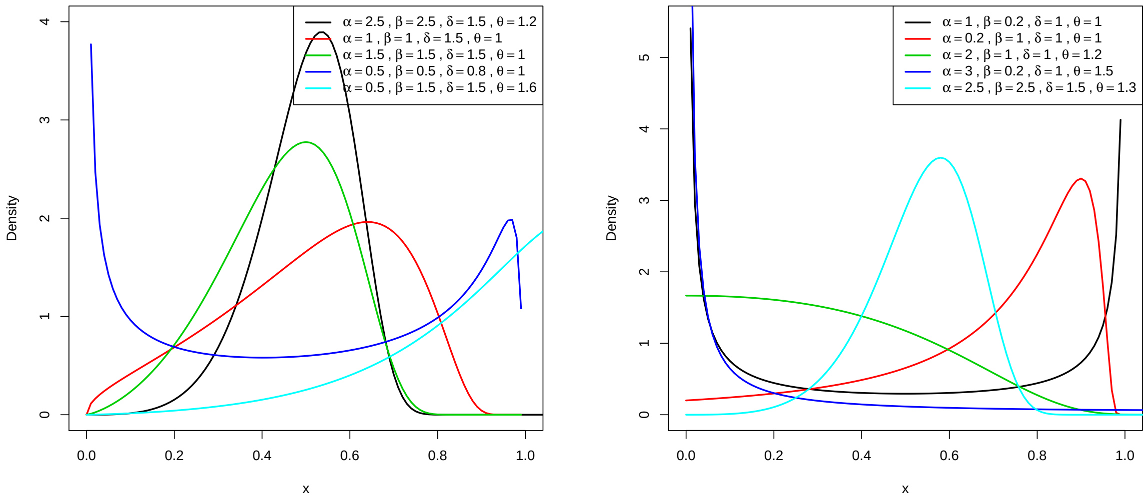


Figure 1: Graphs of GWU pdf with selected parameters

where $E(Z_{i,j}^s)$ denotes the s^{th} moment of $Z_{i,j}$, which follows the EU distribution with power $\beta(\delta+i)+j$ and $\omega_{i,j}$ is defined as (30). Similarly, the incomplete moments and mgf are given by

$$I_X(y) = \int_0^y x^s f_{GWG}(x)dx = \sum_{i,j=0}^{\infty} \omega_{i,j} I_{i,j}(y),$$

where $I_{i,j}(y) = \int_0^y x^s h_{\beta(\delta+i)+j}(x,\theta)dx$ and

$$M_X(t) = \sum_{i,j=0}^{\infty} \omega_{i,j} E(e^{tZ_{i,j}}),$$

where $E(e^{tZ_{i,j}})$ is the mgf of the EU distribution with power $\beta^* = \beta(\delta+i)+j$ and $\omega_{i,j}$ is defined as (30). The characteristic function is given by $\phi(t) = E(e^{itX})$, where $i = \sqrt{-1}$. Thus, we have

$$\phi(t) = \sum_{i,j=0}^{\infty} \omega_{i,j} \phi_{\beta(\delta+i)+j}(t),$$

where $\phi_{\beta(\delta+i)+j}(t)$ denotes the characteristic function of the EU distribution with power $\beta(\delta+i)+j$, and $\omega_{i,j}$ is defined as (30).

The first six moments, standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS), and coefficient of kurtosis (CK) for selected values of the parameters of the *GWU* distribution are listed in Tables 2 below. The variance (σ^2), CV, CS and CK are given by

$$\sigma^2 = \mu_2' - \mu^2, \quad CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu_2' - \mu^2}}{\mu} = \sqrt{\frac{\mu_2'}{\mu^2} - 1},$$

$$CS = \frac{E\left[(X-\mu)^3\right]}{[E(X-\mu)^2]^{3/2}} = \frac{\mu_3' - 3\mu\mu_2' + 2\mu^3}{(\mu_2' - \mu^2)^{3/2}},$$

and

$$CK = \frac{E\left[(X-\mu)^4\right]}{[E(X-\mu)^2]^2} = \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4}{(\mu_2' - \mu^2)^2},$$

respectively. The graphs of CS and CK versus δ show the dependence of skewness and kurtosis measures on the shape parameter.

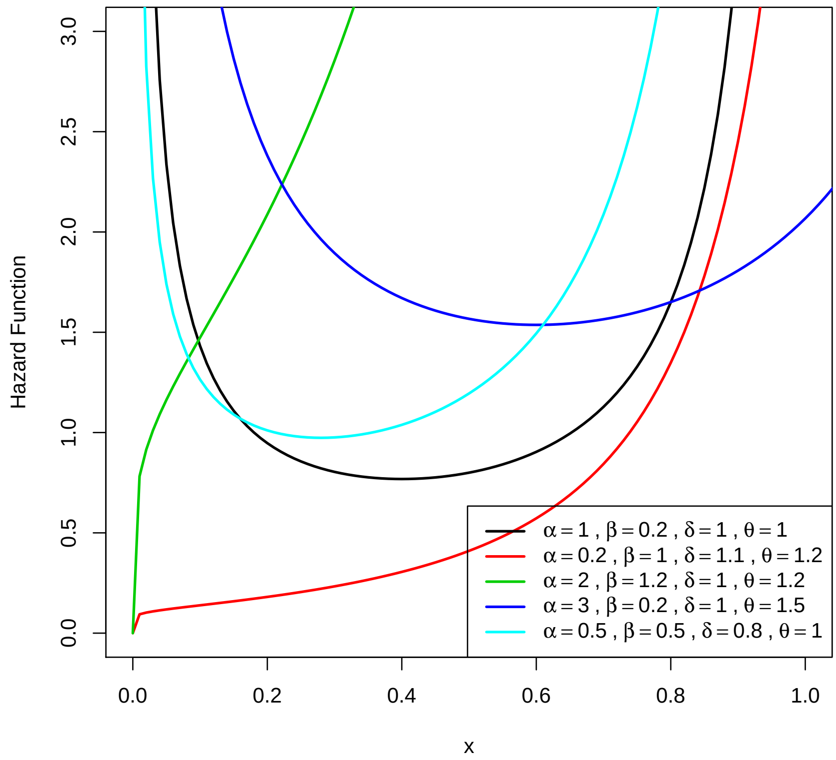


Figure 2: Graphs of GWU hrf with selected parameters

Table 2: GWU moments for selected values

	$(\alpha,\beta,\delta,\theta)$			
Moments	(5.0,3.0,2.0,1.0)	(1.0,1.0,1.0,1.0)	(0.5,1.5,1.5,1.6)	(1.0,1.0,1.5,1.5)
$E(X)$	0.4042	0.4037	0.9802	0.7736
$E(X^2)$	0.1673	0.2110	1.0133	0.6865
$E(X^3)$	0.0706	0.1237	1.0852	0.6560
$E(X^4)$	0.0303	0.0778	1.1930	0.6578
$E(X^5)$	0.0132	0.0512	1.3386	0.6831
$E(X^6)$	0.0058	0.0350	1.5273	0.7286
SD	0.0620	0.2191	0.2290	0.2967
CV	0.1534	0.5429	0.2336	0.3835
CS	-0.5254	-0.0171	-0.9034	-0.4321
CK	3.3243	1.9815	3.6357	2.3932

Moreover, the conditional expectations of the GWU distribution can also be obtained from the EU (Cordeiro *et al.* 2013b) distribution with power $\beta(\delta+i)+j$, and we present it as follow:

$$E(X^t \mid X > x) = \frac{\Gamma(\delta)}{\Gamma(\delta) - \gamma\left\{\delta, \alpha\left(\frac{x}{\theta - x}\right)^{\beta}\right\}} \times \sum_{i,j=0}^{\infty} \omega_{i,j} I_{i,j}(y),$$

where $I_{i,j}(y) = \int_0^y x^s h_{\beta(\delta+i)+j}(x,\theta)dx$, and $\omega_{i,j}$ is defined as (30).

5.5. Mean deviation, Bonferroni and Lorenz curves

Let $X \sim GWU(\alpha,\beta,\delta,\theta), 0 < x < \theta, \alpha,\beta,\delta > 0$, $\mu = E(X)$ be the mean and $M =$

