AUSTRIAN JOURNAL OF STATISTICS
Volume 28 (1999), Number 4, 203–214


# A Multiepoch Regression Model used in Geodesy[1]

Lubomír Kubáček, Ludmila Kubáčková
Palacký University, Olomouc, Czech Republic

**Abstract**: An investigation of the deformations of large buildings (bridges, dams, etc.) needs replicated measurements in special types of geodetical networks. They are characterized by two groups of points creating the network; one group is formed by points with stable positions and the other one is formed by points located on the building and characterizing its deformations. A statistical analysis of measurement results is done after each epoch of measurement and also after several epochs. It is of a practical importance to develop an algorithm of estimation which enables us to use the partial results obtained after each epoch for results after several epochs.

**Zusammenfassung**: Die Erforschung von Deformationen von großen Bauten (Brücken, Dämme etc.) benötigt wiederholte Messungen eines speziellen Typs von geodätischen Netzwerken. Diese werden durch zwei Gruppen von Punkten charakterisiert, die das Netzwerk definieren; eine Gruppe besteht aus Punkten auf stabilen Lagen und die andere aus Punkten, die an den Gebäuden plaziert sind und ihre Deformationen charakterisieren. Eine statistische Analyse von Meßergebnissen wird dargestellt nach jeder Epoche von Messungen und auch nach mehreren Epochen. Es ist von großer praktischer Bedeutung, einen Algorithmus der Schätzung zu entwickeln, welcher uns in die Lage versetzt, Teilresultate nach jeder Epoche für Ergebnisse nach mehreren Epochen zu verwenden.

**Keywords**: Multiepoch Regression Model, Variance Components.

## 1 Introduction

Deformations of large buildings (bridges, dams, gas holders, etc.) can be studied by the help of special geodetic networks constructed for this purpose. Let us consider a network which consists of two groups of points: the first one consists of points with stable positions which are located in a neighbourhood of the investigated object and the other one which consists of points located on the object. The movement of the points of the object characterize the investigated deformations.

The first measurement (the first epoch) of the positions of all mentioned points is made before the first loading, the second measurement (the second epoch) is made after the first loading, the third measurement (the third epoch) after the second loading, etc.

After each measurement set (epoch) we obtain approximately the same positions of stable points and new positions of the points located on the investigated object. Positions

[1]Supported by the grant No. 201/99/0327 of the Grant Agency of the Czech Republic and the project of the European Union "Erasmus–Socrates" common with the Department of Statistics, Probability and Actuarial Mathematics, Vienna University of Technology.

of the last points create trajectories of each single point and they enable us to study the deformation and rotation tensor on the investigated object.

The aim of the paper is to study algorithms for estimating the point positions and their accuracy in the framework of a single epoch and the same when the first $m$ $(> 1)$ epochs are realized.

The results after $m$ epochs should be expressed, if possible, as a function of results of separate epochs.

## 2 Notation and Auxiliary Statements

Let $\mathbf{Y}_j$ be the $n$-dimensional observation vector normally distributed, i.e. $\mathbf{Y}_j \sim N_n[\mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_{2,j}, \boldsymbol{\Sigma}]$ in the $j$-th epoch, $j = 1, \ldots, m$. Here $(\mathbf{X}_1, \mathbf{X}_2)$ is an $n \times (k_1 + k_2)$ design matrix which is considered to be the same in all epochs $j = 1, \ldots, m$. The $k_1$-dimensional vector $\boldsymbol{\beta}_1$ characterizes the actual positions of the stable points, the $k_2$-dimensional vector $\boldsymbol{\beta}_{2,j}$ characterizes the actual positions of the points located on the investigated object in the $j$th epoch. The $n \times n$ matrix $\boldsymbol{\Sigma}$ is assumed to be positive definite and the rank of the matrix $(\mathbf{X}_1, \mathbf{X}_2)$ is assumed to be $r(\mathbf{X}_1, \mathbf{X}_2) = k_1 + k_2 < n$. Further $\boldsymbol{\beta}_{2,(\cdot)} = (\boldsymbol{\beta}'_{2,1}, \boldsymbol{\beta}'_{2,2}, \ldots, \boldsymbol{\beta}'_{2,m})'$, $\mathbf{1} = (1, \ldots, 1)' \in \mathcal{R}^m$, $\mathbf{P}_m = (1/m)\mathbf{1}\mathbf{1}'$, $\mathbf{M}_m = \mathbf{I} - \mathbf{P}_m$.

