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On Estimation and Decrease of the Dispersion in GPS Data Processing

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1 Introduction

The aim of this paper is to make one keep in view that the geographical coordinates, obtained with the help of a GPS receiver cannot be regarded as accurate data. Based on the results of one exemplary measurement, we will show that it is always necessary to take into account an uncertainty of data acquired from the GPS receiver. The user of the GPS receiver should always consider carefully if the measured values are sufficiently accurate with respect to the particular purposes. This conclusion can be drawn only in cases when an estimation of a dispersion of the GSP receiver is known in a given place and time.

In order to lower the uncertainty of the measurement, various measuring approaches are used. A repeated (multistage) measurement is one of such procedures. In addition, it is also well-known how to determine the estimation of the dispersion of the GPS receiver. However, a possible situation can arise when the user of the device is not in a position to repeat the measurement several times during longer time interval. This can be caused either by a physical principle of a given design of the measurement or by practical aspects (e.g. expensiveness of the repeated measurement carried out for several days). To avoid this difficulty, we will show another possible approach which leads to the estimation of the dispersion of the GPS receiver. Moreover, the presented method can serve for an improvement of the accuracy of data acquired from the GPS receiver.

In the following text, an algorithm based on the theory of estimation is introduced which would eventually decrease the uncertainty of the coordinates obtained from the GPS receiver with an utilization of an additional measurement (in our case, by a measuring tape). Even for an amateur measurement, the dispersion of the measured lengths is approximately about 0.1^2 m^2. From here and on, the uncertainty of the first-stage measurement is considered as the B-type uncertainty (in our case, the B-type uncertainty represents the uncertainty of the measurement by the measuring tape) and the lengths obtained in the first-stage measurement are denoted by a symbol Θ. On the contrary, the uncertainty of the second-stage measurement is considered as the A-type uncertainty (in our case, the A-type uncertainty represents the uncertainty of the measurement by the GPS receiver) and the coordinates acquired in the second-stage measurement are denoted by a symbol β.

Motivation: Let us suppose the following situation. The goal was to determine a stochastic distribution of a chemical element in the soil. The coordinates of the positions, where the value of the chemical element was intended to be measured, have been acquired by the GPS receiver. The obtained values are depicted in Figure 1 where every point corresponds to the place where the sample was taken. According to the design of the measurement and principle of the utilized device, it was then expected that the acquired data would create an accumulation in the form of a ring.

As it is evident from Figure 1, the ring was generated from data for one "locality". However, the expected ring for the second locality was extended in comparison with the previous one. One may therefore ask the following questions. What were the reasons for such an anomalous behavior of the measured data? Was it a consequence of the uncertainty of the acquired coordinates?

In the next example from another area of interest, it will be shown that the estimation of the dispersion of the GPS device is 0.354^2 m^2. This value may greatly differ depending on a number of available satellites, surrounding landscape and sedulity of the person performing the measurement. Therefore, the values acquired by the GPS receiver always exhibit different accuracy.

In the above-discussed example it was found out that the student carrying out the measurement did not respect the instructions for a given measurement. The measurement was not performed all at once but there was a time delay between particular steps of the measurement.

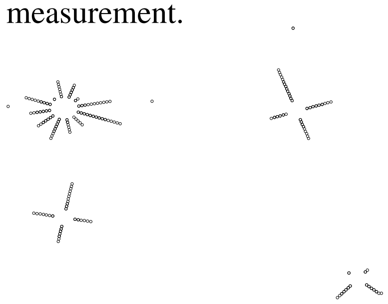


Figure 1: Coordinates of the measured points.

If Ivan Klíma, the well-known Czech writer, was present personally during this measurement, he would surely not write the story called The Geodetic Story where the following thought about an accuracy appears:
From time immemorial a man has been striving for perfection. One of the devices through which we explore mystical places, is measurement. We measure the Earth, the Universe, velocities, times and depths. Depths both uncomprehensibly big and uncomprehensibly small. We draw schemes, maps and plans that are getting more and more accurate. Accuracy has put a spell on us, we have stepped into a dimension where our senses may be deluding us.

