Representations for Integral Functionals of Kernel Density Estimators

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Abstract: We establish a representation as a sum of independent random variables, plus a remainder term, for estimators of integral functionals of the density function, which have a certain simple structure. From this representation we derive a central limit theorem, a law of large numbers and a law of the iterated logarithm.

Keywords: Kernel Density Estimators, Integral Functionals, Law of Large Numbers, Central Limit Theorem, Law of the Iterated Logarithm.

1 Introduction

For several decades there has been considerable interest in the problem of estimating integral functionals of the density function of the form

\[ T(f) = \int_{\mathbb{R}} \varphi \left( f(x), f'(x), \ldots, f^{(k)}(x), x \right) \, dx, \]  

where \( f \) is a Lebesgue density. Such integral functionals appear in the asymptotic variance of certain nonparametric statistics. They also arise in a number of bandwidth selection procedures for density estimators. (Refer, especially, to Bickel and Ritov, 1988; Birgé and Massart, 1995; Hall and Marron, 1987; Laurent, 1996, 1997; Levit, 1978; Nadaraya, 1989, along with the references therein.). Some motivating special cases are

\[ T_1(f) := \int_{\mathbb{R}} \left( f'(x) \right)^2 \, dx, \quad T_2(f) := \int_{\mathbb{R}} f^2(x) \, dx \quad \text{and} \quad T_3(f) := \int_{\mathbb{R}} f(x) \log(f(x)) \, dx. \]

Let \( \hat{f}_h(x) \) denote the kernel density estimator

\[ \hat{f}_h(x) = n^{-1} \sum_{i=1}^{n} h^{-1} K \left( \frac{x-X_i}{h} \right) := n^{-1} \sum_{i=1}^{n} K_h \left( x-X_i \right), \]  

where \( K \) is a kernel satisfying conditions (K.i) and (K.ii) below and \( h_n = h \) is a sequence of positive constants converging to zero. Set

\[ f_h(x) = E\hat{f}_h(x) = f * K_h(x). \]

We will assume that \( K \) satisfies

(K.i) \( K \) is non-negative and bounded by some \( 0 < \kappa < \infty \);

(K.ii) \( \int_{-\infty}^{\infty} K(u) \, du = 1. \)

Sometimes we will also impose the condition

(K.iii) \( \int_{\mathbb{R}} \Psi(x) \, dx < \infty \), where \( \Psi(x) = \sup_{|y| \geq |x|} |K(y)|. \)
Here are some examples of kernels that satisfy these conditions.

1. \( K(u) = 1 \{ u \in [-1/2, 1/2] \} \) (Uniform Kernel);
2. \( K(u) = \frac{3}{4} (1 - u^2) \) (Epanechnikov Kernel);
3. \( K(u) = \frac{1}{2} \exp(-|u|) \) (Double exponential Kernel);
4. \( K(u) = \frac{1}{\sqrt{2\pi}} \exp(-u^2/2) \) (Standard Normal Kernel).

Assuming a smooth enough density \( f \), Eggermont and LaRiccia (1999) (also see their monograph Eggermont and LaRiccia, 2001) established an almost sure representation for the kernel density function negative entropy estimator

\[
T(\hat{f}_h) = \int_{\mathbb{R}} \hat{f}_h(x) \log(\hat{f}_h(x)) \, dx,
\]

(4)

where \( K(u) = 2^{-1} \exp(-|u|), u \in \mathbb{R} \), is the double exponential kernel. Namely they proved under suitable regularity conditions on \( h = h_n \) and \( f \) that the following representation as a sum of independent random variables, plus a remainder term, holds for \( T(\hat{f}_h) \):

\[
T(\hat{f}_h) - T(f) = n^{-1} \sum_{i=1}^{n} \{ \log f(X_i) - E \log f(X_i) \} + R_n^*,
\]

where \( R_n^* = o\left(n^{-1/2}\right), \ a.s. \) From this representation they inferred a law of the iterated logarithm [LIL] for \( T(\hat{f}_h) \), along with best asymptotic normality by applying a result of Levit (1978). Their approach was based upon a special submartingale structure of the estimator, which follows from a Green’s function property possessed by the double exponential kernel. For other approaches towards proving consistency and best asymptotic normality of estimators of integral functionals of \( f \) consult Laurent (1996), who concentrates on the special cases \( T_2(f) \) and \( T_3(f) \), and the references therein.

