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Sequential Point Estimation of a Function of the Exponential Scale Parameter

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Abstract: We consider sequential point estimation of a function of the scale parameter of an exponential distribution subject to the loss function given as a sum of the squared error and a linear cost. For a fully sequential sampling scheme, we present a sufficient condition to get a second order approximation to the risk of the sequential procedure as the cost per observation tends to zero. In estimating the mean, our result coincides with that of Woodroofe (1977). Further, in estimating the hazard rate for example, it is shown that our sequential procedure attains the minimum risk associated with the best fixed sample size procedure up to the order term.

Zusammenfassung: Wir betrachten die sequentielle Punktschätzung einer Funktion des Skalenparameters einer Exponentialverteilung bezüglich einer Verlustfunktion, welche als Summe des quadratischen Fehlers und einer linearen Kostenfunktion gegeben ist. Für einen vollständigen sequentiellen Stichprobenplan presentieren wir eine hinreichende Bedingung, um für das Risiko der sequentiellen Prozedur eine Approximation zweiter Ordnung zu bekommen, wobei die Kosten je Beobachtung gegen Null streben. Bei der Schätzung des Erwartungswertes stimmt unser Ergebnis mit jenem in Woodroofe (1977) überein. Schätzt man beispielsweise die Hazardrate, so wird gezeigt, dass unsere sequentielle Prozedur bis auf den Ordnungsterm das minimale Risiko erreicht, welches zur besten Prozedur bei festem Stichprobenumfang gehört.

Keywords: Stopping Rule, Second Order Approximation, Regret, Uniform Integrability, Hazard Rate.

1 Introduction

Let $X_1, X_2, \ldots$ be independent and identically distributed random variables according to an exponential distribution having the probability density function

$$f_\sigma(x) = \frac{1}{\sigma}\exp\left(-\frac{x}{\sigma}\right), \quad x > 0,$$

where the scale parameter $\sigma \in (0, \infty)$ is unknown. It is interesting to estimate the mean σ and the variance σ^2. One may like to estimate the hazard rate σ^{-1} and the reliability parameter, that is, $P(X_1 > b) = \exp(-b/\sigma)$ for some fixed $b\,(> 0)$. For this reason, we consider the estimation of a function of the scale parameter.

Suppose that $\theta(x)$ is a positive-valued and three times continuously differentiable function on $x > 0$ and that $\theta'(x) \neq 0$ for $x > 0$, where θ' stands for the first derivative of θ. Let θ'' and $\theta^{(3)}$ denote the second and third derivatives of θ, respectively. Given a sample $X_1, \ldots, X_n$ of size n, one wishes to estimate a function $\theta = \theta(\sigma)$ by $\hat{\theta}_n = \theta(\overline{X}_n)$, subject to the loss function

$$L(\hat{\theta}_n) = \left(\hat{\theta}_n - \theta\right)^2 + cn\,,$$

where $\overline{X}_n = n^{-1}\sum_{i=1}^n X_i$ and $c > 0$ is the known cost per unit sample. The risk is given by $R_n = \mathrm{E}\{L(\hat{\theta}_n)\} = \mathrm{E}(\hat{\theta}_n - \theta)^2 + cn$. We want to find an appropriate sample size that will minimize the risk. By Taylor's expansion and the Hölder inequality we can show that under a certain condition, $R_n \approx \sigma^2\{\theta'(\sigma)\}^2 n^{-1} + cn$ for sufficiently large n. Thus, R_n is approximately minimized at

$$n_0 \approx \frac{\sigma|\theta'(\sigma)|}{\sqrt{c}} = n^* \quad \text{(say)} \tag{1}$$

with $R_{n_0} \approx 2cn^*$. Since σ is unknown, however, we can not use the best fixed sample size procedure n_0. Further, there is no fixed sample size procedure that will attain the minimum risk R_{n_0} (see Takada, 1986). Thus, it is necessary to find a sequential sampling rule.