Notation The following notations will be used throughout the paper:

$\mathbb{R}^n$	space of all n-dimensional real vectors
Θ	real column vector – from the first stage
β	real column vector – from the second stage
$\mathbf{I}_{m,n}$, $\mathbf{A}_{m,n}$	$m \times n$ identity matrix; the real $m \times n$ matrix
$\mathbf{A}_{r_1:s_1,r_2:s_2}$	$(s_1 - r_1) \times (s_2 - r_2)$ block matrix with elements of $\mathbf{A}$
$\mathbf{A}', r(\mathbf{A}), \mathrm{Tr}(\mathbf{A})$	transpose, the rank and the trace of the matrix $\mathbf{A}$
$\mathbf{A} = \mathrm{Diag}(\mathbf{u})$	diagonal matrix with diagonal equal elements of vector $\mathbf{u}$
$\mathcal{M}(\mathbf{A})$	column space of the matrix $\mathbf{A}$; $\mathcal{M}(\mathbf{A}) = \{\mathbf{Au} : \mathbf{u} \in \mathbb{R}^n\} \subset \mathbb{R}^m$
$\mathrm{Ker}(\mathbf{A})$	null space of the matrix $\mathbf{A}$; $\mathrm{Ker}(\mathbf{A}) = \{\mathbf{u} : \mathbf{u} \in \mathbb{R}^n, \mathbf{Au} = \mathbf{0}\} \subset \mathbb{R}^n$
$\mathbf{A}^-$	generalized inverse of the matrix $\mathbf{A}$ (satisfying $\mathbf{AA}^-\mathbf{A} = \mathbf{A}$) see Rao and Mitra (1971)
$\mathrm{P}_{\mathbf{A}}$	orthogonal projector onto $\mathcal{M}(\mathbf{A})$ in Euclidean norm $\mathrm{P}_{\mathbf{A}} = \mathbf{A}(\mathbf{A}'\mathbf{A})^-\mathbf{A}'$
$\mathrm{M}_{\mathbf{A}}$	orthogonal projector onto $\mathcal{M}^{\perp}(\mathbf{A}) = \mathrm{Ker}(\mathbf{A}')$ in Euclidean norm $\mathrm{M}_{\mathbf{A}} = \mathbf{I} - \mathrm{P}_{\mathbf{A}}$
$\mathbf{Y} \sim (\mathbf{A}\Theta, \mathbf{T})$	observation vector $\mathbf{Y}$ with mean value $\mathbf{A}\Theta$ and covariance matrix $\mathbf{T}$.

2 Model of Measurements

Definition 2.1 Let us consider the linear model $\mathbf{Y} - \mathbf{D}\hat{\Theta} \sim_n (\mathbf{X}\beta, \Sigma_0)$, where $\mathbf{\Sigma}_0 = \sigma^2\mathbf{V}_1 + \mathbf{D}\mathbf{V}_0\mathbf{D}'$ and $\mathbf{Y} \sim_n (\mathbf{D}\Theta + \mathbf{X}\beta, \sigma^2\mathbf{V}_1)$ is a random observation vector, $\beta \in \mathbb{R}^k$ stands for a vector of the useful parameters and $\mathbf{X}_{n,k}$ denotes a design matrix belonging to the vector β. We suppose that an estimator $\widehat{\Theta} \sim_{k_1} (\Theta, \mathbf{V}_0)$ of Θ is at our disposal only.
Theorem 2.1 *The standard estimator $\hat{\sigma}^2$ of the parameter σ^2 for the model defined in Definition 2.1 is given by*

$$\hat{\sigma}^2 = \lambda[(\mathbf{Y} - \mathbf{D}\widehat{\Theta})'(\mathrm{M}_\mathbf{A}\mathbf{\Sigma}_0\mathrm{M}_\mathbf{A})^+\mathbf{V}_1(\mathrm{M}_\mathbf{A}\mathbf{\Sigma}_0\mathrm{M}_\mathbf{A})^+(\mathbf{Y} - \mathbf{D}\widehat{\Theta})] - \\ -\mathrm{Tr}[(\mathrm{M}_\mathbf{A}\mathbf{\Sigma}_0\mathrm{M}_\mathbf{A})^+\mathbf{V}_1(\mathrm{M}_\mathbf{A}\mathbf{\Sigma}_0\mathrm{M}_\mathbf{A})^+\mathbf{V}_0]\,,$$