It would be very useful to develop a general approach to establish the asymptotic distribution and exact rate of consistency under minimal assumptions of estimators of \( T(f) \) of the form

\[
\int_{\mathbb{R}} \varphi \left( \hat{f}_h(x), \hat{f}_h^{(1)}(x), \ldots, \hat{f}_h^{(k)}(x), x \right) \, dx,
\]

(5)

where for each \( 1 \leq i \leq k \), \( \hat{f}_h^{(i)} \) is the \( i \)th derivative of \( \hat{f}_h \), when \( K \) is \( i \) times continuously differentiable. However, at present this goal appears to be much too ambitious. In this note we obtain a representation in terms of a sum of independent random variables, plus a remainder term, for integral functionals of density estimators having a certain simple structure. From our representation we will derive a central limit theorem, a law of large numbers and an LIL for \( T(\hat{f}_h) \). In the last section we will apply our results to the special case of the kernel density estimator \( T_2(\hat{f}_h) \) of the integrated squared density \( T_2(f) \).

2 Representations of Integral Functionals of Density Estimators

In this note we shall restrict ourselves to integral functionals of the following form:

\[
T(f) = \int_{\mathbb{R}} f(x) \phi(f(x)) \, dx,
\]

(6)
where \( \phi \) is a twice continuously differentiable function defined on an open set containing the closure of the image of \( f \), \( \{ f(x) : x \in R \} \). The motivating special cases are \( T_2(f) \) and \( T_3(f) \). Set \( \Phi(u) = u\phi(u) \),

\[
\Phi_1(u) = \frac{d}{dx} \Phi(x)|_{x=u} \quad \text{and} \quad \Phi_2(u) = \frac{d}{dx} \Phi_1(x)|_{x=u}. \tag{7}
\]

By Taylor’s formula we have

\[
T(\hat{f}_h) - T(f_h) = \int_R \Phi_1(f_h(x)) \left( \hat{f}_h(x) - f_h(x) \right) dx + R_n, \tag{8}
\]

where

\[
R_n = \int_R \left\{ \int_{f_h(x)}^{\hat{f}_h(x)} \left( \hat{f}_h(x) - u \right) \Phi_2(u) du \right\} dx. \tag{9}
\]

We shall confine ourselves to determining the rate at which the remainder term \( R_n \) in (9) converges to zero, when there exists a positive nondecreasing function \( \varphi \) with derivative \( \varphi' \) such that

\[
(\varphi.i) |\Phi_2(u)| \leq \varphi'(u);

(\varphi.ii) \varphi \text{ satisfies a Hölder condition with exponent } 0 < \alpha \leq 1.
\]

Notice that the class of integral functionals satisfying these conditions includes the \( T_2(f) \) functional, but not the negative entropy functional \( T_3(f) \). To establish a useful representation for \( T_3(f) \) under minimal assumptions requires too much space and therefore will be presented elsewhere.

**Theorem 1.** Assume that \( K \) satisfies (K.i) and (K.ii) and that (\( \varphi.\text{i} \)) and (\( \varphi.\text{ii} \)) hold. Then for \( R_n \) in (9) and any sequence of positive \( h = h_n \leq 1 \), we have

\[
R_n = O \left( \left( \frac{\log n}{n} \right)^{(1+\alpha)/2} \frac{1}{(nh^\alpha)} \right), \quad \text{a.s.} \tag{10}
\]

and

\[
R_n = O_P \left( \frac{1}{(nh^\alpha)} \right). \tag{11}
\]

**Corollary 1.** In addition to assumptions (K.i), (K.ii), (\( \varphi.\text{i} \)) and (\( \varphi.\text{ii} \)), assume that there exists a \( C_\alpha > 0 \) such that

\[
\sup_{0 < h \leq 1} \int_R \{ f_h(x) \}^{(1+\alpha)/2} dx \leq C_\alpha. \tag{12}
\]

Then for \( R_n \) in (9) and any sequence of positive \( h = h_n \leq 1 \), we have

\[
R_n = O \left( \left( \frac{\log n}{n} \right)^{(1+\alpha)/2} \frac{1}{(nh^\alpha)} \right), \quad \text{a.s.} \tag{13}
\]

and

\[
R_n = O_P \left( \frac{1}{(nh)^{(1+\alpha)/2}} \right). \tag{14}
\]
Remark 1. Note that for $0 < \alpha < 1$, the $R_n$ in (10) is $= O \left( 1 / (nh)^\alpha \right)$, a.s., and when $\alpha = 1$, this $R_n = O \left( \log n / (nh) \right)$, a.s.