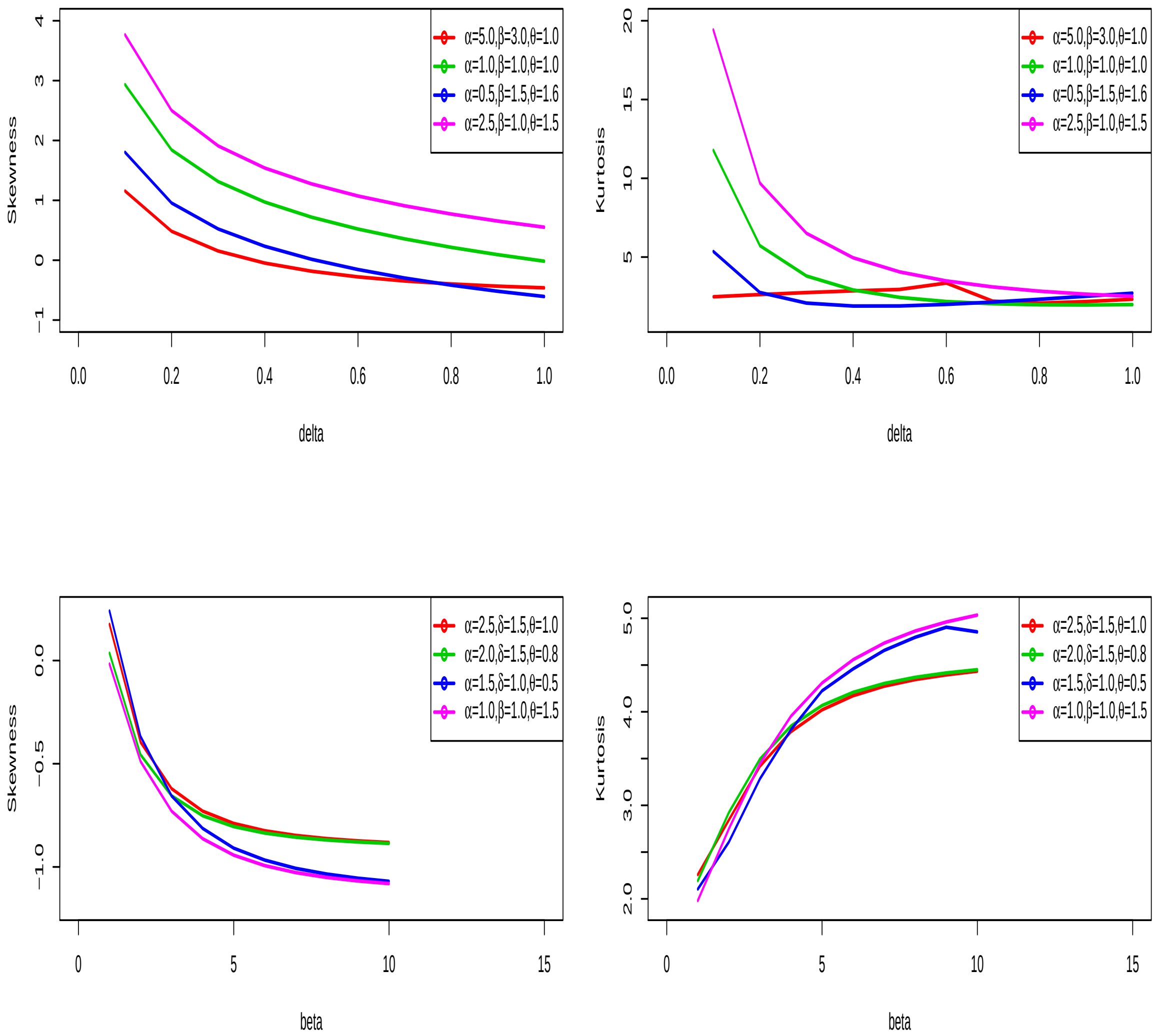


Figure 3: Graphs of GWU skewness and kurtosis with selected parameters

$Median(X)$ the median. Then the mean deviation about the mean and the mean deviation about the median can be obtained from

$$\delta_1(X) = 2\mu F_{GWU}(\mu) - 2\int_0^{\mu} x f_{GWU}(x)dx, \quad \text{and} \quad \delta_2(X) = \mu - 2\int_0^{M} x f_{GWU}(x)dx,$$

respectively, where $m(z) = \int_0^z x f_{GWU}(x)dx$ is the first incomplete moment. Bonferroni and Lorenz curves are given by

$$B(p) = \frac{1}{p\mu}\int_0^q x f(x)dx = \frac{1}{p\mu}\sum_{i,j=0}^{\infty} \omega_{i,j} I_{\beta(\delta+i)+j}(i,j),$$

and

$$L(p) = \frac{1}{\mu}\int_0^q x f(x)dx = \frac{1}{\mu}\sum_{i,j=0}^{\infty} \omega_{i,j} I_{\beta(\delta+i)+j}(i,j),$$

respectively, where $I_{\beta(\delta+i)+j}(i,j) = \int_0^q x f_{\beta(\delta+i)+j}(x,\theta)dx$, is the first incomplete moment of the EU distribution with power $\beta(\delta+i)+j$ and $\omega_{i,j}$ is defined as (30).

5.6. Order statistics

The distribution of the i^{th} order statistic for the GWU distribution is presented in this section. Let $X_1, \ldots, X_n$ be i.i.d random variables distributed according to (29). The pdf of the i^{th} order statistic, for example $X_{i:n}$, is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\sum_{q=0}^{n-i}\sum_{m=0}^{\infty}\binom{n-i}{q}\frac{(-1)^q d_{m,q+i-1}\Gamma(\delta i+\delta q+m)}{[\Gamma(\delta)]^{i+k}} f_{GWU}(x;\alpha,\beta,\delta_*),$$

that is, the pdf of the i^{th} order statistic is a linear combination of GWU densities with parameters $(\alpha,\beta,\delta_*,\theta)$, where $\delta_* = \delta i+\delta q+m$. This is a very useful result since properties of the order statistics of the GWU distribution can be obtained from those of EU distribution. For instance, we can obtain

$$E(X_{i:n}^t) = \sum_{l,j=0}^{\infty}\sum_{q=0}^{n-i}\sum_{m=0}^{\infty}\frac{n!\omega_{i,j}}{(i-1)!(n-i)!}\binom{n-i}{q}\frac{(-1)^q d_{m,q+i-1}\Gamma(\delta i+\delta q+m)}{[\Gamma(\delta)]^{i+q}} E(Z_{l,j}(x;\alpha,\beta,\delta_*)^t),$$

where $E(Z_{l,j}(x;\alpha,\beta,\delta_*)^t)$ is the t^{th} moment of EU distribution with power $\beta(\delta_*+i)+j$, where $\delta_* = \delta i+\delta q+m$ and $\omega_{i,j}$ is given by equation (30).

5.7. Rényi entropy

By equation (21), Rényi entropy for the GWU distribution is given by

$$I_R(\nu) = (1-\nu)^{-1}\Bigg\{\nu\log(\beta)+\delta\nu\log(\alpha)-\nu\log(\Gamma(\delta))+\log\Bigg[\sum_{i=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^i(\alpha\nu)^i}{i!} \times b_{s,(\beta\delta+1)\nu+\beta i}\frac{\nu^{\nu}}{[(\beta\delta-1)\nu+\beta i+s+\nu]^{\nu}}\times e^{(1-\nu)I_{REU}}\Bigg]\Bigg\}, \quad (34)$$

where I_{REU} denotes the Rényi entropy for the exponentiated uniform (EU) distribution with parameter $\beta^* = \frac{(\beta\delta-1)\nu+\beta i+s+\nu}{\nu}$ and $b_{s,m} = s^{-1}\sum_{l=1}^{s}\Big[m(l+1)-s\Big]b_{s-l,m}$ with $b_{0,m}=1, m = (\beta\delta+1)\nu+\beta i$.

6. The gamma-Weibull-Weibull distribution

Suppose that the baseline distribution is the Weibull distribution with parameters λ and k. Then, $g(x;\lambda,k)=\frac{k}{\lambda}(\frac{x}{\lambda})^{k-1}e^{-(\frac{x}{\lambda})^k}$ and $G(x;\lambda,k)=1-e^{-(\frac{x}{\lambda})^k}$, where $x\geq 0$. The cdf and pdf of gamma-Weibull-Weibull (GWW) distribution are given by

$$F_{GWW}(x)=\frac{1}{\Gamma(\delta)}\gamma\Big[\delta,\alpha\big(e^{(\frac{x}{\lambda})^k}-1\big)^{\beta}\Big],$$

and

$$f_{GWW}(x) \;=\; \frac{\beta\alpha^{\delta}}{\Gamma(\delta)}\times\frac{kx^{k-1}e^{(\frac{x}{\lambda})^k}}{\lambda^k}\big[e^{(\frac{x}{\lambda})^k}-1\big]^{\beta\delta-1}\exp\Big[-\alpha\big(e^{(\frac{x}{\lambda})^k}-1\big)^{\beta}\Big],$$

respectively, where $x\geq 0,\alpha,\beta,\delta,\lambda,k>0$.

Graphs of the GWW density functions with several combinations of parameter values are given by Figure 4, which shows different shapes including negative skew, positive skew, monotonically decreasing and almost symmetric bell-shape.

6.1. Some sub-models of the GWW distribution

When $\alpha=\beta=\delta=k=1$, the GWW distribution reduces to the exponential exponential (EE) distribution with pdf as follows:

$$f(x) \;=\; \frac{1}{\lambda}\times\exp\Big(1+\frac{x}{\lambda}-e^{\frac{x}{\lambda}}\Big).$$

By letting $\alpha=\beta=\delta=1$, we obtain the exponential power (EP) distribution (Smith and Bain 1975) as a special case of the GWW distribution, whose pdf is given by

$$f(x) \;=\; \frac{kx^{k-1}e^{(\frac{x}{\lambda})^k}}{\lambda^k}\exp\Big[1-e^{(\frac{x}{\lambda})^k}\Big].$$

We can also obtain Chen distribution (Chen 2000) by letting $\beta=\delta=\lambda=1$ with pdf given by

$$f(x) \;=\; \alpha kx^{k-1}e^{x^k}\exp\Big[-\alpha\big(e^{x^k}-1\big)\Big].$$

When $\beta=\delta=k=1$, the GWW distribution reduces to Gompertz distribution (Gompertz 1895) with pdf given by

$$f(x) \;=\; \frac{\alpha e^{\frac{x}{\lambda}}}{\lambda}\exp\Big[-\alpha\big(e^{\frac{x}{\lambda}}-1\big)\Big].$$

By setting $\alpha=\beta=1$, we obtain the gamma exponential power (GEP) distribution. The pdf of GEP distribution is given by

$$f(x) \;=\; \frac{1}{\Gamma(\delta)}\times\frac{kx^{k-1}e^{(\frac{x}{\lambda})^k}}{\lambda^k}\big[e^{(\frac{x}{\lambda})^k}-1\big]^{\delta-1}\exp\Big[1-e^{(\frac{x}{\lambda})^k}\Big].$$

Moreover, when $\alpha=\delta=1$, we obtain the exponentiated exponential power (EEP) distribution as a special case of the GWW distribution, whose pdf is given by

$$f(x) \;=\; \frac{\beta kx^{k-1}e^{(\frac{x}{\lambda})^k}}{\lambda^k}\big[e^{(\frac{x}{\lambda})^k}-1\big]^{\beta-1}\exp\Big[-\big(e^{(\frac{x}{\lambda})^k}-1\big)^{\beta}\Big].$$