where the parameter λ is expressed by

$$\mathbf{S}_{(\mathrm{M}_\mathbf{A}\mathbf{\Sigma}_0\mathrm{M}_\mathbf{A})^+}\lambda = 1\,,$$

where the 1×1 matrix $\mathbf{S}_{(\mathrm{M}_\mathbf{A}\mathbf{\Sigma}_0\mathrm{M}_\mathbf{A})^+}$ is

$$\mathbf{S}_{(\mathrm{M}_\mathbf{A}\mathbf{\Sigma}_0\mathrm{M}_\mathbf{A})^+} = \mathrm{Tr}[(\mathrm{M}_\mathbf{A}\Sigma_0\mathrm{M}_\mathbf{A})^+\mathbf{V}_1(\mathrm{M}_\mathbf{A}\Sigma_0\mathrm{M}_\mathbf{A})^+\mathbf{V}_1]\,.$$

Proof. The idea is given in Kubáček and Kubáčková, 2000, p. 102. Details are given in Tuček and Marek (2006). □

Hereafter we will focus on the same model but from a different point of view. We will consider the model of the measurement and then we will present how to determine the estimators of the fundamental parameters.

Definition 2.2 The model of connecting measurement will be represented by

$$\text{(i)} \qquad \begin{pmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{pmatrix} \sim \left[\begin{pmatrix}\mathbf{X}_1, & \mathbf{0}\\ \mathbf{D}, & \mathbf{X}_2\end{pmatrix}\begin{pmatrix}\Theta\\ \beta\end{pmatrix}, \begin{pmatrix}\mathbf{\Sigma}_{1,1}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{\Sigma}_{2,2}\end{pmatrix}\right],$$

where $\mathbf{X}_1, \mathbf{D}, \mathbf{X}_2$ are known $n_1 \times k_1, n_2 \times k_1, n_2 \times k_2$ matrices, respectively, such that $\mathrm{M}(\mathbf{D}') \subset \mathrm{M}(\mathbf{X}_1')$; Θ and β are unknown k_1- and k_2-dimensional vectors; $\mathbf{\Sigma}_{2,2} = \sigma^2\mathbf{V}_1$, where $\mathbf{\Sigma}_{1,1}$ and $\mathbf{V}_1$ are known matrices.

In this model the parameter Θ is estimated on the basis of the vector $\mathbf{Y}_1$ of the first stage and parameter β on the basis of the vectors $\mathbf{Y}_2 - \mathbf{D}\hat{\Theta}$ and $\hat{\Theta}$. At this point, it should be mentioned that the results of the measurement from the second stage (i.e. $\mathbf{Y}_2$) cannot be used for a modification of the estimator $\hat{\Theta}$.

The parametric space $\underline{\Theta}$ of this model of connecting measurement $\mathbf{Y}$ is defined as

$$\text{(ii)} \qquad \underline{\Theta} = \{(\Theta', \beta')' : \mathbf{B}\beta + \mathbf{C}\Theta + \mathbf{a} = \mathbf{0}\}\,,$$

where $\mathbf{B}$ and $\mathbf{C}$ are $q \times k_2$ and $q \times k_1$ matrices, $\mathbf{a}$ is q-dimensional vector, $r(\mathbf{B}) = q < k_2$.

Definition 2.3 The model in the parametric space $\underline{\Theta}$ (see Definition 2.2) is regular provided that $r(\mathbf{X}_1) = k_1$, $r(\mathbf{X}_2) = k_2$, $\mathbf{\Sigma}_{1,1}, \mathbf{\Sigma}_{2,2}$ are positively definite matrices, $r(\mathbf{B}) = q$.

Remark 2.1 Θ represents the parameter of the first stage (connecting) whereas the vector β denotes the parameter of the second stage (connected). In the second stage, we then start with the unbiased estimator $\widehat{\Theta} = (\mathbf{X}_1'\mathbf{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{\Sigma}_{1,1}^{-1}\mathbf{Y}_1$ originating from the first stage whose covariance matrix is expressed in the form of $\mathrm{var}(\widehat{\Theta}) = V_0 = (\mathbf{X}_1'\mathbf{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}$.