**Definition 2.1** The $m$-epoch regression model with the fixed design matrix is

$$\begin{pmatrix} \mathbf{Y}_1 \\ \vdots \\ \mathbf{Y}_m \end{pmatrix} = \mathbf{Y}^{(m)} = N_{mn}\left[(\mathbf{1} \otimes \mathbf{X}_1, \mathbf{I} \otimes \mathbf{X}_2)\begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_{2,(\cdot)} \end{pmatrix}, \mathbf{I} \otimes \boldsymbol{\Sigma}\right],$$

$$\boldsymbol{\beta}_1 \in R^{k_1}, \quad \boldsymbol{\beta}_{2,j} \in R^{k_2}. \tag{1}$$

**Lemma 2.2** Let $\boldsymbol{\eta} \sim N_n(\mathbf{A}\boldsymbol{\Theta}, \boldsymbol{\Sigma})$, $\mathbf{A}$ be an $n \times k$ matrix with the rank $r(\mathbf{A}) = k < n$ and $\boldsymbol{\Sigma}$ be a positive definite matrix.

(i) If $\boldsymbol{\Sigma}$ is given, then the BLUE (best linear unbiased estimator) of $\boldsymbol{\Theta}$ is

$$\hat{\boldsymbol{\Theta}} = (\mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A})^{-1}\mathbf{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\eta} \sim N_k\left(\boldsymbol{\Theta}, (\mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A})^{-1}\right).$$

(ii) If $\boldsymbol{\Sigma} = \sigma^2\mathbf{V}$, where $\sigma^2 \in (0, \infty)$ is an unknown parameter and $\mathbf{V}$ is a known $n \times n$ positive definite matrix, the UBLUE (uniformly best linear unbiased estimator) of $\boldsymbol{\Theta}$ is

$$\hat{\boldsymbol{\Theta}} = (\mathbf{A}'\mathbf{V}^{-1}\mathbf{A})^{-1}\mathbf{A}'\mathbf{V}^{-1}\boldsymbol{\eta} \sim N_k\left(\boldsymbol{\Theta}, \sigma^2(\mathbf{A}'\mathbf{V}^{-1}\mathbf{A})^{-1}\right);$$

the uniformly best estimator of $\sigma^2$ is

$$\hat{\sigma}^2 = \frac{\boldsymbol{\eta}'(\mathbf{M}_A\mathbf{V}\mathbf{M}_A)^+\boldsymbol{\eta}}{n-k} \sim \sigma^2\frac{\chi^2_{n-k}(0)}{n-k}.$$

Here $\mathbf{M}_A = \mathbf{I} - \mathbf{P}_A, \mathbf{P}_A = \mathbf{A}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'$ and $^+$ denotes the Moore–Penrose generalized inverse (see, e.g., Rao and Mitra, 1971).

(iii) Let $\boldsymbol{\Sigma}(\boldsymbol{\vartheta}) = \sum_{i=1}^{p}\vartheta_i\mathbf{V}_i$, $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_p)' \in \underline{\vartheta}$ (open set in $R^p$), where $\boldsymbol{\vartheta}$ is an unknown vector parameter and $\mathbf{V}_i = \mathbf{V}'_i$, $i = 1, \ldots, p$, be known matrices and let the

the implication "$\boldsymbol{\vartheta} \in \underline{\vartheta} \Rightarrow \boldsymbol{\Sigma}(\boldsymbol{\vartheta})$ is positive definite" be valid. Let further the matrix $\mathbf{S}_{(M_A\Sigma_0 M_A)^+}$ (3) be regular. Then