We also obtain the gamma exponentiated exponential power (GEEP) distribution by setting $\alpha=1$ and the corresponding pdf is given by

$$f(x) \;=\; \frac{\beta}{\Gamma(\delta)}\times\frac{kx^{k-1}e^{(\frac{x}{\lambda})^k}}{\lambda^k}\big[e^{(\frac{x}{\lambda})^k}-1\big]^{\beta\delta-1}\exp\Big[-\big(e^{(\frac{x}{\lambda})^k}-1\big)^{\beta}\Big].$$

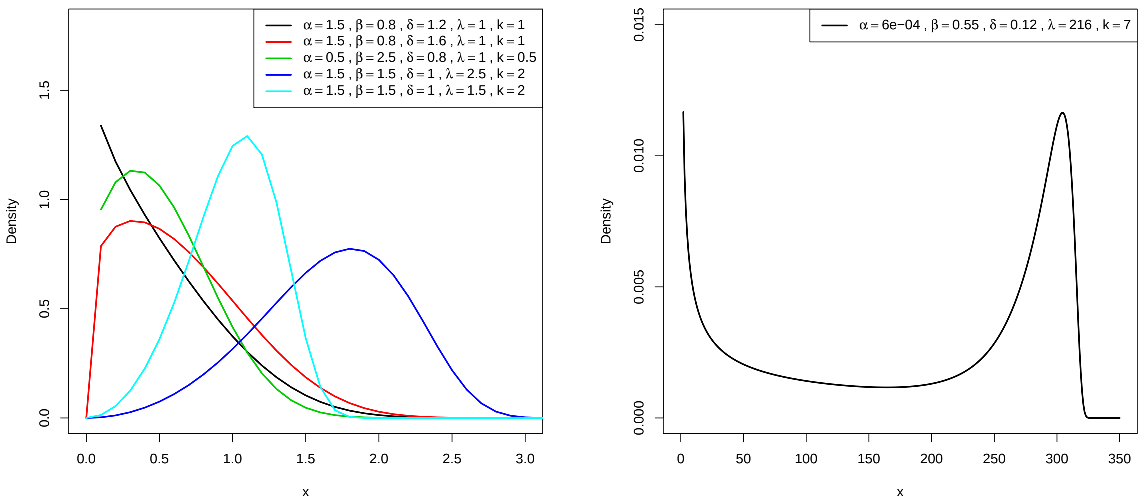


Figure 4: Graphs of GWW pdf with selected parameters. Note that, the focus and originality of this paper is that the GWG family of distributions are capable of generating bathtub-shaped probability density functions (pdf). Because of its importance and in a different scale, we are presenting a bathtub-shaped pdf in a separated plot (right panel) as above.

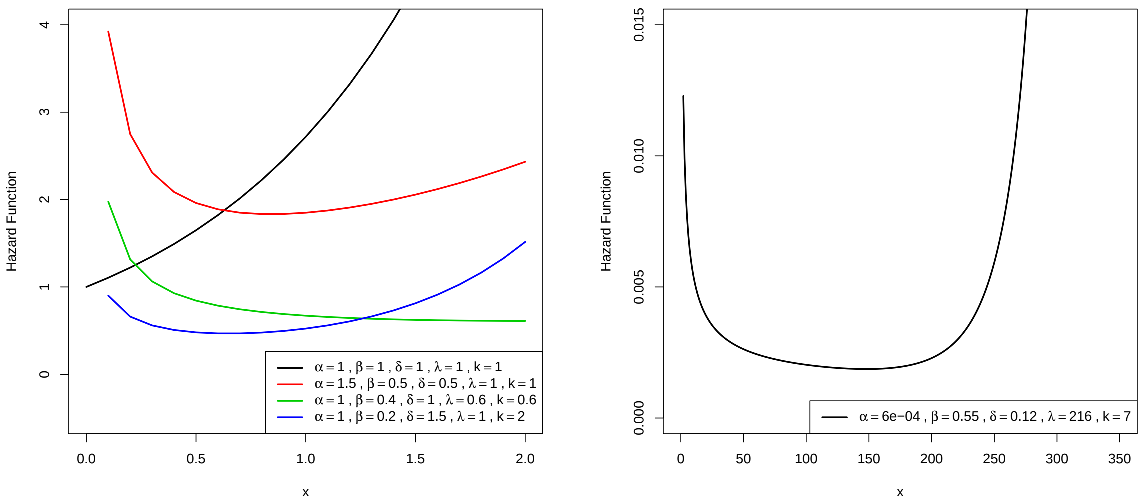


Figure 5: Graphs of GWW hazard rate functions (hrf) with selected parameters. Similar with pdf of GWW, we are presenting a bathtub-shaped hrf separately in the right panel.

6.2. Expansion of the GWW density

From equation (9), we can obtain

$$f_{GWG}(x) \quad = \quad \sum_{i,j=0}^{\infty} \omega_{i,j} h_{\beta(\delta+i)+j}(x;\lambda,k), \tag{35}$$

where

$$\omega_{i,j} = \frac{(-1)^i \beta \alpha^{\delta+i} \Gamma[\beta(\delta+i)+j+1]}{i!j!\Gamma(\delta)\Gamma[\beta(\delta+i)+1][\beta(\delta+i)+j]}, \tag{36}$$

and

$$h_{\beta(\delta+i)+j}(x;\lambda,k) = \big[\beta(\delta+i)+j\big]\frac{k}{\lambda}\Big(\frac{x}{\lambda}\Big)^{k-1} e^{-(\frac{x}{\lambda})^k}\big[1-e^{-(\frac{x}{\lambda})^k}\big]^{\beta(\delta+i)+j-1}$$

denotes pdf of the exponentiated Weibull (EW) (Nassar and Eissa 2003) distribution with power $\beta(\delta+i)+j$. Therefore, we can obtain the statistical properties of the GWW distribution from those of the EW distribution.

6.3. Hazard, reverse hazard and quantile functions

The hazard and reverse hazard functions of the GWW distribution can be obtained as

$$h_{GWW}(x) \quad = \quad \frac{\beta\alpha^{\delta}\frac{kx^{k-1}e^{(\frac{x}{\lambda})^k}}{\lambda^k}\big[e^{(\frac{x}{\lambda})^k}-1\big]^{\beta\delta-1}\exp\Big[-\alpha\big(e^{(\frac{x}{\lambda})^k}-1\big)^{\beta}\Big]}{\Gamma(\delta)-\gamma\Big[\delta,\alpha\big(e^{(\frac{x}{\lambda})^k}-1\big)^{\beta}\Big]},$$

and

$$\tau_{GWW}(x) \quad = \quad \frac{\beta\alpha^{\delta}\frac{kx^{k-1}e^{(\frac{x}{\lambda})^k}}{\lambda^k}\big[e^{(\frac{x}{\lambda})^k}-1\big]^{\beta\delta-1}\exp\Big[-\alpha\big(e^{(\frac{x}{\lambda})^k}-1\big)^{\beta}\Big]}{\gamma\Big[\delta,\alpha\big(e^{(\frac{x}{\lambda})^k}-1\big)^{\beta}\Big]},$$

respectively. The graphs of the hrf for selected parameters are given in Figure 5. The plots show various shapes including monotonically decreasing, monotonically increasing, and bathtub shapes for several combinations of parameters values. This attractive flexibility makes the GWW hazard function be useful and suitable for monotonic and non-monotonic empirical hazard behaviors which are more likely to be encountered in real life situations.

In addition, we can also obtain the quantile function of GWW distribution from equation (14) and (15) as

$$x_u \quad = \quad \lambda\big[-\log\left(1-q\right)\big]^{\frac{1}{k}} = \lambda\Bigg[-\log\Bigg(1-\frac{\Big[\gamma^{-1}\big(\delta,u\Gamma(\delta)\big)\Big]^{\frac{1}{\beta}}}{\Big[\gamma^{-1}\big(\delta,u\Gamma(\delta)\big)\Big]^{\frac{1}{\beta}}+\alpha^{\frac{1}{\beta}}}\Bigg)\Bigg]^{\frac{1}{k}}.$$

6.4. Moments, incomplete moments, moment generating and characteristic functions

Let $X \sim GWW(\alpha,\beta,\delta,\lambda,k), x \geq 0, \alpha,\beta,\delta,\lambda,k > 0$, the s^{th} moment of X can be obtained from (35) as

$$E(X^s) = \sum_{i,j=0}^{\infty} \omega_{i,j} E(Z_{i,j}^s),$$

where $E(Z_{i,j}^s)$ denotes the s^{th} moment of $Z_{i,j}$, which follows the EW distribution (Nassar and Eissa 2003), with power $\beta(\delta+i)+j$ and $\omega_{i,j}$ is defined as (36). The s^{th} moment of $Z_{i,j}$ is given by (Pal, Ali, and Woo 2006) and we present it as follows:

$$E(Z_{i,j}^s) = \beta(\delta+i)+j]\lambda^s \Gamma\big(\frac{s}{k}+1\big) \sum_{t=0}^{\beta(\delta+i)+j-1} \binom{\beta(\delta+i)+j-1}{i} (-1)^t (t+1)^{\frac{s}{k}-1},$$

if $(\beta(\delta+i)+j) \in N$,

$$E(Z_{i,j}^s) = [\beta(\delta+i)+j]\lambda^s \Gamma\big(\frac{s}{k}+1\big) \sum_{t=0}^{\beta(\delta+i)+j-1} \frac{P_t^{\beta(\delta+i)+j-1}}{t!} (-1)^t (t+1)^{\frac{s}{k}-1},$$

if $(\beta(\delta+i)+j) \notin N$, for $k=0,1,2,...$, where

$$P_t^\alpha = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-t+1),\ for\ any\ \alpha \notin N.$$

Similarly, the incomplete moments and mgf can be obtained as

$$I_X(y) = \int_0^y x^s f_{GWW}(x)dx = \sum_{i,j=0}^{\infty} \omega_{i,j} I_{i,j}(y),$$

where $I_{i,j}(y) = \int_0^y x^s h_{\beta(\delta+i)+j}(x,\underline{\theta})dx$ and $M_X(t) = \sum_{i,j=0}^{\infty} \omega_{i,j} E(e^{tZ_{i,j}})$, where $E(e^{tZ_{i,j}})$ is the mgf of the EW (Cordeiro *et al.* 2013b) distribution with power $\beta(\delta+i)+j$ and $\omega_{i,j}$ is defined as (36). The characteristic function is given by $\phi(t) = E(e^{itX})$, where $i=\sqrt{-1}$. Thus, the characteristic function is given by

$$\phi(t) = \sum_{i,j=0}^{\infty} \omega_{i,j} \phi_{\beta(\delta+i)+j}(t),$$

where $\phi_{\beta(\delta+i)+j}(t)$ denotes the characteristic function of the EW distribution with power $\beta(\delta+i)+j$ and $\omega_{i,j}$ is defined as (36).