Definition 2.4 The least-square estimator of the parameter β, obtained under the condition that $\mathbf{\Sigma}_{1,1} = \mathbf{0}$ ($\Rightarrow$ var$(\widehat{\Theta}) = \mathbf{0}$), is called the standard estimator if the vector Θ is substituted by $\widehat{\Theta}$ in this estimator.

Theorem 2.2 *The standard estimator $\widehat{\beta}$ of the parameter β in the model (i) and (ii) postulated in Definition 2.2 and given by*

$$\widehat{\beta} = (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}(\mathbf{Y}_2 - \mathbf{D}\widehat{\Theta}) - (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\times$$
$$\times \{\mathbf{a} + \mathbf{C}\widehat{\Theta} + \mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}(\mathbf{Y}_2 - \mathbf{D}\widehat{\Theta})\},$$

is unbiased.

Proof. See Marek (2003, p. 72-73). □

Theorem 2.3 *If var$(\widehat{\Theta}) \neq \mathbf{0}$ then the covariance matrix of the standard estimator $\widehat{\beta}$ is composed of two uncertainties, i.e. the "uncertainty of type A" and "uncertainty of type B", as*

$$\begin{aligned} var(\widehat{\beta}) = \underbrace{var_0(\widehat{\beta})}_{\text{uncertainty of type A}} \underbrace{\begin{aligned} &+\langle\{\mathbf{I} - (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\mathbf{B}\} \\ &\times (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{D} - (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1} \\ &\times \mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\mathbf{C}\rangle \\ &\times var(\widehat{\Theta}) \\ &\times \langle\{\mathbf{I} - (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\mathbf{B}\} \\ &\times (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{D} - (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1} \\ &\times \mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\mathbf{C}\rangle' \end{aligned}}_{\text{uncertainty of type B}} \end{aligned}$$

where $var_0(\widehat{\beta}) = (\mathrm{M}_{\mathbf{B}'}\mathbf{X}_2'\mathbf{\Sigma}_{22}\mathbf{X}_2\mathrm{M}_{\mathbf{B}'})^{+} = (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1} - (\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}' \times[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{22}^{-1}\mathbf{X}_2)^{-1}.$

Proof. See Marek (2003, p. 74). □

Theorem 2.4 *The $(1-\alpha)$-confidence domain for the parameter β, $\beta \in \underline{\Theta}$ (see Definition 2.2), based on the standard BLUE $\widehat{\beta}$, is a set expressed by*

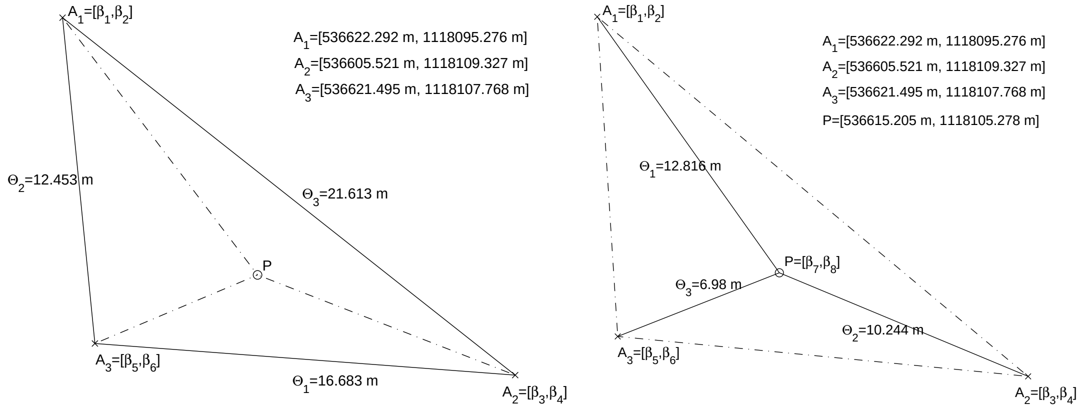
$$\mathcal{E}_{1-\alpha}(\beta) = \left\{\mathbf{u} : \mathbf{u} \in \underline{\Theta}_\beta \subset \mathbb{R}^{k_2}, (\mathbf{u} - \widehat{\beta})'[var(\widehat{\beta})]^{-}(\mathbf{u} - \widehat{\beta}) \leq \chi^2_{k_2-q+r(\mathbf{C})}(1-\alpha)\right\}.$$

The symbol $\chi^2_{k_2-q+r(\mathbf{C})}(1-\alpha)$ denotes the $(1-\alpha)$-quantile of a χ^2-distribution with $k_2 - q + r(\mathbf{C})$ degrees of freedom.