For the estimation of the mean $\theta = \sigma$, Woodroofe (1977) proposed a fully sequential procedure and gave a second order approximation to the risk. Mukhopadhyay et al. (1997) considered the sequential estimation of the reliability parameter $\theta = \exp(-b/\sigma)$ for some fixed $b\,(> 0)$. For the normal case, Takada (1997) constructed sequential confidence intervals for a function of normal parameters and Uno and Isogai (2002) considered the sequential estimation of the powers of a normal scale parameter. In this paper, motivated by (1), we propose the following stopping rule:

$$N = N_c = \inf\left\{n \geq m:\ n \geq \frac{\overline{X}_n|\theta'(\overline{X}_n)|}{\sqrt{c}}\right\}\,, \tag{2}$$

where $m\,(\geq 1)$ is the pilot sample size. By the strong law of large numbers we have $P(N < +\infty) = 1$. In estimating $\theta = \theta(\sigma)$ by $\hat{\theta}_N = \theta(\overline{X}_N)$, the risk is given by $R_N = \mathrm{E}(\hat{\theta}_N - \theta)^2 + c\mathrm{E}(N)$. The performance of the procedure is measured by the regret $R_N - 2cn^*$. The purpose of this paper is to derive second order approximations to the expected sample size $\mathrm{E}(N)$ and the risk of the above sequential procedure R_N as $c \to 0$. In Section 2, we present a sufficient condition to get an asymptotic expansion of the risk. In Section 3, as an example of the function $\theta(x)$ we consider the estimation of the hazard rate $\theta(\sigma) = \sigma^{-1}$ with simulation experiments and show that our sequential procedure attains the minimum risk $2cn^*$ up to the order term.

2 Main Results

In this section, we shall investigate second order asymptotic properties of the sequential procedure. Let

$$h(x) = \frac{1}{x\sqrt{\{\theta'(x)\}^2}} \quad \text{for } x > 0\,.$$

The stopping rule N defined by (2) becomes

$$N = \inf\{n \geq m :\ Z_n \geq n^*\}\,, \quad \text{where } Z_n = n\frac{h(\overline{X}_n)}{h(\sigma)}\,.$$

Let $Y_i = (X_i/\sigma) - 1$, for $i = 1, 2, \ldots$, $S_n = \sum_{i=1}^n Y_i$ and $\overline{Y}_n = n^{-1}S_n$. By Taylor's theorem,

$$h(\overline{X}_n) = h(\sigma) + h'(\sigma)(\overline{X}_n - \sigma) + \tfrac{1}{2}(\overline{X}_n - \sigma)^2 h''(\eta_n)\,,$$

where η_n is a random variable between σ and $\overline{X}_n$. Then we have $Z_n = n + \alpha S_n + \xi_n$ with

$$\alpha = -\left(1 + \frac{\sigma\theta''(\sigma)}{\theta'(\sigma)}\right)\,, \qquad \xi_n = n(\overline{X}_n - \sigma)^2 \frac{h''(\eta_n)}{2h(\sigma)} \tag{3}$$

and

$$h''(x) = \frac{\theta'(x)}{|\theta'(x)|}\left\{2\frac{\{\theta'(x) + x\theta''(x)\}^2}{x^3\{\theta'(x)\}^3} - \frac{2\theta''(x) + x\theta^{(3)}(x)}{x^2\{\theta'(x)\}^2}\right\}\,.$$

Let

$$T = \inf\{n \geq 1 :\ n + \alpha S_n > 0\} \quad \text{and} \quad \rho = \frac{\mathrm{E}(T + \alpha S_T)^2}{2\mathrm{E}(T + \alpha S_T)}\,. \tag{4}$$

Consider the following assumptions:

(A1) $\left\{\left[\left(Z_n - \dfrac{n}{\varepsilon_0}\right)^+\right]^3,\ n \geq m\right\}$ is uniformly integrable for some $0 < \varepsilon_0 < 1$, where $x^+ = \max(x, 0)$.

(A2) $\displaystyle\sum_{n=m}^{\infty} nP\{\xi_n < -\varepsilon_1 n\} < \infty$ for some $0 < \varepsilon_1 < 1$.

Then we obtain the following approximation to the expected sample size for all $\sigma \in (0, \infty)$ but not uniformly in σ.