In addition, the conditional expectations of the GWW distribution can also be obtained from the EW (Nassar and Eissa 2003) distribution as follow:

$$E(X^t \mid X > x) = \frac{\Gamma(\delta)}{\Gamma(\delta) - \gamma\Big\{\delta, \alpha\Big[e^{(\frac{x}{\lambda})^k} - 1\Big]^\beta\Big\}} \times \sum_{i,j=0}^{\infty} \omega_{i,j} I_{i,j}(y),$$

where $I_{i,j}(y) = \int_0^y x^t h_{\beta(\delta+i)+j}(x,\theta)dx$ is the t^{th} incomplete moment of EW and $\omega_{i,j}$ is defined as (36).

The first six moments, SD, CV, CS, and CK for different selected values of the parameters of the GWW distribution are listed in the Tables 3 below. The graphs of CS and CK versus the shape parameters show the dependence of skewness and kurtosis measures on these parameters, β and δ. Also, plots of CS and CK versus the shape parameter k can be readily obtained.

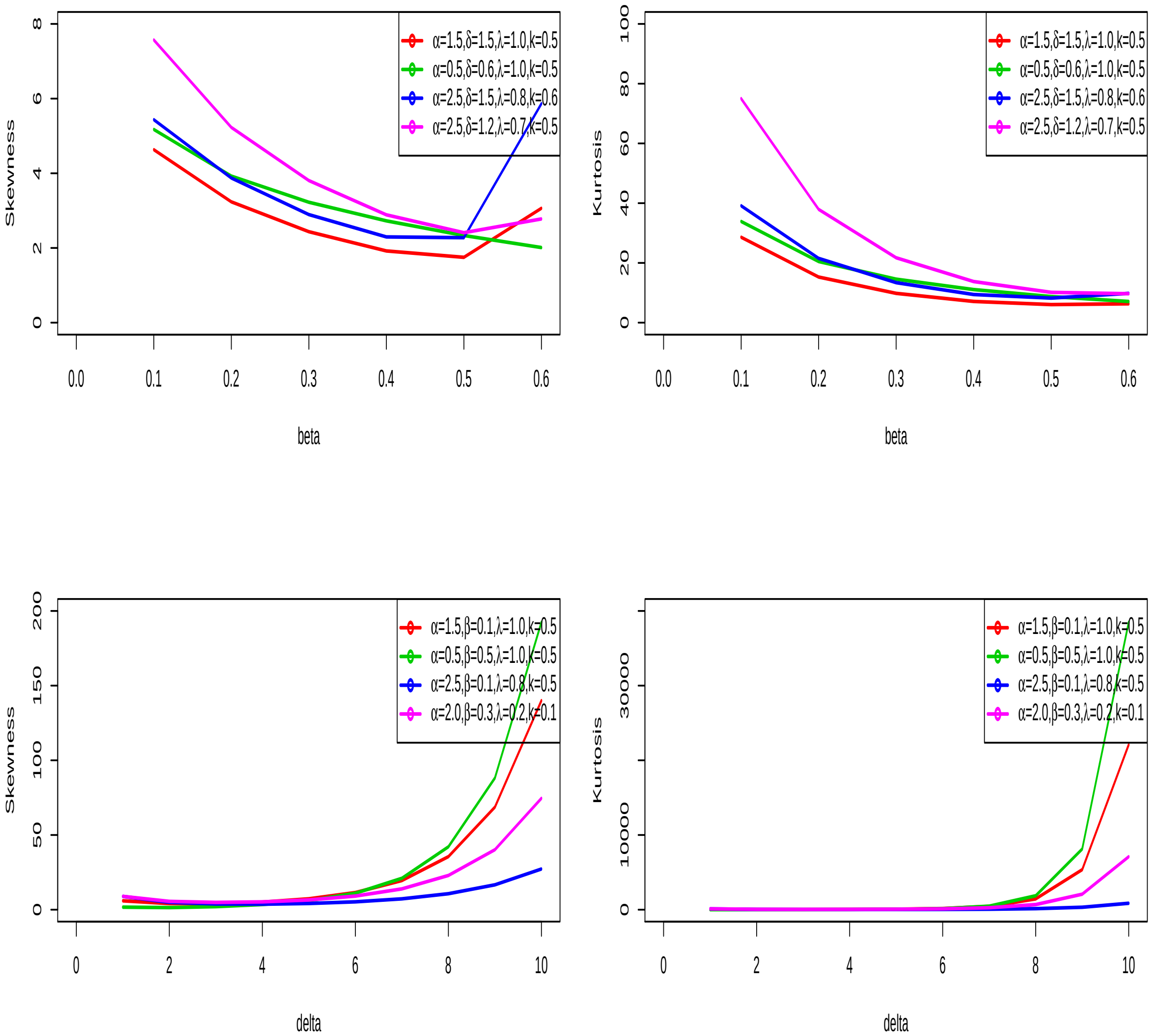


Figure 6: Graphs of GWW skewness and kurtosis with selected parameters

Table 3: GWW moments for selected values

	$(\alpha, \beta, \delta, \lambda, k)$			
Moments	(1.5,0.8,1.5,1.0,0.5)	(0.5,2.5,0.6,1.0, 0.5)	(2.5,1.0,1.5,0.8,0.6)	(0.5,3.0,0.8,0.8, 0.8)
$E(X)$	0.5911	0.4297	0.2391	0.5010
$E(X^2)$	0.8997	0.2921	0.1137	0.2869
$E(X^3)$	2.0769	0.2486	0.0770	0.1792
$E(X^4)$	6.2139	0.2450	0.0658	0.1193
$E(X^5)$	22.3739	0.2691	0.0666	0.0836
$E(X^6)$	92.9078	0.3221	0.0770	0.0610
SD	0.7419	0.3278	0.2377	0.1892
CV	1.2551	0.7628	0.9937	0.3777
CS	2.1910	0.8710	1.6992	-0.0662
CK	9.3210	3.3866	6.6864	2.5279

6.5. Mean deviation, Bonferroni and Lorenz curves

Let $X \sim GWW(\alpha, \beta, \delta, \lambda, k), x \geq 0, \alpha, \beta, \delta, \lambda, k > 0,\ \mu = E(X)$ be the mean and $M = Median(X)$ be the median. Then the mean deviation about the mean and the mean deviation about the median can be obtained as

$$\delta_1(X) = 2\mu F_{GWW}(\mu) - 2\int_0^{\mu} x f_{GWW}(x)dx, \quad \text{and} \quad \delta_2(X) = \mu - 2\int_0^{M} x f_{GWW}(x)dx,$$

respectively, and $m(z) = \int_0^z x f_{GWW}(x)dx$ is the first incomplete moment. Bonferroni and Lorenz curves are given by

$$B(p) = \frac{1}{p\mu}\int_0^q x f(x)dx = \frac{1}{p\mu}\sum_{i,j=0}^{\infty} \omega_{i,j} I_{\beta(\delta+i)+j}(i,j),$$

and

$$L(p) = \frac{1}{\mu}\int_0^q x f(x)dx = \frac{1}{\mu}\sum_{i,j=0}^{\infty} \omega_{i,j} I_{\beta(\delta+i)+j}(i,j),$$

respectively, where $I_{\beta(\delta+i)+j}(i,j) = \int_0^q x f_{\beta(\delta+i)+j}(x;\lambda,k)dx$, is the first incomplete moment of the EW (Nassar and Eissa 2003) distribution with power $\beta(\delta+i)+j$ and $\omega_{i,j}$ is given by (36).

6.6. Order statistics

The distribution of the i^{th} order statistic for the GWW distribution is presented in this section. Let $X_1, \ldots, X_n$ be i.i.d random variables distributed according to (35). The pdf of the i^{th} order statistics, for example $X_{i:n}$, is given by

$$f_{i:n}(x) \;=\; \frac{n!}{(i-1)!(n-i)!}\sum_{q=0}^{n-i}\sum_{m=0}^{\infty}\binom{n-i}{q}\frac{(-1)^q d_{m,q+i-1}\Gamma(\delta i+\delta q+m)}{[\Gamma(\delta)]^{i+k}} f_{GWW}(x;\alpha,\beta,\delta_*),$$

that is, the pdf of the i^{th} order statistic is a linear combination of GWW densities with parameters $(\alpha, \beta, \delta_*, \lambda, k)$, where $\delta_* = \delta i + \delta q + m$.