Proof. See Kubáčková (1996, p. 158-159). □

Example 2.1 The first aim of this example is to find a dispersion for a GARMIN GPS 12XL navigator, the second aim is to estimate plane coordinates β of points A_1, A_2, A_3 in situation I and plane coordinates of points A_1, A_2, A_3 and P in situation II (see Figure 2). We have given geographical coordinates of these points, i.e. their latitudes and longitudes, which have been obtained from a navigator. For our purposes, the geographical coordinates were transformed to the plane system known as S-JTSK (where $+x$-axes … south, $+y$-axes … west)— Ryšavý (1953). So we have estimated values $A_i = (Y_{2i-1}, Y_{2i})$, $i = 1, 2, 3$ and measured values $\widehat{\Theta}^I = (\widehat{\Theta}_1^I, \widehat{\Theta}_2^I, \widehat{\Theta}_3^I)'$ in situation I or we have estimated values $A_i = (Y_{2i-1}, Y_{2i})$ and $P = (Y_7, Y_8)$ and measured values $\widehat{\Theta}^{II} = (\widehat{\Theta}_1^{II}, \widehat{\Theta}_2^{II}, \widehat{\Theta}_3^{II})'$ in situation II.

Let the result from the first and the second stage of measurement in situation I be $(\widehat{\Theta}_1^I, \widehat{\Theta}_2^I, \widehat{\Theta}_3^I)' = (16.683\text{m}, 12.453\text{m}, 21.613\text{m})'$ and
$\mathbf{Y}^{Ig} = (49°38'02.2'', 17°23'35.1'', 49°38'01.8'', 17°23'36.0'', 49°38'01.8'', 17°23'35.2'')' \rightarrow$
$\mathbf{Y}^I = (536622.29\text{m}, 1118095.28\text{m}, 536605.52\text{m}, 1118109.33\text{m}, 536621.49\text{m}, 1118107.77\text{m})'$
or in situation II be $(\widehat{\Theta}_1^{II}, \widehat{\Theta}_2^{II}, \widehat{\Theta}_3^{II})' = (12.816\text{m}, 10.244\text{m}, 6.980\text{m})'$ and
$\mathbf{Y}^{IIg} = ((\mathbf{Y}^{Ig})', 17°23'35.5'', 49°38'01.9'')' \rightarrow \mathbf{Y}^{II} = ((\mathbf{Y}^I)', 536614.79\text{m}, 1118105.88\text{m})'$.



situation I *situation II*

Figure 2: The polygonometric measurement.

We have the model given by

$$\mathbf{Y} = \mathbf{f}(\beta) + \varepsilon = \begin{pmatrix} \sqrt{(\beta_1 - \beta_3)^2 + (\beta_2 - \beta_4)^2} \\ \sqrt{(\beta_3 - \beta_5)^2 + (\beta_4 - \beta_6)^2} \\ \sqrt{(\beta_1 - \beta_5)^2 + (\beta_2 - \beta_6)^2} \\ \beta_1 \\ \vdots \\ \beta_6 \end{pmatrix} + \varepsilon\,, \tag{1}$$

where $\text{var}(\varepsilon) = \mathbf{\Sigma}_0 = \sigma^2 \mathbf{W}_1 + \mathbf{W}_0$.

In our case, we will consider the covariance matrices $\mathbf{W}_0 = (0.1)^2 \times \begin{pmatrix} \mathbf{I}_{3,3}, \mathbf{0}_{3,6} \\ \mathbf{0}_{6,3}, \mathbf{0}_{6,6} \end{pmatrix}$, $\sigma^2 \mathbf{W}_1 = \sigma^2 \times \text{Diag}((0,0,0,1,\cos^2\varphi, 1, \cos^2\varphi, 1, \cos^2\varphi)')$, $\sigma^2 = 3.1^2\text{m}^2$, $\cos(\varphi) = \cos(49°) = 0.6564$.