Theorem 1 *If (A1) and (A2) hold, then*

$$\mathrm{E}(N) = n^* + \rho - l + o(1) \quad \textit{as } c \to 0\,,$$

where

$$l = 1 + \frac{\sigma\theta''(\sigma)}{\theta'(\sigma)} + \frac{\sigma^2\{\theta''(\sigma)\}^2}{\{\theta'(\sigma)\}^2} - \frac{\sigma^2\theta^{(3)}(\sigma)}{2\theta'(\sigma)}\,.$$

Proof. Let W be distributed according to a standard normal distribution $N(0,1)$. Then, from (3)

$$\xi_n \xrightarrow{d} \xi \equiv l \cdot W^2 \quad \text{as } n \to \infty\,,$$

where '$\xrightarrow{d}$' denotes convergence in distribution. We shall check conditions (C1)–(C6) of Aras and Woodroofe (1993). Clearly, (C1) holds. (C2) with $p = 3$ and (C3) are identical with (A1) and (A2), respectively. Letting $g(y) = h(\sigma y + \sigma)/h(\sigma)$, (C4), (C5) and (C6) follow from Proposition 4 of Aras and Woodroofe (1993). Hence, from Theorem 1 of Aras and Woodroofe (1993),

$$\mathrm{E}(N) = n^* + \rho - \mathrm{E}(\xi) + o(1) = n^* + \rho - l + o(1) \quad \text{as } c \to 0\,,$$

which concludes the theorem. □

The proposition below gives sufficient conditions for (A2) which are useful in actual estimation problems.

Proposition 1

(i) *If* $h''(\eta_n) \geq 0$ *for all* $n \geq m$*, then (A2) holds.*
(ii) *If* $\sup_{n\geq m}\mathrm{E}|h''(\eta_n)|^s < \infty$ *for some* $s > 2$*, then (A2) holds.*

Proof. From (3), (i) implies (A2). Suppose that (ii) holds. For $0 < \varepsilon < 1$ and $q > 2$,

$$\begin{aligned} P(\xi_n < -\varepsilon n) &= P\{-h''(\eta_n)(\overline{X}_n - \sigma)^2 > 2h(\sigma)\varepsilon\} \\ &\leq (2h(\sigma)\varepsilon)^{-q}\mathrm{E}|(\overline{X}_n - \sigma)^2 h''(\eta_n)|^q \\ &\leq (2h(\sigma)\varepsilon)^{-q}\left\{\mathrm{E}|\overline{X}_n - \sigma|^{2qu}\right\}^{1/u}\left\{\mathrm{E}|h''(\eta_n)|^{qv}\right\}^{1/v}\,, \end{aligned}$$

where $u^{-1} + v^{-1} = 1$ and $u > 1$. Choose (u, v) and $2 < q < s$ such that $s = qv$. By the Marcinkiewicz-Zygmund inequality (see Chow and Teicher, 1988) we get $\mathrm{E}|\overline{X}_n - \sigma|^{2qu} = O(n^{-qu})$. Thus we have $nP(\xi_n < -\varepsilon n) = O(n^{1-q})$ as $n \to \infty$, which implies (A2). □

We shall now assess the regret $R_N - 2cn^*$. By Taylor's theorem,

$$\theta(\overline{X}_N) - \theta(\sigma) = \theta'(\sigma)(\overline{X}_N - \sigma) + \tfrac{1}{2}\theta''(\sigma)(\overline{X}_N - \sigma)^2 + \tfrac{1}{6}(\overline{X}_N - \sigma)^3\theta^{(3)}(\phi_c)\,, \qquad (5)$$

where ϕ_c is a random variable between σ and $\overline{X}_N$. We impose the following assumption:

(A3) For some $a > 1,\ u > 1$ and $c_0 > 0$,

$$\sup_{0<c\leq c_0}\{c^{-au}\mathrm{E}|\overline{X}_N - \sigma|^{4au}\} < \infty \qquad \text{and} \qquad \sup_{0<c\leq c_0}\mathrm{E}|\theta^{(3)}(\zeta_c)|^{2au/(u-1)} < \infty\,,$$

where ζ_c is any random variable between σ and $\overline{X}_N$.