Remark 2. Assume $h = h_n = n^{-\gamma}$ for some $\gamma > 0$. Notice that under the assumptions of Theorem 1, if $1/2 < \alpha \leq 1$ and $1 - \frac{1}{2\alpha} > \gamma > 0$, then $R_n = o(n^{-1/2})$, a.s., and under the assumptions of Corollary 1, if we choose $0 < \gamma < \alpha/(1 + \alpha)$ then $R_n = o(n^{-1/2})$, a.s.

Remark 3. Condition (12) is satisfied whenever for some $\lambda > (1 - \alpha)/(1 + \alpha)$,

$$\int_{\mathbb{R}} |x|^\lambda f(x)dx + \int_{\mathbb{R}} |y|^\lambda K(y)dy < \infty.$$ 

Consult Devroye (1987) for a proof of this fact.

2.1 Proof of Theorem 1

Notice that by ($\varphi_i$), the remainder term $R_n$ defined in (9) satisfies

$$|R_n| \leq \int_{\mathbb{R}} \left( \hat{f}_h(x) - f_h(x) \right) \left( \varphi(\hat{f}_h(x)) - \varphi(f_h(x)) \right) dx,$$

which by ($\varphi_{ii}$) is, for some $C > 0$,

$$\leq C \int_{\mathbb{R}} \left( |\hat{f}_h(x) - f_h(x)|^{1+\alpha} \right) dx =: C \Delta_n(\alpha).$$ (15)

Write for $i = 1, 2, \ldots, n$,

$$Y_i(x) = n^{-1} \{ K_h(x - X_i) - f_h(x) \},$$

so that

$$(\Delta_n(\alpha))^{1/(1+\alpha)} = \left\| \sum_{i=1}^{n} Y_i \right\|_{1+\alpha}. \quad (16)$$

Furthermore, for each $i = 1, 2, \ldots, n$, we get

$$\left\| n^{-1} K_h (\cdot - X_i) \right\|_{1+\alpha} = \frac{1}{nh^{\alpha/(1+\alpha)}} \left( \int_{\mathbb{R}} h^{-1} K \left( \frac{x-X_i}{h} \right)^{1+\alpha} dx \right)^{1/(1+\alpha)} = \left\| K \right\|_{1+\alpha}.$$ 

Thus by applying the McDiarmid inequality, exactly as on page 39 of Devroye (1991) (also refer to Pinelis (1990)), we get for any $t > 0$,

$$P \left\{ \left\| \sum_{i=1}^{n} Y_i \right\|_{1+\alpha} - E \left\| \sum_{i=1}^{n} Y_i \right\|_{1+\alpha} > t \right\} \leq 2 \exp \left( -\frac{t^2 n h^{2\alpha/(1+\alpha)}}{2 \left\| K \right\|^2_{1+\alpha}} \right).$$ (17)