Therefore, properties of the order statistics of the GWW distribution can be obtained from those of EW (Nassar and Eissa 2003) distribution. For instance, we can obtain the the t^{th} moment of the i^{th} order statistics and is given by

$$\begin{aligned} E(X_{i:n}^t) \;&=\; \sum_{l,j=0}^{\infty}\sum_{q=0}^{n-i}\sum_{m=0}^{\infty}\frac{n!\omega_{i,j}}{(i-1)!(n-i)!}\binom{n-i}{q}\frac{(-1)^q d_{m,q+i-1}\Gamma(\delta i+\delta q+m)}{[\Gamma(\delta)]^{i+q}} \\ &\times\; E(Z_{l,j}(x;\alpha,\beta,\delta_*)^t), \end{aligned}$$

where $E(Z_{l,j}(x;\alpha,\beta,\delta_*)^t)$ is the t^{th} moment of EW distribution with power $\beta(\delta_*+i)+j$, where $\delta_* = \delta i + \delta q + m$ and $\omega_{i,j}$ is defined as (36).

6.7. Rényi entropy

By equation (21), we obtain Rényi entropy for the GWW distribution as

$$\begin{aligned} I_R(\nu) \;&=\; (1-\nu)^{-1}\Big\{\nu\log(\beta) + \delta\nu\log(\alpha) - \nu\log(\Gamma(\delta)) + \log\Big[\sum_{i=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^i(\alpha\nu)^i}{i!} \\ &\times\; b_{s,(\beta\delta+1)\nu+\beta i}\frac{\nu^{\nu}}{[(\beta\delta-1)\nu+\beta i+s+\nu]^{\nu}} \times e^{(1-\nu)I_{REW}}\Big]\Big\}, \end{aligned} \tag{37}$$

where I_{REW} denotes the Rényi entropy for the exponentiated Weibull distribution (Nassar and Eissa 2003) with parameter $\beta^* = \frac{(\beta\delta-1)\nu+\beta i+s+\nu}{\nu}$ and $b_{s,m} = s^{-1}\sum_{l=1}^{s}\Big[m(l+1)-s\Big]b_{s-l,m}$ with $b_{0,m} = 1, m = (\beta\delta+1)\nu+\beta i$.

7. Monte Carlo simulation study

In this section, a simulation study is conducted to assess the performance and examine the mean estimate, average bias, root mean square error of the maximum likelihood estimators for each parameter for both GWU and GWW distributions. We study the performance of the GWU and GWW distributions by conducting various simulations for different parameter values. The simulation study is repeated for $N = 5000$ times each with sample size $n = 25, 50, 75, 100, 200, 400, 800, 1600, 3200$ and parameter values $I : \alpha = 0.5, \beta = 1.5, \delta = 1.5, \theta = 1.6, II : \alpha = 1, \beta = 1, \delta = 1.5, \theta = 1, III : \alpha = 2.5, \beta = 1.5, \delta = 0.4, \theta = 0.5, IV : \alpha = 1.2, \beta = 1.2, \delta = 0.8, \theta = 1.5$, for the GWU distribution and $I : \alpha = 0.8, \beta = 1.6, \delta = 0.3, \lambda = 0.4, k = 1, II : \alpha = 0.9, \beta = 1.2, \delta = 0.5, \lambda = 0.6, k = 1.2, III : \alpha = 1.0, \beta = 1.0, \delta = 0.8, \lambda = 1.0, k = 1.5, IV : \alpha = 0.5, \beta = 0.8, \delta = 0.5, \lambda = 0.8, k = 0.5$, for the GWW distribution, respectively.

Tables 4, 5 and 6, 7 lists the means MLEs of the model parameters along with the respective average bias and root mean squared errors (RMSE) for the GWU and GWW distributions, respectively. From the results, we can verify that as the sample size n increases, the mean estimates of the parameters tend to be closer to the true parameter values, since RMSEs decay toward zero. The bias and RMSEs are given by:

$$Bias(\hat{\theta}) = \frac{\sum_{i=1}^{n}\hat{\theta}_i}{n} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{n}(\hat{\theta}_i-\theta)^2}{n}},$$

respectively.

8. Applications

In this section, we present two examples to illustrate the flexibility of the GWU and GWW distributions and their sub-models for data modeling.

The first data consists of the lifetimes of $n = 50$ devices given by Aarset data (Aarset 1987). It is known to have a bathtub-shaped hazard function, thus been widely studied and is given by: 0.1 0.2 1.0 1.0 1.0 1.0 1.0 2.0 3.0 6.0 7.0 11.0 12.0 18.0 18.0 18.0 18.0 18.0 21.0 32.0 36.0 40.0 45.0 46.0 47.0 50.0 55.0 60.0 63.0 63.0 67.0 67.0 67.0 67.0 72.0 75.0 79.0 82.0 82.0 83.0 84.0 84.0 84.0 85.0 85.0 85.0 85.0 85.0 86.0 86.0.

The second data gives failure and running times of a sample of $n = 30$ devices given by Meeker and Escobar (Meeker and Escobar 1998) and this data also has a bathtub shaped hazard function and is given by: 2 10 13 23 23 28 30 65 80 88 106 143 147 173 181 212 245 247 261 266 275 293 300 300 300 300 300 300 300 300 .

Some descriptive statistics of these two data sets are given in Table 8.

The maximum likelihood estimates (MLEs) of the GWU and GWW parameters α, β, δ, and $\underline{\theta}$ are computed by maximizing the objective function via the subroutine NLMIXED in SAS. The estimated values of the parameters (standard error in parenthesis), -2log-likelihood statistic, Akaike Information Criterion, $AIC = 2p - 2\ln(L)$, Bayesian Information Criterion, $BIC = p\ln(n) - 2\ln(L)$, and Consistent Akaike Information Criterion, $AICC = AIC + 2\frac{p(p+1)}{n-p-1}$, where $L = L(\hat{\mathbf{\Delta}})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters, and Kolmogorov-Smirnov (KS) statistic ($KS = \max_{1\le i\le n}\{G_{GWG}(x_i) - \frac{i-1}{n}, \frac{i}{n} - G_{GWG}(x_i)\}$) are presented in Table 9 and 10. In order to compare the models, we use the criteria stated above. Note that for the value of the log-likelihood function at its maximum (ℓ_n), larger value is good and preferred,

Table 4: Monte Carlo simulation results for GWU distribution: mean estimate, average bias, and RMSE

		I			II		
Parameter	n	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
α	25	1.2113	0.7139	3.1626	1.7216	0.7216	3.5384
	50	1.0893	0.5893	2.3287	1.5490	0.5490	2.6918
	75	0.8740	0.3740	1.7007	1.4273	0.4273	2.2506
	100	0.8345	0.3345	1.4842	1.2994	0.2994	1.8730
	200	0.6818	0.1818	0.9301	1.1211	0.1211	1.3929
	400	0.5626	0.0626	0.5420	1.0271	0.0271	1.1389
	800	0.5132	0.0132	0.3916	0.9001	-0.0999	0.7646
	1600	0.4848	-0.0152	0.2803	0.8124	-0.1876	0.4785
	3200	0.4687	-0.0313	0.2021	0.7909	-0.2091	0.2800
β	25	3.0323	1.5323	2.0056	2.3244	1.3244	1.6974
	50	2.8732	1.3732	2.0006	2.1701	1.1701	1.6716
	75	2.7714	1.2714	1.9362	2.0924	1.0924	1.6628
	100	2.7162	1.2162	1.9034	2.0632	1.0632	1.6511
	200	2.6277	1.1277	1.8255	1.8542	0.8542	1.4436
	400	2.4775	0.9775	1.6645	1.7263	0.7263	1.3110
	800	2.3439	0.8439	1.5235	1.5232	0.5232	1.0124
	1600	2.1536	0.6536	1.2958	1.3518	0.3518	0.6608
	3200	2.0095	0.5095	1.1140	1.2358	0.2358	0.3515
δ	25	1.0949	-0.4051	1.5079	1.0751	-0.4249	1.7584
	50	1.3423	-0.1577	1.7580	1.2335	-0.2665	1.7850
	75	1.3513	-0.1487	1.6424	1.2455	-0.2545	1.6163
	100	1.3641	-0.1340	1.5612	1.1761	-0.3239	1.2890
	200	1.2844	-0.2156	1.1153	1.1413	-0.3587	0.9076
	400	1.2375	-0.2625	0.7970	1.1190	-0.3810	0.7008
	800	1.2370	-0.2630	0.6618	1.1364	-0.3636	0.5612
	1600	1.2675	-0.2325	0.5496	1.1605	-0.3395	0.4537
	3200	1.2930	-0.2070	0.4623	1.1992	-0.3008	0.3670
θ	25	1.8684	0.2684	0.4773	1.2085	0.2085	0.3601
	50	1.8534	0.2534	0.4296	1.1904	0.1904	0.3158
	75	1.8234	0.2234	0.3772	1.1796	0.1796	0.2930
	100	1.8216	0.2216	0.3691	1.1728	0.1728	0.2834
	200	1.8091	0.2091	0.3484	1.1405	0.1405	0.2474
	400	1.7791	0.1791	0.3130	1.1198	0.1198	0.2244
	800	1.7534	0.1534	0.2823	1.0840	0.0840	0.1717
	1600	1.7190	0.1190	0.2419	1.0546	0.0546	0.1102
	3200	1.6922	0.0922	0.2086	1.0350	0.0350	0.0555

Table 5: Monte Carlo simulation results for GWU distribution: mean estimate, average bias, and RMSE