Remark 1 *If* $|\theta^{(3)}(x)|$ *is bounded, then the second part of* (A3) *is satisfied. If* $|\theta^{(3)}(x)|$ *is convex on* $(0,\infty)$ *and* $\sup_{0<c\leq c_0}\mathrm{E}|\theta^{(3)}(\overline{X}_N)|^{2au/(u-1)} < \infty$*, then the second part of* (A3) *is satisfied. Let* $\theta(x) = x^r$ *for* $x > 0$ *with any fixed* r*, for instance. Then* $|\theta^{(3)}(x)|$ *is convex unless* $3 < r < 4$*.*

The main theorem of this paper is as follows.

Theorem 2 *If (A1), (A2) and (A3) hold, then*

$$R_N - 2cn^* = \left\{3 + 2\frac{\sigma\theta''(\sigma)}{\theta'(\sigma)} + \frac{7\sigma^2\{\theta''(\sigma)\}^2}{4\{\theta'(\sigma)\}^2} - \frac{\sigma^2\theta^{(3)}(\sigma)}{\theta'(\sigma)}\right\} c + o(c) \quad \textit{as} \ \ c \to 0\,.$$

Remark 2 *Theorem 2 shows that in estimating the mean $\theta = \sigma$, the regret becomes $3c + o(c)$, which coincides with the result of Woodroofe (1977).*

Proof of Theorem 2. From (5),

$$\begin{aligned} R_N - 2cn^* &= \mathrm{E}(\hat{\theta}_N - \theta)^2 + c\mathrm{E}(N) - 2cn^* \\ &= \{\theta'(\sigma)\}^2\mathrm{E}(\overline{X}_N - \sigma)^2 + c\mathrm{E}(N) - 2cn^* \\ &\quad + \theta'(\sigma)\theta''(\sigma)\mathrm{E}(\overline{X}_N - \sigma)^3 + \tfrac{1}{4}\{\theta''(\sigma)\}^2\mathrm{E}(\overline{X}_N - \sigma)^4 \\ &\quad + \tfrac{1}{3}\theta'(\sigma)\mathrm{E}\{(\overline{X}_N - \sigma)^4\theta^{(3)}(\phi_c)\} + \tfrac{1}{6}\theta''(\sigma)\mathrm{E}\{(\overline{X}_N - \sigma)^5\theta^{(3)}(\phi_c)\} \\ &\quad + \tfrac{1}{36}\mathrm{E}[(\overline{X}_N - \sigma)^6\{\theta^{(3)}(\phi_c)\}^2]\,. \end{aligned} \tag{6}$$

From Theorems 2 and 3 of Aras and Woodroofe (1993), we obtain (7)–(9) below, as $c \to 0$.

$$\begin{aligned} &\{\theta'(\sigma)\}^2\mathrm{E}(\overline{X}_N - \sigma)^2 + c\mathrm{E}(N) - 2cn^* \\ &= c\{(n^*)^2\mathrm{E}(\overline{Y}_N)^2 + \mathrm{E}(N) - 2n^*\} \\ &= \{2\mathrm{E}(\xi W^2) - 2\mathrm{E}(\xi) + 3\alpha^2 + 2\alpha\mathrm{E}(Y_1)^3\}c + o(c) \\ &= \{4l + 3\alpha^2 + 4\alpha\}c + o(c) \\ &= \left\{3 + 6\frac{\sigma\theta''(\sigma)}{\theta'(\sigma)} + 7\frac{\sigma^2\{\theta''(\sigma)\}^2}{\{\theta'(\sigma)\}^2} - 2\frac{\sigma^2\theta^{(3)}(\sigma)}{\theta'(\sigma)}\right\} c + o(c)\,. \end{aligned} \tag{7}$$

$$\begin{aligned} \theta'(\sigma)\theta''(\sigma)\mathrm{E}(\overline{X}_N - \sigma)^3 &= \frac{\sigma\theta''(\sigma)}{\theta'(\sigma)} c(n^*)^2\mathrm{E}(\overline{Y}_N)^3 \\ &= \frac{\sigma\theta''(\sigma)}{\theta'(\sigma)}\{6\alpha + \mathrm{E}(Y_1)^3\}c + o(c) \\ &= -\left\{4\frac{\sigma\theta''(\sigma)}{\theta'(\sigma)} + 6\frac{\sigma^2\{\theta''(\sigma)\}^2}{\{\theta'(\sigma)\}^2}\right\} c + o(c)\,. \end{aligned} \tag{8}$$