From this inequality we readily conclude via the Borel-Cantelli lemma that

$$\left\| \sum_{i=1}^{n} Y_i \right\|_{1+\alpha} = E \left\| \sum_{i=1}^{n} Y_i \right\|_{1+\alpha} + O \left( \frac{\sqrt{\log n}}{\sqrt{nh^{\alpha/(1+\alpha)}}} \right), \text{ a.s.}$$ (18)
Now
\[
\left( E \left\| \sum_{i=1}^{n} Y_i \right\|_{1+\alpha} \right)^{1+\alpha} \leq E \left\| \sum_{i=1}^{n} Y_i \right\|_{1+\alpha}^{1+\alpha} = \int_{\mathbb{R}} E \left\| \sum_{i=1}^{n} Y_i(x) \right\|_{1+\alpha}^{1+\alpha} dx,
\]
which by the moment inequality of von Bahr and Esseen (1965) is
\[
\leq 2 \sum_{i=1}^{n} \int_{\mathbb{R}} E|Y_i(x)|^{1+\alpha} dx.
\]
Next observe that by Jensen’s inequality for each \( i = 1, \ldots, n \),
\[
\sum_{i=1}^{n} \int_{\mathbb{R}} E|Y_i(x)|^{1+\alpha} dx \leq 2\sum_{i=1}^{n} \frac{2^n}{n^{1+\alpha}} \left[ \int_{\mathbb{R}} E(K_h (x - X_i))^{1+\alpha} dx + \int_{\mathbb{R}} \left( \int_{\mathbb{R}} K_h (x - y) f(y) dy \right)^{1+\alpha} dx \right],
\]
which, in turn, again by Jensen’s inequality and \( |K| \leq \kappa \), is
\[
\leq \frac{2^{\alpha+1} \kappa^n}{n^{\alpha} h^{\alpha} h^{\alpha}} \int_{\mathbb{R}^2} K_h (x - y) f(y) dy dx = \frac{2^{\alpha+1} \kappa^{\alpha}}{n^{\alpha} h^{\alpha}}. \tag{19}
\]
Thus we have by (15), (16), (18) and (19),
\[
\Delta_n(\alpha) = O \left( \frac{\log n}{n^{(1+\alpha)/2} h^{\alpha}} + \frac{1}{(nh)^{\alpha}} \right), \text{ a.s.}
\]
Assertion (11) follows in the same way from (17). \( \square \)

### 2.2 Proof of Corollary 1

We will only prove (13). Assertion (14) is proved similarly. Notice that
\[
E \left\| \sum_{i=1}^{n} Y_i \right\|_{1+\alpha}^{1+\alpha} = \int_{\mathbb{R}} E \left| \sum_{i=1}^{n} Y_i(x) \right|^{1+\alpha} dx \leq \int_{\mathbb{R}} \left[ E \left( \sum_{i=1}^{n} Y_i(x) \right)^2 \right]^{(1+\alpha)/2} dx,
\]
which by using \( |K| \leq \kappa \), is
\[
\leq \frac{\kappa^{(1+\alpha)/2}}{(nh)^{(1+\alpha)/2}} \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} K_h (x - y) f(y) dy \right]^{(1+\alpha)/2} dx.
\]
By assumption (12) this last bound is \( \leq \kappa^{(1+\alpha)/2} C_{\alpha}/(nh)^{(1+\alpha)/2} \). Hence from (18) we conclude that whenever (12) holds we have almost surely that
\[
\Delta_n(\alpha) = O \left( \frac{\log n}{n^{(1+\alpha)/2} h^{\alpha}} + \frac{1}{(nh)^{(1+\alpha)/2}} \right). \quad \square
\]
3 Central Limit Theorem and Law of Large Numbers

Our next theorem, when combined with Theorem 1 and Corollary 1, provides conditions under which the central limit theorem and law of large numbers hold for \( T(\hat{f}_h) \). Introduce the assumptions

(\( \Phi. i \)) \( f \) is bounded by some constant \( M > 0 \);  
(\( \Phi. ii \)) \( \Phi_1 \) is continuous and bounded by a constant \( L \) on \([0, M]\).

Write for \( h > 0 \),

\[
S_n(h) := \int_{\mathbb{R}} \Phi_1(f_h(x))(\hat{f}_h(x) - f_h(x))\,dx.
\]

**Theorem 2.** Assume that \( K \) satisfies (K.i), (K.ii) and (K.iii) and (\( \Phi. i \)) and (\( \Phi. ii \)) hold. Then for any sequence of positive constants \( h = h_n \leq 1 \) converging to zero and satisfying

\[
T(\hat{f}_h) - T(f_h) = S_n(h) + R_n, 
\]

with \( R_n = o(1) \) almost surely, we have, with probability 1, as \( n \to \infty \),

\[
T(\hat{f}_h) \to \int_{\mathbb{R}} \Phi_1(f(y))f(y)dy, 
\]

and whenever \( R_n = o_p\left(n^{-1/2}\right) \) in (12), we have, as \( n \to \infty \),

\[
\sqrt{n}\left\{T(\hat{f}_h) - T(f_h)\right\} \to d N \left(0, \sigma_f^2(\Phi_1)\right), 
\]

where

\[
\sigma_f^2(\Phi_1) := \int_{\mathbb{R}} \Phi_1^2(f(y))f(y)dy - \left(\int_{\mathbb{R}} \Phi_1(f(y))f(y)dy\right)^2. 
\]

**Remark 4.** Applying the results of Levit (1978) (also consult Laurent, 1996) we see that \( \sigma_f^2(\Phi_1) \) is the variance of any best asymptotically normal estimator of \( T(f) \).