Note that the value of 0.1m, is usually used for the value of the standard deviation of the measuring tape. For the parameter σ^2 we will use the following value $\sigma^2_{GPS} = \frac{2\pi 6378 \cdot 1000}{360 \cdot 60 \cdot 60 \cdot 10} = 3.1^2\text{m}^2$, where the expression above, especially the value of 3.1 m, denotes the standard deviation, derived from the smallest decimal digit which the GPS receiver displays. The angle $\varphi = 49°$ stands for the value of the latitude where the measurement has been carried out.

We will determine a linear model for the function $\mathbf{f}$ above. We generate the Taylor expansion at the suitable point which is given by $\mathbf{f}(\beta) = \mathbf{f}(\beta^0) + \mathbf{A}(\beta - \beta^0)$. According to the theory of the measurement, we have to define the matrix $\mathbf{A} = (\partial \mathbf{f} / \partial \Theta')$, for example $A_{3,6} = (\beta_6 - \beta_2)/\sqrt{\beta_5^2 - 2\beta_5\beta_1 + \beta_1^2 + \beta_6^2 - 2\beta_6\beta_2 + \beta_2^2}$.

Now we will determine the estimator $\hat{\sigma}^2$ of the parameter σ^2 according to the Theorem 2.1. The whole process of determining the estimator $\hat{\sigma}^2$ can be now, according to the Theorem 2.1, written as

$$\hat{\sigma}^2 = \lambda \Big\{ [(\mathbf{Y} - \mathbf{D}\widehat{\Theta})'(\mathrm{M}_{\mathbf{A}}\boldsymbol{\Sigma}_0\mathrm{M}_{\mathbf{A}})^{+}\mathbf{W}_1(\mathrm{M}_{\mathbf{A}}\boldsymbol{\Sigma}_0\mathrm{M}_{\mathbf{A}})^{+}(\mathbf{Y} - \mathbf{D}\widehat{\Theta})] - \\ -\mathrm{Tr}[(\mathrm{M}_{\mathbf{A}}\boldsymbol{\Sigma}_0\mathrm{M}_{\mathbf{A}})^{+}\mathbf{W}_1(\mathrm{M}_{\mathbf{A}}\boldsymbol{\Sigma}_0\mathrm{M}_{\mathbf{A}})^{+}\mathbf{W}_0] \Big\} , \quad (2)$$

where λ is expressed by the equation

$$\mathbf{S}_{(\mathrm{M}_{\mathbf{A}}\boldsymbol{\Sigma}_0\mathrm{M}_{\mathbf{A}})^{+}}\lambda = 1 , \quad (3)$$

where the 1×1 matrix $\mathbf{S}_{(\mathrm{M}_{\mathbf{A}}\boldsymbol{\Sigma}_0\mathrm{M}_{\mathbf{A}})^{+}}$ is (for details see Tuček and Marek, 2006)

$$\mathbf{S}_{(\mathrm{M}_{\mathbf{A}}\boldsymbol{\Sigma}_0\mathrm{M}_{\mathbf{A}})^{+}} = \mathrm{Tr}[(\mathrm{M}_{\mathbf{A}}\boldsymbol{\Sigma}_0\mathrm{M}_{\mathbf{A}})^{+}\mathbf{W}_1(\mathrm{M}_{\mathbf{A}}\boldsymbol{\Sigma}_0\mathrm{M}_{\mathbf{A}})^{+}\mathbf{W}_1] . \quad (4)$$

By solving equations (2),(3), and (4) we obtain $\lambda = 4.1751e^{-27}$ and $\hat{\sigma}^2 = (0.3540\mathrm{m})^2$. We can say that the estimator of the uncertainty in GPS-coordinates is $\hat{\sigma}^2 = 0.3540^2\mathrm{m}^2$.