$$\begin{aligned} \tfrac{1}{4}\{\theta''(\sigma)\}^2\mathrm{E}(\overline{X}_N - \sigma)^4 &= \frac{\sigma^2\{\theta''(\sigma)\}^2}{4\{\theta'(\sigma)\}^2} c(n^*)^2\mathrm{E}(\overline{Y}_N)^4 \\ &= \frac{3}{4}\cdot\frac{\sigma^2\{\theta''(\sigma)\}^2}{\{\theta'(\sigma)\}^2} c + o(c)\,. \end{aligned} \tag{9}$$

For $b > 1$, $p > 1$, $q = p/(p-1)$ and $v = u/(u-1)$, we have

$$\begin{aligned} &\mathrm{E}\left|\left\{(n^*)^{1/2}\overline{Y}_N\right\}^4\{\theta^{(3)}(\phi_c)\}^2(\overline{Y}_N)^2\right|^b \\ &\le \left\{\mathrm{E}\left|(n^*)^{1/2}\overline{Y}_N\right|^{4bpu}\right\}^{1/pu}\left\{\mathrm{E}\left|\theta^{(3)}(\phi_c)\right|^{2bpv}\right\}^{1/pv}\left\{\mathrm{E}(\overline{Y}_N)^{2bq}\right\}^{1/q} \end{aligned}$$

and by Doob's maximal inequality,

$$\mathrm{E}(\overline{Y}_N)^{2bq} \le \mathrm{E}\left\{\sup_{n\ge 1}(\overline{Y}_n)^{2bq}\right\} \le \left(\tfrac{2bq}{2bq-1}\right)^{2bq} \mathrm{E}(Y_1)^{2bq} < \infty\,.$$

Choosing b and p such that $a = bp$, (A3) yields the uniform integrability of

$$\left\{\{(n^*)^{1/2}\overline{Y}_N\}^4\{\theta^{(3)}(\phi_c)\}^2(\overline{Y}_N)^2\,,\ 0 < c \le c_0\right\}\,.$$

Since $\{(n^*)^{1/2}\overline{Y}_N\}^4\{\theta^{(3)}(\phi_c)\}^2 \xrightarrow{d} \{\theta^{(3)}(\sigma)\}^2W^4$ and $(\overline{Y}_N)^2 \longrightarrow 0$ a.s. as $c \to 0$, we get

$$\begin{aligned}&\tfrac{1}{36}\mathrm{E}[(\overline{X}_N-\sigma)^6\{\theta^{(3)}(\phi_c)\}^2]\\ &= \frac{\sigma^4}{36\{\theta'(\sigma)\}^2}c\,\mathrm{E}\left[\left\{(n^*)^{1/2}\overline{Y}_N\right\}^4\{\theta^{(3)}(\phi_c)\}^2(\overline{Y}_N)^2\right] = o(c)\,.\end{aligned} \tag{10}$$

By arguments similar to (10), we obtain

$$\begin{aligned}\tfrac{1}{3}\theta'(\sigma)\mathrm{E}\left\{(\overline{X}_N-\sigma)^4\theta^{(3)}(\phi_c)\right\} &= \frac{\sigma^2}{3\theta'(\sigma)}c\,\mathrm{E}\left[\left\{(n^*)^{1/2}\overline{Y}_N\right\}^4\theta^{(3)}(\phi_c)\right]\\ &= \frac{\sigma^2}{3\theta'(\sigma)}c\left\{3\theta^{(3)}(\sigma)+o(1)\right\} = \frac{\sigma^2\theta^{(3)}(\sigma)}{\theta'(\sigma)}c + o(c)\end{aligned} \tag{11}$$

and

$$\begin{aligned}&\tfrac{1}{6}\theta''(\sigma)\mathrm{E}\{(\overline{X}_N-\sigma)^5\theta^{(3)}(\phi_c)\}\\ &= \frac{\sigma^3\theta''(\sigma)}{6\{\theta'(\sigma)\}^2}c\,\mathrm{E}\left[\left\{(n^*)^{1/2}\overline{Y}_N\right\}^4\theta^{(3)}(\phi_c)\overline{Y}_N\right] = o(c)\,.\end{aligned} \tag{12}$$