		III			IV		
Parameter	n	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
α	25	3.7320	1.2320	5.2669	3.1961	1.9961	5.5193
	50	3.8880	1.3880	4.8103	3.0213	1.8213	4.5538
	75	3.6292	1.1292	4.2663	2.6633	1.4633	3.4775
	100	3.6382	1.1382	4.2407	2.5454	1.3454	3.3138
	200	3.3322	0.8322	3.5639	2.1108	0.9108	2.3186
	400	3.2340	0.7340	3.3193	1.7823	0.5823	1.6210
	800	3.0544	0.5544	2.7256	1.5654	0.3654	1.2227
	1600	2.9853	0.4853	2.2392	1.3730	0.1730	0.6680
	3200	2.9198	0.4198	1.8403	1.2753	0.0753	0.3757
β	25	1.5649	0.0649	1.0530	1.7431	0.5431	1.0095
	50	1.3337	-0.1663	0.8907	1.5984	0.3984	1.0152
	75	1.2332	-0.2668	0.7684	1.5318	0.3318	0.9746
	100	1.2347	-0.2653	0.7381	1.4812	0.2812	0.9184
	200	1.2450	-0.2550	0.6390	1.4129	0.2129	0.8179
	400	1.3000	-0.2000	0.5450	1.3378	0.1378	0.6711
	800	1.3560	-0.1440	0.4534	1.2818	0.0818	0.5374
	1600	1.4206	-0.0794	0.3687	1.2569	0.0569	0.3955
	3200	1.4728	-0.0272	0.2983	1.2285	0.0285	0.2700
δ	25	0.8864	0.4864	1.5351	1.1253	0.3253	1.8677
	50	0.9760	0.5760	1.5304	1.2626	0.4626	1.8562
	75	0.8957	0.4957	1.1348	1.2134	0.4134	1.5772
	100	0.8127	0.4127	0.8361	1.1591	0.3591	1.2916
	200	0.6743	0.2743	0.4993	1.0007	0.2007	0.7172
	400	0.5714	0.1714	0.3164	0.9299	0.1299	0.4838
	800	0.5078	0.1078	0.2136	0.8887	0.0887	0.3575
	1600	0.4586	0.0586	0.1388	0.8436	0.0436	0.2551
	3200	0.4279	0.0279	0.0942	0.8238	0.0238	0.1859
θ	25	0.4319	-0.0681	0.2054	1.7041	0.2041	0.6352
	50	0.4428	-0.0572	0.1887	1.6819	0.1819	0.5482
	75	0.4388	-0.0612	0.1788	1.6539	0.1539	0.4784
	100	0.4469	-0.0531	0.1756	1.6482	0.1482	0.4626
	200	0.4528	-0.0472	0.1567	1.6139	0.1139	0.3977
	400	0.4672	-0.0328	0.1413	1.5799	0.0799	0.3236
	800	0.4783	-0.0217	0.1209	1.5487	0.0487	0.2582
	1600	0.4916	-0.0084	0.1037	1.5318	0.0318	0.1835
	3200	0.5012	0.0012	0.0859	1.5150	0.0150	0.1209

Table 6: Monte Carlo simulation results for GWW distribution: mean estimate, average bias, and RMSE

		I			II		
Parameter	n	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
α	25	1.4111	1.6111	2.0556	1.2597	0.3597	2.0332
	50	1.3657	0.5657	1.9041	1.8731	0.9731	2.3628
	75	1.3151	0.5151	1.7345	1.8999	0.9999	2.2978
	100	1.2305	0.4305	1.6311	1.8312	0.9312	2.1015
	200	1.0764	0.2764	1.3353	1.6426	0.7426	1.7880
	400	0.9667	0.1667	1.0912	1.4320	0.5320	1.4453
	800	0.8323	0.0323	0.8472	1.3082	0.4082	1.2432
	1600	0.8298	0.0298	0.7923	1.1277	0.2277	0.9012
	3200	0.8283	0.0283	0.7736	1.0537	0.1537	0.7300
β	25	1.5221	-0.0779	1.0899	0.9488	-0.2512	1.0695
	50	1.5254	-0.0746	1.2401	1.1865	-0.0135	1.0872
	75	1.5237	-0.0763	1.3291	1.1436	-0.0564	1.0509
	100	1.5279	-0.0721	1.2235	1.1340	-0.0660	1.0689
	200	1.5322	-0.0678	1.2059	1.0849	-0.1151	0.9382
	400	1.5774	-0.0226	1.1630	1.0811	-0.1189	0.8791
	800	1.5734	-0.0266	1.0250	1.0590	-0.1410	0.7351
	1600	1.5838	-0.0162	0.9319	1.0521	-0.1479	0.6227
	3200	1.5438	-0.0562	0.6773	1.0517	-0.1483	0.4862
δ	25	0.7776	0.4776	1.3797	0.8674	0.3674	1.5471
	50	0.9079	0.6079	1.5213	1.4396	0.9396	2.0780
	75	0.9202	0.6202	1.4821	1.4847	0.9847	2.0539
	100	0.8887	0.5887	1.4373	1.4782	0.9782	2.0212
	200	0.8051	0.5051	1.2080	1.4481	0.9481	1.9149
	400	0.7035	0.4035	0.9875	1.3136	0.8136	1.6746
	800	0.5588	0.2588	0.6596	1.2048	0.7048	1.5002
	1600	0.4572	0.1572	0.4272	1.0225	0.5225	1.1389
	3200	0.3928	0.0928	0.2306	0.8521	0.3521	0.7663
λ	25	0.3232	-0.0768	0.2692	0.3881	-0.2119	0.4425
	50	0.3003	-0.0997	0.2543	0.5002	-0.0998	0.3366
	75	0.2979	-0.1021	0.2551	0.5003	-0.0997	0.3436
	100	0.2909	-0.1091	0.2485	0.4953	-0.1047	0.3337
	200	0.2880	-0.1119	0.2517	0.4680	-0.1320	0.3255
	400	0.3032	-0.0968	0.2421	0.4617	-0.1383	0.3143
	800	0.3116	-0.0884	0.2253	0.4649	-0.1351	0.3021
	1600	0.3385	-0.0615	0.2100	0.4704	-0.1296	0.2719
	3200	0.3496	-0.0504	0.1891	0.4952	-0.1048	0.2500
k	25	1.3177	0.3177	0.9108	1.3263	0.1263	1.3839
	50	1.1464	0.1464	0.6429	1.5864	0.3864	1.0178
	75	1.0753	0.0753	0.5442	1.4788	0.2788	0.8604
	100	1.0128	0.0128	0.4760	1.4348	0.2348	0.7814
	200	0.9259	-0.0741	0.3826	1.2766	0.0766	0.5867
	400	0.8912	-0.1088	0.3273	1.1838	-0.0162	0.4632
	800	0.8785	-0.1215	0.2823	1.1273	-0.0727	0.3594
	1600	0.9030	-0.0970	0.2442	1.1089	-0.0911	0.2926
	3200	0.9250	-0.0750	0.1962	1.1245	-0.0755	0.2342

Table 7: Monte Carlo simulation results for GWW distribution: mean estimate, average bias, and RMSE

		III			IV		
Parameter	n	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
α	25	0.7654	-0.2346	1.7530	1.5137	1.0137	2.2141
	50	2.2911	1.2911	2.7660	1.6133	1.1133	2.1858
	75	2.2837	1.2837	2.5529	1.6818	1.1818	2.1972
	100	2.2296	1.2296	2.4651	1.6428	1.1428	2.1245
	200	1.9894	0.9894	2.0170	1.4936	0.9994	1.8406
	400	1.8281	0.8281	1.7085	1.2771	0.7771	1.4535
	800	1.6887	0.6887	1.4409	1.0323	0.5323	1.0468
	1600	1.6107	0.6107	1.2653	0.7962	0.2962	0.6006
	3200	1.4834	0.4834	1.0017	0.6491	0.1491	0.2891
β	25	0.4491	-0.5509	1.0208	0.8165	0.0165	0.8797
	50	0.9048	-0.0952	0.9894	0.7427	-0.0573	0.8801
	75	0.8385	-0.1615	0.8989	0.6953	-0.1047	0.8374
	100	0.8427	-0.1573	0.8459	0.6918	-0.1082	0.8448
	200	0.8445	-0.1555	0.7775	0.6691	-0.1309	0.7426
	400	0.8337	-0.1663	0.6693	0.6458	-0.1542	0.6458
	800	0.8233	-0.1767	0.5808	0.6562	-0.1438	0.5133
	1600	0.7954	-0.2046	0.5142	0.6921	-0.1079	0.3945
	3200	0.7884	-0.2116	0.4255	0.7244	-0.0756	0.3123
δ	25	0.6440	-0.1560	1.4363	1.4641	0.9641	2.0198
	50	1.9927	1.1927	2.4202	1.6718	1.1718	2.1626
	75	2.0496	1.2496	2.3840	1.7934	1.2934	2.2456
	100	2.0282	1.2282	2.3704	1.7924	1.2924	2.2401
	200	1.9370	1.1370	2.2145	1.7205	1.2205	2.0941
	400	1.9110	1.1110	2.1317	1.5382	1.0382	1.7831
	800	1.8432	1.0432	2.0047	1.2565	0.7565	1.3622
	1600	1.8388	1.0388	1.9322	0.9572	0.4572	0.8627
	3200	1.6646	0.8646	1.6220	0.7516	0.2516	0.4760
λ	25	0.3752	-0.6248	0.8168	0.7368	-0.0632	0.9234
	50	0.8789	-0.1211	0.4086	0.6406	-0.1594	0.8107
	75	0.8689	-0.1311	0.3924	0.5883	-0.2117	0.7975
	100	0.8729	-0.1271	0.3811	0.5814	-0.2186	0.7938
	200	0.8539	-0.1461	0.3682	0.5414	-0.2586	0.7352
	400	0.8330	-0.1670	0.3677	0.5471	-0.2529	0.7351
	800	0.8241	-0.1759	0.3657	0.5901	-0.2099	0.6990
	1600	0.8079	-0.1921	0.3652	0.6731	-0.1269	0.6978
	3200	0.8201	-0.1799	0.3542	0.7212	-0.0788	0.6526
k	25	1.1718	-0.3282	1.9945	0.7013	0.2013	0.4864
	50	2.4561	0.9561	1.8343	0.5924	0.0924	0.3084
	75	2.2968	0.7968	1.5723	0.5536	0.0536	0.2489
	100	2.1698	0.6698	1.3819	0.5310	0.0310	0.2161
	200	1.8856	0.3856	1.0197	0.4885	-0.0115	0.1609
	400	1.6780	0.1780	0.7698	0.4650	-0.0350	0.1248
	800	1.5614	0.0614	0.5897	0.4592	-0.0408	0.1013
	1600	1.4836	-0.0164	0.4575	0.4621	-0.0379	0.0805
	3200	1.4661	-0.0339	0.3724	0.4694	-0.0306	0.0639