3.1 Proof of Theorem 2

Our proof requires the following fact and lemma. First we need a special case of a result in Stein (1970) (also consult Devroye and Györfi, 1985).

**Fact 1.** (Stein) Let \( \varphi \) be a measurable function on \( \mathbb{R} \) such that \( \int_{\mathbb{R}} \varphi(x)dx = 1 \) and set \( \varphi_\varepsilon(x) = \varepsilon^{-1}\varphi(\varepsilon^{-1}x), \varepsilon > 0 \). Further let \( \Psi(x) = \sup_{|y| \geq |x|} |\varphi(y)| \) and assume that \( \int_{\mathbb{R}} \Psi(x)dx < \infty \). Then for all functions \( g \) in \( L_1(\mathbb{R}) \) and almost all \( x \in \mathbb{R} \),

\[
\lim_{\varepsilon \to 0} g * \varphi_\varepsilon(x) = g(x). 
\]

For for \( h > 0 \), define

\[
Z(h) = \int_{\mathbb{R}} \Phi_1(f_h(x))K_h(x-X)dx, 
\]

where \( X \) has density \( f \).
Lemma 1. Under the assumptions of Theorem 2, as \( h \downarrow 0 \),
\[
E(Z(h)) = \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} \Phi_1(f_h(s))K_h(s-y) \, ds \right] f(y) \, dy \rightarrow \int_{\mathbb{R}} \Phi_1(f(y))f(y) \, dy \tag{26}
\]
and
\[
E(Z(h))^2 = \int_{\mathbb{R}} \left[ \int_{\mathbb{R}^2} \Phi_1(f_h(s))\Phi_1(f_h(t))K_h(s-y)K_h(t-y) \, ds \, dt \right] f(y) \, dy
\rightarrow \int_{\mathbb{R}} \Phi_1^2(f(y))f(y) \, dy < \infty. \tag{27}
\]

Proof. First consider (27). Notice that after a change of variables the left hand side of (27) is equal to
\[
\int_{\mathbb{R}} \left[ \int_{\mathbb{R}^2} \Phi_1(f_h(y+hu))\Phi_1(f_h(y+hv))K(u)K(v) \, dudv \right] f(y) \, dy.
\]
Writing
\[
f_h(y+hu) = \int_{\mathbb{R}} h^{-1}K \left( \frac{y-t}{h} + u \right) f(t) \, dt,
\]
we see by the Stein Fact 1 that for almost every \( y \) and all \( u \), as \( h \downarrow 0 \),
\[
f_h(y+hu) \rightarrow f(y).
\]
Therefore, since \(|f_h| \leq M\), we have by (\( \Phi.ii \)) that for almost every \( y \) and every \( u \) and \( v \),
\[
\Phi_1(f_h(y+hu))\Phi_1(f_h(y+hv)) \rightarrow \Phi_1^2(f(y)).
\]
Thus by the bounded convergence theorem for almost every \( y \),
\[
\varphi_h(y) := \int_{\mathbb{R}^2} \Phi_1(f_h(y+hu))\Phi_1(f_h(y+hv))K(u)K(v) \, dudv \rightarrow \Phi_1^2(f(y)).
\]
Moreover, since each \( \varphi_h \) is bounded, we have as \( h \downarrow 0 \),
\[
\int_{\mathbb{R}} \varphi_h(y)f(y) \, dy \rightarrow \int_{\mathbb{R}} \Phi_1^2(y)f(y) \, dy.
\]
This proves (27). Assertion (26) is proved similarly. \( \square \)

Turning to the proof of Theorem 2, write for \( i = 1, \ldots, n \),
\[
Z_i(h) := \int_{\mathbb{R}} \Phi_1(f_h(x))K_h(x-X_i) \, dx, \tag{28}
\]
where each \( X_1, \ldots, X_n \) are i.i.d. with density \( f \). Note that since each \(|Z_i(h)| \leq L\) and
\[
S_n(h) = n^{-1} \sum_{i=1}^n \{Z_i(h) - EZ_i(h)\},
\]
we can apply Hoeffding’s inequality to get for all \( t > 0 \),
\[
P \left\{ \left| S_n(h) \right| > t / \sqrt{n} \right\} \leq 2 \exp \left( -2t^2 / L^2 \right),
\]
from which we readily check using the Borel–Cantelli lemma that almost surely
\[
S_n(h) = O \left( \sqrt{\log n / n} \right). \tag{29}
\]
Thus (21) is a consequence of (26).