Substituting (7)–(12) into (6), we get

$$\begin{aligned}&R_N - 2cn^*\\ &= \left\{3 + (6-4)\frac{\sigma\theta''(\sigma)}{\theta'(\sigma)} + (7-6+\tfrac{3}{4})\frac{\sigma^2\{\theta''(\sigma)\}^2}{\{\theta'(\sigma)\}^2} + (-2+1)\frac{\sigma^2\theta^{(3)}(\sigma)}{\theta'(\sigma)}\right\}c + o(c)\\ &= \left\{3 + 2\frac{\sigma\theta''(\sigma)}{\theta'(\sigma)} + \frac{7\sigma^2\{\theta''(\sigma)\}^2}{4\{\theta'(\sigma)\}^2} - \frac{\sigma^2\theta^{(3)}(\sigma)}{\theta'(\sigma)}\right\}c + o(c) \quad \text{as } c \to 0\,,\end{aligned}$$

which concludes the theorem. □

3 Example

As an example of the function $\theta(x)$, we consider the estimation of the hazard rate $\theta = \theta(\sigma) = \sigma^{-1}$. Ali and Isogai (2003) considered the bounded risk point estimation problem for the power of scale parameter σ^r of a negative exponential distribution. In this case $\theta(x) = x^r$. In estimating $\theta = \sigma^{-1}$ by $\hat\theta_n = \overline{X}_n^{-1}$, the risk is given by

$$R_n = \mathrm{E}\{L(\hat\theta_n)\} = \mathrm{E}\{(\hat\theta_n-\theta)^2 + cn\} = \mathrm{E}(\overline{X}_n^{-1} - \sigma^{-1})^2 + cn\,,$$

which is finite for $n > 2$. In fact,

$$R_n = \frac{n+2}{(n-1)(n-2)}\sigma^{-2} + cn = \sigma^{-2}n^{-1} + cn + O(n^{-2}) \quad \text{as } n \to \infty\,.$$

Then, $n^* = c^{-1/2}\sigma^{-1}$ and the stopping rule N in (2) becomes

$$N = \inf\left\{n \ge m:\ n \ge c^{-1/2}\overline{X}_n^{-1}\right\}\,. \tag{13}$$

A second order approximation to the expected sample size is given in the next theorem.

Theorem 3 *Suppose* $m \ge 1$. *Then,*

$$\mathrm{E}(N) = n^* + 1 + o(1) \quad \textit{as } c \to 0\,.$$

Proof. From (13), $N = \inf\{n \ge m:\ Z_n \ge n^*\}$ where

$$Z_n \equiv n\overline{X}_n/\sigma = ng(\overline{Y}_n) \quad \text{and} \quad g(x) = x + 1\,.$$

Since $g(x)$ is convex and $\mathrm{E}[\{g(Y_1)\}^+]^3 = \mathrm{E}(X_1/\sigma)^3 < \infty$, from Proposition 5 of Aras and Woodroofe (1993), (A1) and (A2) hold. Since $\theta = \sigma^{-1}$, we have

$$\alpha = -\left(1 + \frac{\sigma\theta''(\sigma)}{\theta'(\sigma)}\right) = 1 \quad \text{and} \quad l = 1 + \frac{\sigma\theta''(\sigma)}{\theta'(\sigma)} + \frac{\sigma^2\{\theta''(\sigma)\}^2}{\{\theta'(\sigma)\}^2} - \frac{\sigma^2\theta^{(3)}(\sigma)}{2\theta'(\sigma)} = 0\,.$$

The stopping time T in (4) becomes $T = \inf\{n \ge 1:\ n + S_n > 0\} = 1$, so that

$$\rho = \frac{\mathrm{E}(T + \alpha S_T)^2}{2\mathrm{E}(T + \alpha S_T)} = \frac{\mathrm{E}(1 + Y_1)^2}{2\mathrm{E}(1 + Y_1)} = 1\,.$$