Hereafter, we will focus on the same model but from a different point of view. We will consider the model of the measurement (i) and condition (ii) from Definition 2.2. Finally, we have in situation I the model given by

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} = \begin{pmatrix} \Theta_1 \\ \widehat{\Theta}_2 \\ \widehat{\Theta}_3 \\ Y_1 \\ \vdots \\ Y_6 \end{pmatrix} \sim N_9 \left[\begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{0}, & \mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \\ \beta_1 \\ \vdots \\ \beta_6 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_{11}, & \mathbf{0} \\ \mathbf{0}, & \boldsymbol{\Sigma}_{22} \end{pmatrix} \right] .$$

In our case $\mathbf{X}_1 = \mathbf{I}$, $\mathbf{X}_2 = \mathbf{I}$, $\boldsymbol{\Sigma}_{11} = (\boldsymbol{\Sigma}_0)_{1:3,1:3}$, $\boldsymbol{\Sigma}_{22} = (\boldsymbol{\Sigma}_0)_{4:9,4:9}$ (see (1).

One can observe from Figure 2 in situation I that the condition $\mathbf{g}(\Theta, \beta) = 0$ is implied for the parameters Θ and β, where $g_1(\Theta, \beta) = (\beta_5 - \beta_3)^2 + (\beta_6 - \beta_4)^2 - \Theta_1^2$, $g_2(\Theta, \beta) = (\beta_5 - \beta_1)^2 + (\beta_6 - \beta_2)^2 - \Theta_2^2$, $g_3(\Theta, \beta) = (\beta_3 - \beta_1)^2 + (\beta_4 - \beta_2)^2 - \Theta_3^2$.

The linear version of the condition $\mathbf{g}(\Theta, \beta) = 0$, obtained using the Taylor expansion at the approximate point $(\Theta^0, \beta^0) = (\widehat{\Theta}_1, \widehat{\Theta}_2, \widehat{\Theta}_3, Y_1, Y_2, \ldots, Y_6)$, is in the form of $\mathbf{B}\delta\beta + \mathbf{C}\delta\Theta + \mathbf{a} = \mathbf{0}$, where $\delta\beta = \beta - \beta^0$, $\delta\Theta = \Theta - \Theta^0$, $\mathbf{B} = \partial g(\Theta^0, \beta^0)/\partial\beta'$, $\mathbf{C} = \partial g(\Theta^0, \beta^0)/\partial\Theta'$, $\mathbf{a} = \mathbf{g}(\beta^0, \Theta^0)$.

Here we present values of the vector of the estimator $\widehat{\beta}^I$ (calculated according to Theorem 2.2) based on the model with the measurement of all triangular lengths by the measuring tape. They are: (536621.93m,1118095.92m,536604.64m, 1118108.12m,536622.73m, 1118108.32m)$'$. Its covariance matrix was calculated (see Theorem 2.3) leading to

$$\mathrm{var}(\widehat{\beta}^I) = \begin{pmatrix} 1.2455 & 0.5064 & 0.0300 & -0.9437 & 0.1645 & 0.4373 \\ 0.5064 & 3.3361 & -0.2980 & 2.3743 & -0.2084 & 3.2896 \\ 0.0300 & -0.2980 & 0.7453 & 0.5556 & 0.6647 & -0.2576 \\ -0.9437 & 2.3743 & 0.5556 & 4.1672 & 0.3881 & 2.4585 \\ 0.1645 & -0.2084 & 0.6647 & 0.3881 & 0.6108 & -0.1797 \\ 0.4373 & 3.2896 & -0.2576 & 2.4585 & -0.1797 & 3.2519 \end{pmatrix} .$$

Furthermore in the same way we will find estimator $\widehat{\beta}^{II}$ for model for situation II.

As we can see, it is possible to use estimator $\widehat{\beta}^{I}$ from model for situation I for finding estimator in model for situation II. We will denote this estimator $\widehat{\widehat{\beta}}^{II'}$. All estimators and their uncertainty are shown in the following tables.

$\mathbf{Y}$	$\text{var}(\mathbf{Y})_{i,i}^{1/2}$	$\widehat{\beta}^{I}$	$\text{var}(\widehat{\beta}^{I})_{i,i}^{1/2}$	$\mathbf{Y}-\widehat{\beta}^{I}$	$\widehat{\beta}^{II}$	$\text{var}(\widehat{\beta}^{II})_{i,i}^{1/2}$	$\mathbf{Y}-\widehat{\beta}^{II}$
536622.29	2.03	536621.93	1.12	0.362	536622.42	1.17	−0.124
1118095.28	3.10	1118095.92	1.83	−0.647	1118094.18	1.86	1.092
536605.52	2.03	536604.64	0.86	0.878	536605.58	1.00	−0.057
1118109.33	3.10	1118108.12	2.04	1.204	1118109.18	2.46	0.149
536621.49	2.03	536622.73	0.78	−1.240	536621.75	1.02	−0.257
1118107.77	3.10	1118108.32	1.80	−0.556	1118108.40	2.57	−0.635
536615.20	2.03	–	–	–	536614.77	0.84	0.437
1118105.28	3.10	–	–	–	1118105.88	1.76	−0.607