Table 8: Descriptive statistics of application data sets

Data	n	Mean	Median	Minimum	Maximum	Variance	SD
Aarset	50	45.686	48.5	0.1	86.0	1078.2	32.8352
Meeker	30	177.03	196.5	2	300	13223	114.9913

Table 9: GWU estimation for Aarset data

Model	α	β	δ	θ	-2 Log Likelihood	AIC	BIC	AICC	KS	SS
EU(θ)	1	1	1	118.59	462.3	464.3	466.2	464.4	0.1962	0.4998
	-	-	-	(5.2776)						
WU(θ,β)	1	0.2745	1	86.1605	423.8	427.8	431.6	428.0	0.2404	1.1142
	-	(0.03494)	-	(0.1979)						
GEU(θ,δ)	1	1	0.6870	128.72	453.3	457.3	461.1	457.5	0.1948	0.4963
	-	-	(0.09336)	(8.8515)						
EU(θ,α)	0.7038	1	1	108.12	461.5	465.5	469.3	465.8	0.2298	0.5739
	(0.2671)	-	-	(9.9097)						
Phani	0.5455	0.3478	1	86.1591	410.3	416.3	422.0	416.8	0.1104	0.0743
	(0.09907)	(0.04773)	-	(0.1684)						
GWU(θ,β,δ)	1	0.2684	1.5248	86.1002	408.8	414.8	420.6	415.4	0.1121	0.0754
	-	(0.03266)	(0.1487)	(0.1220)						
GU(θ,α,δ)	0.02046	1	0.2750	86.7543	415.8	421.8	427.5	422.3	0.1553	0.1952
	(0.01124)	-	(0.05022)	(0.4914)						
GWU	44.0462	0.03578	45.3699	86.0003	398.2	406.3	413.9	407.1	0.0645	0.0336
	(0.00713)	(0.000897)	(0.1289)	(0.0144)						
BW	k	λ	a	b						
	0.9653	2.0997	0.4511	0.04673	477.5	485.5	486.4	493.2	0.1516	0.4494
	(0.02915)	(0.2233)	(0.1527)	(0.006841)						
GMW	β	θ	λ	δ						
	0.003631	0.4785	0.03249	2.7121	453.3	461.3	462.2	468.9	0.1275	0.2723
	(0.01222)	(0.2023)	(0.01365)	(2.2044)						

and for the Kolmogorov-Smirnov test statistic (K-S), smaller value is preferred. The GWU distribution is fitted to the Aarset data (Aarset 1987) set and these fits are compared to the fits using the EU(θ), WU(θ,β), GEU(θ,δ), EU(θ,α), Phani, GWU(θ,β,δ), GEU(θ,α,δ) and GWU distributions. Similarly, the GWW distribution is also fitted to the Meeker and Escobar (Meeker and Escobar 1998) data set and these fits are compared to the fits using the EE, EP, Chen, Gompertz, GEP, EEP and GEEP distributions. The GWW and GWU distributions was compared with the non-nested beta exponentiated Weibull (BEW) (Cordeiro, Gomes, da Silva, and Ortega 2013a), and beta Weibull (BW) distributions as well. The pdf of the BEW distribution is given by

$$g(x) = \frac{\alpha k \lambda^k}{B(a,b)} x^{k-1} e^{(\lambda x)^k} (1 - e^{(\lambda x)^k})^{a\alpha - 1} [1 - (1 - e^{(\lambda x)^k})^{\alpha}]^{b-1}, \quad x > 0. \tag{38}$$

When $\alpha = 1$, we have the BW distribution. We also compared the GWW and GWU distributions with the gamma generalized modified Weibull (GGMW) (Oluyede, Huang, and Yang (2015)) and gamma modified Weibull (GMW) distributions, respectively. The pdf of the GGMW distribution (Oluyede *et al.* 2015) is given by

$$g_{GGMW}(x) \;=\; \frac{1}{\Gamma(\delta)}[-\log(1 - e^{-\alpha x - \beta x^{\theta} e^{\lambda x}})]^{\delta - 1} (\alpha + \beta x^{\theta - 1} e^{\lambda x}[\theta + \lambda x]) e^{-\alpha x - \beta x^{\theta} e^{\lambda x}}. \tag{39}$$

When $\alpha = 0$, we have the GMW distribution.

We can use the likelihood ratio (LR) test to compare the fit of the GWU and GWW distribution with their sub-models for a given data set. For example, to test $\lambda = \delta = 1$, for the GWW distribution,the LR statistic is $\omega = 2[\ln(L(\hat{\alpha}, \hat{\beta}, \hat{k}, \hat{\lambda}, \hat{\delta})) - \ln(L(\tilde{\alpha}, \tilde{\beta}, \tilde{k}, 1, 1))]$, where $\hat{\alpha}$, $\hat{\beta}$, $\hat{\lambda}$, $\hat{k}$ and $\hat{\delta}$, are the unrestricted estimates, and $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{k}$ are the restricted estimates. The LR test rejects the null hypothesis if $\omega > \chi^2_{\epsilon}$, where χ^2_{ϵ} denote the upper $100\epsilon\%$ point of the χ^2 distribution with 2 degrees of freedom.

We also computed a measure of closeness of each plot to the diagonal line. This measure of closeness (Chambers, Cleveland, Kleiner, and Tukey 1983) is given by the sum of squares (SS)

$$SS = \sum_{j=1}^{n} \left[G_{GWG}(x_{(j)}; \hat{\alpha}, \hat{\beta}, \hat{\delta}, \underline{\hat{\theta}}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2,$$

which is also listed in Table 9 and Table 10.

The 95% confidence intervals for the GWU model (Aarset Data) parameters are: $\alpha \in (44.032, 44.06)$, $\beta \in (0.034022, 0.037538)$, $\delta \in (45.1159, 45.6239)$, $\theta \in (85.972, 86.029)$.

The 95% confidence intervals for the GWW model (Meeker and Escobar Data) parameters are: $\alpha \in (0.0003, 0.0009)$, $\beta \in (0.4986, 0.5874)$, $\delta \in (0.0275, 0.2207)$, $k \in (7.2786, 8.1472)$, $\lambda \in (199.0110, 235.4490)$.

Table 10: GWW estimation for Meeker and Escobar data

Model	α	β	δ	k	λ	-2 Log Likelihood	AIC	BIC	AICC	KS	SS
EE	1	1	1	1	274.87	362.1	364.1	365.6	364.2	0.2294	0.3300
	-	-	-	-	(31.2195)						
EP	1	1	1	1.1028	274.71	361.8	365.8	368.6	366.2	0.2236	0.3206
	-	-	-	(0.1902)	(28.2535)						
Chen	0.005091	1	1	0.3125	1	362.0	366.0	368.8	366.5	0.2059	0.2773
	(0.003423)	-	-	(0.02054)	-						
Gompertz	0.2496	1	1	1	135.09	359.0	363.0	365.8	363.4	0.1889	0.2966
	(0.1795)	-	-	-	(40.6757)						
GEP	1	1	0.1359	6.9362	341.45	350.9	356.9	361.1	357.8	0.2422	0.3943
	-	-	(0.02541)	(0.1927)	(20.2945)						
EEP	1	0.1427	1	7.5321	243.50	341.8	347.8	352.0	348.7	0.2145	0.3033
	-	-	(6.8530)	(0.4399)	(1.4851)						
GEEP	1	13.1977	0.01035	6.9053	324.38	339.8	347.8	353.4	349.4	0.1616	0.1953
	-	(1.6059)	(0.008449)	(0.1905)	(6.5791)						
GWW	0.000610	0.5430	0.1241	7.7129	217.23	326.1	336.1	343.1	338.6	0.1345	0.1404
	(0.000158)	(0.02264)	(0.04927)	(0.2216)	(9.2954)						
BEW	k	λ	α	a	b						
	0.9895	8.0706	0.7834	0.9181	0.0426	371.2	381.2	383.7	388.2	0.1614	0.2904
	(0.03881)	(1.1182)	(0.2485)	(0.5873)	(0.00799)						
GGMW	α	β	θ	λ	δ						
	0.05354	0.004011	0.004549	0.02772	0.06625	345.2	355.2	357.7	362.2	0.1481	0.1885
	(0.00392)	(0.002771)	(0.0893)	(0.002629)	(0.01203)						