Turning to the proof of (22), observe by (26) and (27), as \( h \downarrow 0 \),
\[
\text{Var}(Z(h)) \to \sigma_f^2(\Phi_1).
\]
Thus (22) follows directly from \( |Z_i(h)| \leq L \), combined with Liapounov’s central limit theorem (e.g. Shorack, 2000).

\[\square\]

4 Laws of the Iterated Logarithm

In this section we introduce a general approach to obtain laws of the iterated logarithm [LIL] for \( T(\hat{f}_h) \). This will be based upon the following special case of a LIL stated as Theorem 4.2 in Kuelbs (1976). Let \( C[0,h_0] \), \( h_0 > 0 \), denote the space of continuous functions on \([0, h_0]\) equipped with the supremum norm \( \| \cdot \|_\infty \).

**Proposition 1.** Let \( Z, Z_1, Z_2, \ldots \) be a sequence of i.i.d. \( C[0,h_0] \), \( h_0 > 0 \), valued random variables satisfying for all \( h \in [0, h_0] \),
\[
\text{Var}Z(h) = \sigma^2(h) < \infty. \tag{30}
\]
Also assume there exists a nonnegative random variable \( W \) satisfying \( EW^2 < \infty \) and an \( 0 < \gamma \leq 1 \) such that for all \( h_1, h_2 \in [0, h_0] \)
\[
|Z(h_1) - Z(h_2)| \leq W |h_1 - h_2|^\gamma. \tag{31}
\]
Then
\[
\limsup_{n \to \infty} \sup_{h \in [0,h_0]} \pm \frac{\sum_{i=1}^n \{Z_i(h) - EZ_i(h)\}}{\sqrt{2n \log \log n}} = \tau, \text{ a.s.} \tag{32}
\]
where
\[
\tau = \sup_{h \in [0,h_0]} \sigma(h). \tag{33}
\]
We will soon be applying the following straightforward corollary to Proposition 1.

**Corollary 2.** In addition to the assumptions of Proposition 1, assume that
\[
\lim_{h \downarrow 0} \text{Var}(Z(h) - Z(0)) = 0. \tag{34}
\]
Then for any sequence of positive constants \( h_n \in [0, h_0] \) converging to zero,
\[
\limsup_{n \to \infty} \pm \frac{\sum_{i=1}^n \{Z_i(h_n) - EZ_i(h_n)\}}{\sqrt{2n \log \log n}} = \sigma(0), \text{ a.s.} \tag{35}
\]
4.1 The Aim of All This

The aim of all this is to show, under suitable regularity conditions, that almost surely

\[
\limsup_{n \to \infty} \frac{\sqrt{n} \left( T(\hat{f}_n) - T(f_n) \right)}{\sqrt{2 \log \log n}} = \limsup_{n \to \infty} \frac{\sum_{i=1}^{n} \{ Z_i(h_n) - EZ_i(h_n) \}}{\sqrt{2n \log \log n}} = \sigma(0),
\]

where for \( h > 0 \), \( Z_1(h), Z_2(h), \ldots \), are i.i.d. \( Z(h) \), with \( Z(h) \) as in (25), and \( \sigma^2(0) = \sigma_f^2(\Phi_1) \), whenever \( h = h_n \) is such that \( R_n = o\left(\sqrt{\log \log n/n}\right) \), a.s.