Theorem 1 with $l = 0$ and $\rho = 1$ yields the theorem. □

Remark 3 *The referee pointed out that for this example* $\mathrm{E}(N)$ *can be calculated by the following elementary method. Let* $S_n^* = \frac{1}{\sigma}\sum_{i=1}^n X_i$. *Then*

$$\begin{aligned}
\mathrm{E}(N) &= m + \sum_{k=m}^{\infty} P(N > k) \ = \ m + \sum_{k=m}^{\infty} P(S_k^* < n^*) \\
&= m + \sum_{k=m}^{\infty}\int_0^{n^*} \frac{1}{(k-1)!}\, x^{k-1}e^{-x}\, dx\,, \\
&= m + n^* - \int_0^{n^*}\sum_{k=0}^{m-2}\frac{x^k}{k!}\, e^{-x}\, dx\,, \quad \textit{by interchanging summation and integral} \\
&= n^* + 1 + e^{-n^*}\sum_{k=0}^{m-2}\sum_{i=0}^{k}\frac{(n^*)^i}{i!} = n^* + 1 + o(1) \quad \textit{as } c \to 0\,.
\end{aligned}$$

We cannot always calculate $\mathrm{E}(N)$ *by elementary methods. Take for example, the function* $\theta(x) = x^r$ *with* $r \neq -1$ *and* $r \neq 0$.

In estimating $\theta = \sigma^{-1}$ by $\hat{\theta}_N = \overline{X}_N^{-1}$, the risk is given by

$$R_N = \mathrm{E}\{L(\hat{\theta}_N)\} = \mathrm{E}\{(\overline{X}_N^{-1} - \sigma^{-1})^2 + cN\}\,.$$

The regret of the procedure (13) is given as follows.

Theorem 4 *If* $m > 12$, *then*

$$R_N - 2cn^* = o(c) \quad \textit{as} \;\; c \to 0\,.$$

It follows from Theorem 4 that our procedure attains the minimum risk $2cn^*$ up to the $o(c)$ term. We need the following lemmas to show Theorem 4. Let M stand for a generic positive constant not depending on c and let $c_0 > 0$ be a constant such that $n^* \geq 1$.

Lemma 1 *Let* $q \geq 1$.
(i) $\{(N/n^*)^{-q},\, c > 0\}$ *is uniformly integrable.*
(ii) *If* $m > q$, *then* $\{(N/n^*)^q,\, 0 < c \leq c_0\}$ *is uniformly integrable.*

Proof. From (13), $(N/n^*)^{-q} \leq (\overline{X}_N/\sigma)^q$. Then, $\sup_{c>0}(\overline{X}_N)^q \leq \sup_{n\geq 1}(\overline{X}_n)^q$ and by Doob's maximal inequality,

$$\mathrm{E}\left\{\sup_{n\geq 1}(\overline{X}_n)^q\right\} \leq \left(\tfrac{q}{q-1}\right)^q \mathrm{E}(X_1)^q < \infty\,.$$

Thus (i) holds. To show (ii), observe that $(N-1)\overline{X}_{N-1}/\sigma < n^*$ on $\{N > m\}$, so that for $0 < c < c_0$,

$$\begin{aligned} N/n^* &\leq \{\sigma\overline{X}_{N-1}^{-1} + (1/n^*)\}I_{\{N>m\}} + (m/n^*)I_{\{N=m\}} \\ &\leq (\sigma\overline{X}_{N-1}^{-1})I_{\{N>m\}} + (m+1)\,, \end{aligned}$$

where $I_{\{\,\cdot\,\}}$ denotes the indicator function. Therefore, for $0 < c \leq c_0$,

$$(N/n^*)^q \leq M\left\{(\sigma\overline{X}_{N-1}^{-1})^q I_{\{N>m\}} + (m+1)^q\right\}\,.$$

We have $\sup_{0<c\leq c_0}(\overline{X}_{N-1})^{-q}I_{\{N>m\}} \leq \sup_{n\geq m}(\overline{X}_n)^{-q}$ and by Doob's maximal inequality, $\mathrm{E}\left\{\sup_{n\geq m}(\overline{X}_n)^{-q}\right\} \leq M\mathrm{E}(\overline{X}_m)^{-q} < \infty$ if $m > q$. Thus, the second assertion holds. □

From Theorem 2 of Chow et al. (1979), we have the next lemma.