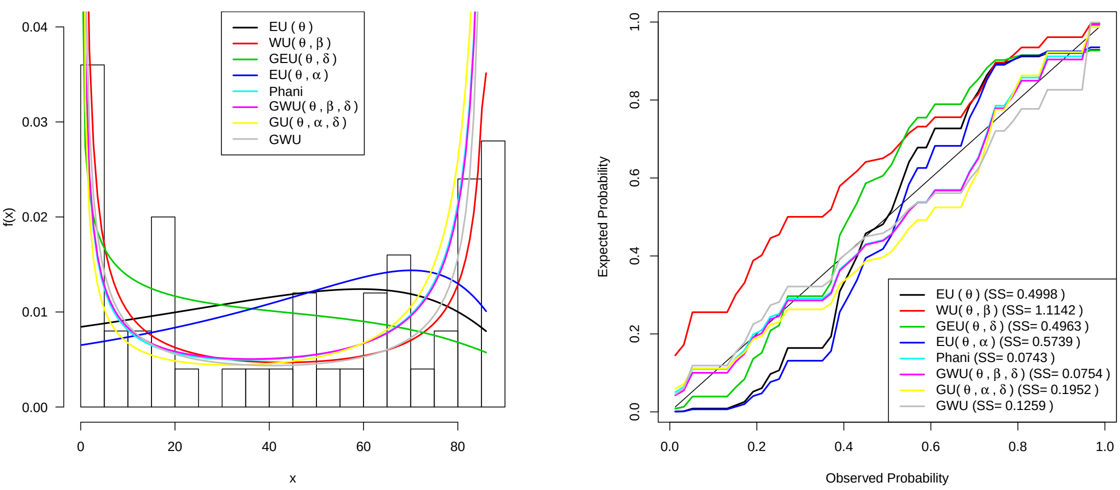
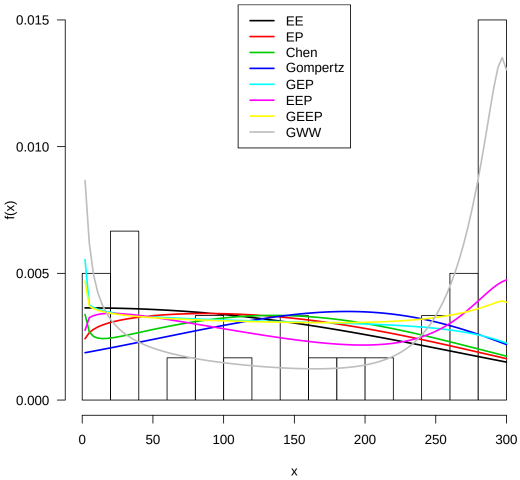


Figure 7: Fitted pdf and observed probabilities of the GWU distribution for Aarset data

Plots of the fitted densities, the histogram of these data and probability plots (Chambers *et al.* 1983) are presented in Figure 7 and Figure 8. For the probability plot, we plotted $G_{GWG}(x_{(j)}; \hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\underline{\theta}})$ against $\dfrac{j-0.375}{n+0.25}$, $j = 1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data.

For the Aarset data, the LR test statistic of the hypothesis H_0: Phani against H_a: GWU and H_0: GEU(θ, α, δ) against H_a: GWU are $\omega_1 = 12.00$ with p-value$= 5.32 \times 10^{-4}$ and $\omega_2 = 17.5$ with p-value$= 2.87 \times 10^{-5}$. Therefore, we reject H_0 in favor of H_a and conclude that the GWU distribution is significant better than the WU and GEPU distributions. Moreover, the values of the statistics AIC, AICC and BIC show that model GWU is a "better" fit for this data. The value of the goodness-of-fit statistic KS is smallest for the GWU distribution. Also, the value of SS for GWU given by the probability plot is smallest for this model. Consequently, the GWU distribution is a "better"fit when compared to the nested distributions and non-nested four parameter GMW and BW distributions.

Similarly, we can also conduct the LR test for the Meeker and Escobar data for the hypothesis H_0: GEP against H_a: GWW and H_0: GEEP against H_a: GWW. The LR test statistic of these hypothesis are $\omega_3 = 24.8$ with p-value$= 4.12 \times 10^{-6}$ and $\omega_4 = 13.7$ with p-value$= 2.14 \times 10^{-4}$, which implies that we should reject H_0 in favor of H_a and conclude that the GWW distribution is a significant better fit for the Meeker and Escobar data. In addition, the values of the statistics AIC, AICC and BIC clearly show that model GWW is a "better"fit for this data.



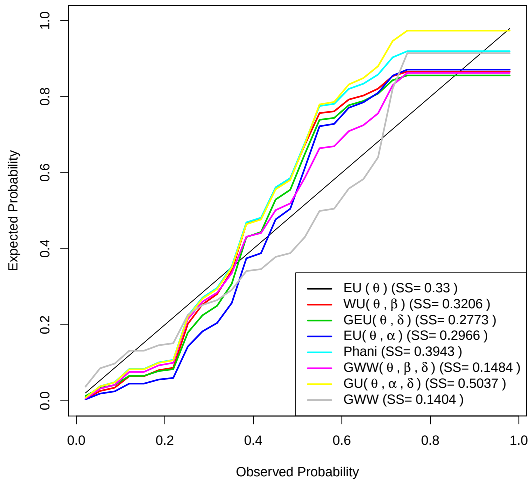


Figure 8: Fitted pdf and observed probabilities of the GWW distribution for Meeker and Escobar data

Furthermore, the value of the statistic KS as well as the value of SS from the probability plot for GWW distribution are the smallest. Clearly the GWW distribution is by far a "better"fit when compared to the nested distributions and the non-nested GGMW and BEW distributions.

9. Concluding remarks

In this paper, the gamma-Weibull-G family of distributions was introduced. This paper also contains the statistical properties of the GWG family such as expansion of density function, hazard and reverse hazard functions, quantile function, moments, incomplete moments, generating functions, mean deviations, Bonferroni and Lorenz curves, order statistics and maximum likelihood estimation for parameters as well as the observed information matrix of the GWG family of distributions. We also discussed two special cases of the GWG family of distributions, namely the gamma-Weibull-uniform and gamma-Weibull-Weibull distributions in detail along with applications of these two special cases to real life data.

Acknowledgements
The authors would like to thank the editor and the referees for carefully reading the paper and for their valuable comments, which greatly improved the presentation in this paper.

References

Aarset M (1987). "How to Identify Bathtub Hazard Rate." *IEEE Transactions on Reliability*, **36**(1), 106–108.

Alzaatreh A, Famoye F, Lee C (2014). "The Gamma-Normal Distribution: Properties and Applications." *Computational Statistics and Data Analysis*, **69**, 67–80.

Alzaatreh A, Ghosh I (2014). "On the Weibull-X Family of Distributions." *Journal of Statistical Theory and Applications*, **14**(2), 169–183.

Bourguignon M, Silva R, Cordeiro G (2014). "The Weibull-G Family of Probability Distributions." *Journal of Data Science*, (12), 53–68.

Chambers JM, Cleveland WS, Kleiner B, Tukey P (1983). *Graphical Methods for Data Analysis.* Wadsworth.

Chen Z (2000). "A New Two Parameter Lifetime Distribution with Bathtub Shape of Increasing Failure Rate Function." *Statistics and Probability Letters*, **49**, 155–161.

Cordeiro G, Gomes A, da Silva C, Ortega M (2013a). "The Beta Exponentiated Weibull Distribution." *Journal of Statistical Computation and Simulations*, **38**(1), 114–138.

Cordeiro GM, Ortega E, Cunha D (2013b). "The Exponentiated Generalized Class of Distributions." *Journal of Data Science*, **11**(11), 1–27.

Gompertz B (1895). "On The Nature of the Function Expressive of The Law of Human Mortality and The New Model of Determining The Value of Life Contingencies." *Philosophical Transactions of the Royal Society of London*, **115**, 513–585.

Gradshteyn IS, Ryzhik IM (2000). *Tables of Integrals, Series and Products.* Academic Press, San Diego.

Meeker W, Escobar L (1998). *Statistical Methods for Reliability Data.* John Wiley.

Nadarajah S, Cordeiro G, Ortega EMM (2015). "The Zografos and Balakhrishnan-G Family of Distributions: Mathematical Properties and Applications." *Communications in Statistics-Theory and Methods*, **44**, 186–215.

Nascimento ADC, Bourguignon M, Zea LM, Santos-Neto M, Silva RB, Cordeiro GM (2014). "The Gamma Extended Weibull Family of Distributions." *Journal of Statistical Theory and Applications*, **13**, 1–16.

Nassar MM, Eissa F (2003). "On the Exponentiated Weibull Distribution." *Communications in Statistics-Theory and Methods*, **32**, 1317–1336.

Oluyede B, Huang S, Yang T (2015). "A New Class of Generalized Modified Weibull Distribution with Applications." *Austrian Journal of Statistics*, **44**(3), 45–68.

Pal M, Ali MM, Woo J (2006). "Exponentiated Weibull Distribution." *Statistica, anno LXVI*, **2**.

Phani KK (1987). "A New Modified Weibull Distribution Function." *Journal of American Ceramic Society*, **70**, 182–184.

Renyi A (1961). "On Measure of Entropy and Information." *Berkeley Symposium on Mathematical Statistics and Probability*, **1**, 547–561.

Shaked M, Shanthikumar J (1994). *Stochastic Orders and Their Applications.* Academic Press.

Smith RM, Bain LJ (1975). "An Exponential Power Life-testing Distribution." *Communications in Statistics-Theory and Methods*, **4**, 469–481.

Tahir MH, Zubair M, Mansoor M, Cordeiro GM, Alizadeh M, Hamedani GG (2015). "A New Weibull-G Family of Distributions." *Hacettepe University Bulletin of Natural Sciences and Engineering, Series B: Mathematics and Statistics*, **45**(57).

Torabi H, Montazeri HN (2012). "The Gamma-uniform Distribution and It's Applications." *Kybernetika*, **48**, 16–30.

Zografos K, Balakrishnan N (2009). "On Families of Beta- and Generalized Gamma-generated Distribution and Associated Inference." *Stat. Method*, **6**, 344–362.

Affiliation:

Broderick O. Oluyede
Department of Mathematical Sciences
Georgia Southern University (GSU)
Statesboro, GA 30460
E-mail: boluyede@georgiasouthern.edu

Shusen Pu
Department of Mathematics and Statistics
Case Western Reserve University (CWRU)
Cleveland, OH

Boikanyo Makubate
Department of Mathematics and Statistical Sciences
Botswana International University of Science and Technology (BIUST)
Palapye, BW

Yuqi Qiu
Department of Biostatistics
University of California San Diego (UCSD)
San Diego, CA

Austrian Journal of Statistics http://www.ajs.or.at/
published by the Austrian Society of Statistics http://www.osg.or.at/
Volume 47 *Submitted:* 2015-12-09
February 2018 *Accepted:* 2017-02-07