$\mathbf{Y}$	$\widehat{\beta}^{I}$	$\widehat{\widehat{\beta}}^{II'}$	$\text{var}(\widehat{\widehat{\beta}}^{II'})_{i,i}^{1/2}$	$\widehat{\beta}^{I}-\widehat{\widehat{\beta}}^{II'}$	$\mathbf{Y}-\widehat{\widehat{\beta}}^{II'}$
	536622.29	536622.47	1.08		−0.057
	1118095.28	1118094.36	1.62		−0.179
	536605.52	536605.36	0.77		0.217
	1118109.33	1118109.34	1.83		−0.163
	536621.49	536622.07	0.69		−0.319
	1118107.77	1118107.49	1.59		0.914
536615.20		536614.61	0.61	0.161	
1118105.28		1118106.18	1.50	−0.298	

We have taken into account three different cases in which we have determined the possible way, how to obtain the coordinates from the GPS receiver, which shows a lesser uncertainty. These results, especially variances and residuals, for the first calculated situation are quite satisfactory. In the second situation we have not obtain better results because we have measured shorter distances. We have corrected this imperfection in situation II', where we have arrived at the best results. The essence of this method is based on the use of outputs of the situation II as the input for the situation II'.

The confidence ellipsoids obtained from calculated covariance matrix (based on Theorem 2.4) for $\alpha = 5\%$ are depicted in Figure 3.

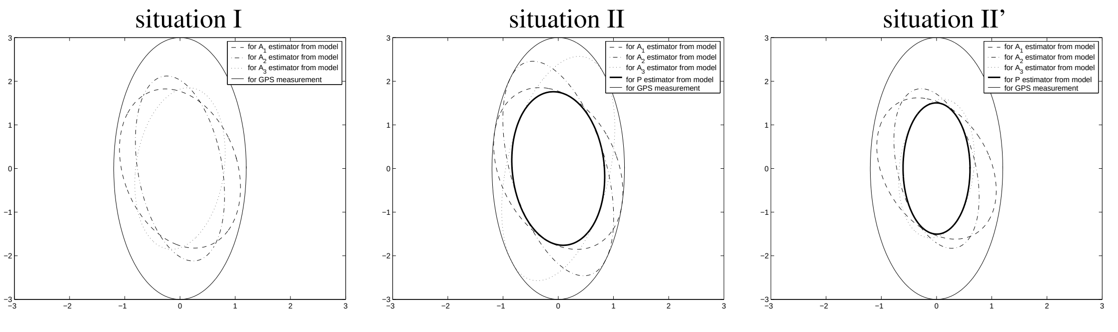


Figure 3: The $(1-\alpha)$ confidence ellipses for points A_1, A_2, and A_3 (solid), for point $\hat{A}_1$ (dashed), $\hat{A}_2$ (dashdotted), and for $\hat{A}_3$ (dotted).

3 Concluding Remarks

We hope that our contribution has evidently pointed out a necessity to investigate the dispersion of the measuring device (the GPS receiver in our case) before the initiation of the measurement itself. In reality, a finding of the estimation of the dispersion can be complicated and infeasible in some cases. It may happen that the measurement cannot be repeated several times. Our proposed procedure, however, allows to estimate the dispersion without the measurement being repeated but with the help of the additional measurement (in our case, by a measure tape).

In the example worked out in this paper, we have calculated the values of the uncertainty of the GPS receiver which may have at the latitude of $\varphi = 49°$. Furthermore, our contribution have shown how the theory of estimation is a powerful tool for a modification of inaccurate data acquired by a measuring device (the GPS receiver in our case) with the utilization of the additional measurement. The example has also demonstrated a possibility of a successive improvement of the estimation by a further additional measurement.

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