Example. Assume that \( \Phi_1 \) is Holder of order \( 0 < \alpha \leq 1 \) and \( f \) satisfies the condition that

\[
|f(x + y) - f(x)| \leq g(x)|y|^\beta,
\]

where \( E(g^\alpha(X))^2 < \infty \). Also assume that

\[
\int_{-\infty}^{\infty} |u|^\beta K(u)du < \infty.
\]

To verify that (31) holds, observe that by two changes of variables

\[
Z(h) = \int_{\mathbb{R}} \Phi_1 \left( \int_{\mathbb{R}} f(X + h (u - v)) K(v)dv \right) K(u)du,
\]

Therefore for some constant \( C > 0 \)

\[
|Z(h_1) - Z(h_2)| \leq C \int_{\mathbb{R}} \left\{ f(X + h_1 (u - v)) - f(X + h_2 (u - v)) \right\} K(v)dv \| u \|^\alpha K(u)du,
\]

which by concavity \( (0 < \alpha \leq 1) \) and (37) is for some constant \( B > 0 \)

\[
\leq B g^\alpha(X) \int_{\mathbb{R}} \left( |u|^\beta + |v|^\beta \right) K(u)K(v)du \| u \|^\alpha |h_1 - h_2|^\alpha \beta =: Ag^\alpha(X)|h_1 - h_2|^\alpha \beta.
\]

Therefore (31) holds with \( W = Ag^\alpha(X) \) and \( \gamma = \alpha \beta \).

Remark 5. Assumption (37) is a generalization of a condition of Bickel and Ritov (1988).

5 Application to the Estimation of the Integrated Squared Density

The functional \( T_2(f) \) serves as a basic quantity that appears in expressions for the asymptotic efficiency of the Wilcoxon signed rank statistic for location hypotheses and the asymptotic variance of the Hodges-Lehmann location estimator. (Refer, for instance, to...
Randles and Wolfe, 1991). In our final section we apply our results to the estimation of \( T_2(f) \). First we employ Theorem 1 with \( \phi(u) = u \) and \( \alpha = 1 \) to give for any density \( f \),

\[
T_2(\hat{f}_h) - T_2(f_h) = 2 \int_{\mathbb{R}} f_{hn}(x) \left( \hat{f}_h(x) - f_h(x) \right) dx + R_n
\]

where

\[
R_n = O \left( \frac{\log n}{nh} \right), \text{ a.s. and } R_n = O_p \left( \frac{1}{nh} \right).
\]

Now assume that \( f \) is bounded. We see that whenever \( h_n \to 0 \) and \( nh_n/\log n \to \infty \), we have \( R_n = o(1) \) a.s., which by Theorem 2, implies

\[
T_2(\hat{f}_h) \to T_2(f), \text{ a.s. (39)}
\]

Further, whenever \( h_n \to 0 \) and \( \sqrt{n}h_n \to \infty \), yielding \( R_n = o_p(n^{-1/2}) \), we get by Theorem 2 that

\[
\sqrt{n} \left\{ T_2(\hat{f}_h) - T_2(f_h) \right\} \to_d N \left( 0, \sigma_f^2(\Phi_1) \right), \tag{40}
\]

where

\[
\sigma_f^2(\Phi_1) = 4 \left[ \int_{\mathbb{R}} f^3(x)dx - \left( \int_{\mathbb{R}} f^2(x)dx \right)^2 \right].
\]

Moreover, whenever \( h_n \to 0 \) and \( h_n \sqrt{n \log \log n} \to \infty \), giving \( R_n = o \left( \sqrt{\log \log n/n} \right) \) a.s., and both (37) and (38) hold for some \( 0 < \beta \leq 1 \), then by Proposition 1 and the Example above,

\[
\limsup_{n \to \infty} \frac{\sqrt{n} \left\{ T_2(\hat{f}_h) - T_2(f_h) \right\}}{\sqrt{2 \log \log n}} = \sigma_f(\Phi_1), \text{ a.s. (41)}
\]

Remark 6. By taking advantage of the \( U \)–statistic structure of \( T_2(f_h) \), Nadaraya (1989) establishes (39) in his Theorem 2.2 under the same assumptions that we impose on \( K \), but with our condition \( ||f||_{\infty} < \infty \) replaced by the weaker assumption that \( T_2(f) < \infty \). However, he requires a stronger condition on \( h_n \), namely he substitutes our \( nh_n/\log n \to \infty \) by the more restrictive condition that \( nh_n^2/\log n \to \infty \). It is also interesting to compare his Theorem 2.3 with our assumptions leading to (40). He shows that with more smoothness assumptions \( T_2(f_h) \) can be replaced by \( T_2(f) \) in (40). Another approach to the LIL for \( T_2(f_h) \), which would require fewer regularity conditions on the density function, can be based on a moderate deviation result for \( T_2(f_h) \). For closely related details consult Giné and Mason (2002).

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