Lemma 2 *Let* $q \geq 1$. *If* $\{(N/n^*)^q,\ 0 < c \leq c_0\}$ *is uniformly integrable, then* $\{|(n^*)^{-1/2}\sum_{i=1}^N (X_i - \sigma)|^q,\ 0 < c \leq c_0\}$ *is uniformly integrable.*

Proof of Theorem 4. We shall show (A3) with $u = 3$. Choose $a > 1$ such that $m > 12a$. For $p > 1$ and $q = p/(p-1)$,

$$c^{-3a}\mathrm{E}|\overline{X}_N - \sigma|^{12a} \le M \left\{ \mathrm{E}\left|(n^*)^{-\frac{1}{2}} \sum_{i=1}^{N}(X_i - \sigma)\right|^{12ap} \right\}^{1/p} \left\{\mathrm{E}(n^*/N)^{12aq}\right\}^{1/q},$$

which, together with Lemmas 1 and 2, implies $\sup_{0<c\le c_0}\{c^{-3a}\mathrm{E}|\overline{X}_N - \sigma|^{12a}\} < \infty$. Since $\theta^{(3)}(x) = -6x^{-4}$, by Doob's maximal inequality and the condition that $m > 12a$ we have

$$\sup_{0<c\le c_0} \mathrm{E}|\theta^{(3)}(\overline{X}_N)|^{3a} \le M \sup_{0<c\le c_0} \mathrm{E}(\overline{X}_N)^{-12a}$$
$$\le M\mathrm{E}\left\{\sup_{n\ge m}(\overline{X}_n)^{-12a}\right\} \le M\mathrm{E}(\overline{X}_m)^{-12a} < \infty .$$

So from Remark 1, (A3) holds. Thus, Theorem 2 with

$$3 + 2\frac{\sigma\theta''(\sigma)}{\theta'(\sigma)} + \frac{7\sigma^2\{\theta''(\sigma)\}^2}{4\{\theta'(\sigma)\}^2} - \frac{\sigma^2\theta^{(3)}(\sigma)}{\theta'(\sigma)} = 0$$

proves Theorem 4. □

Simulation. In order to justify the results of Theorems 3 and 4 we shall give brief simulation results. We are interested in the performance of our sequential procedure for various values of σ, and so we consider the cases when $\sigma = 0.5$, 1 and 2, with corresponding $\theta = 2$, 1 and 0.5. Since the cost c is sufficiently small in our theorems, the values of c are chosen such that $n^* = c^{-1/2}\sigma^{-1} = 20,\ 30$. The pilot sample size is set at $m = 13$ which is sufficient for Theorem 4. The simulation results in Tables 1 and 2 are based on 100,000 repetitions by means of the stopping rule N defined by (13). From Theorems 3 and 4 we have that $\mathrm{E}(N) = n^* + 1 + o(1)$ and $(R_N - 2cn^*)/c = o(1)$ as $c \to 0$. Tables 1 and 2 show that these results are justified. Further, it appears that the estimates $\mathrm{E}(\hat{\theta}_N)$ for θ in both tables are very close to the true values. Therefore, our sequential procedure $\hat{\theta}_N$ seems to be effective and useful.

Table 1: $n^* = 20$

	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 2$
$m = 13$	$c = 0.01$ $\theta = 2$	$c = 0.0025$ $\theta = 1$	$c = 0.000625$ $\theta = 0.5$
$\mathrm{E}(N)$	21.055920	21.032170	21.052520
$\mathrm{E}(\hat{\theta}_N)$	2.007637	1.002746	0.501807
$(R_N - 2cn^*)/c$	−0.166272	−0.312113	−0.298089

Table 2: $n^* = 30$

	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 2$
$m = 13$	$c = 0.0044$	$c = 0.0011$	$c = 0.000277$
	$\theta = 2$	$\theta = 1$	$\theta = 0.5$
$\mathrm{E}(N)$	30.996970	31.031260	30.997910
$\mathrm{E}(\hat{\theta}_N)$	2.001831	1.002166	0.500437
$(R_N - 2cn^*)/c$	-0.104954	-0.201335	-0.222469

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