(a) the $\boldsymbol{\vartheta}_0$-LBLUE (locally best linear unbiased estimator) of $\boldsymbol{\Theta}$ is

$$\hat{\boldsymbol{\Theta}} = (\mathbf{A}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{A})^{-1}\mathbf{A}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\boldsymbol{\eta} \sim N_k\left[\boldsymbol{\Theta}, (\mathbf{A}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{A})^{-1}|\boldsymbol{\vartheta} = \boldsymbol{\vartheta}_0\right]$$

(the notation $N_k\left[\boldsymbol{\Theta}, (\mathbf{A}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{A})^{-1}|\boldsymbol{\vartheta} = \boldsymbol{\vartheta}_0\right]$ means the normal distribution with the mean value equal to $\boldsymbol{\Theta}$ and with the covariance matrix

$$(\mathbf{A}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{A})^{-1}$$

under the assumption that the true value of the vector parameter $\boldsymbol{\vartheta}$ is $\boldsymbol{\vartheta}_0$) and

(b) the $\boldsymbol{\vartheta}_0$-LBQUIE (locally best quadratic unbiased invariant estimator) of $\boldsymbol{\vartheta}$ is

$$\hat{\boldsymbol{\vartheta}}^{(I)} = \mathbf{S}^{-1}_{(M_A\Sigma_0 M_A)^+}\begin{pmatrix} \boldsymbol{\eta}'(\mathbf{M}_A\boldsymbol{\Sigma}_0\mathbf{M}_A)^+\mathbf{V}_1(\mathbf{M}_A\boldsymbol{\Sigma}_0\mathbf{M}_A)^+\boldsymbol{\eta} \\ \vdots \\ \boldsymbol{\eta}'(\mathbf{M}_A\boldsymbol{\Sigma}_0\mathbf{M}_A)^+\mathbf{V}_p(\mathbf{M}_A\boldsymbol{\Sigma}_0\mathbf{M}_A)^+\boldsymbol{\eta} \end{pmatrix} \tag{2}$$

and

$$\mathrm{var}\left(\hat{\boldsymbol{\vartheta}}^{(I)}|\boldsymbol{\vartheta}_0\right) = 2\mathbf{S}^{-1}_{(M_A\Sigma_0 M_A)^+}.$$

Here

$$\boldsymbol{\Sigma}_0 = \sum_{i=1}^{p}\vartheta_{0,i}\mathbf{V}_i, \boldsymbol{\vartheta}_0 = (\vartheta_{0,1}, \ldots, \vartheta_{0,p})' \in \underline{\vartheta}$$

and

$$\begin{aligned}&\left\{\mathbf{S}_{(M_A\Sigma_0 M_A)^+}\right\}_{i,j} = \\ &= \mathrm{Tr}\left[(\mathbf{M}_A\boldsymbol{\Sigma}_0\mathbf{M}_A)^+\mathbf{V}_i(\mathbf{M}_A\boldsymbol{\Sigma}_0\mathbf{M}_A)^+\mathbf{V}_j\right], \quad i,j = 1,\ldots,p.\end{aligned} \tag{3}$$

*Proof.* The statements (i) and (ii) are well known facts. The statement (iii) (b) is proved for instance in Kubáček et al. (1995) or Rao and Kleffe (1988). □

# 3 Estimation in a Separate Epoch

**Theorem 3.1** Let $\boldsymbol{\Sigma}$ be given. Then

$$\begin{aligned}\begin{pmatrix}\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_j) \\ \boldsymbol{\beta}_{2,j}(\mathbf{Y}_j)\end{pmatrix} &= \begin{pmatrix}[\mathbf{X}_1'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}\mathbf{M}_{X_2})^+\mathbf{X}_1]^{-1}\mathbf{X}_1'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}\mathbf{M}_{X_2})^+\mathbf{Y}_j \\ (\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}[\mathbf{Y}_j - \mathbf{X}_1\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_j)]\end{pmatrix} \sim \\ &\sim N_{k_1+k_2}\left[\begin{pmatrix}\boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_{2,j}\end{pmatrix}, \begin{pmatrix}\mathbf{W}_{11}, & \mathbf{W}_{1,2} \\ \mathbf{W}_{2,1}, & \mathbf{W}_{2,2}\end{pmatrix}\right],\end{aligned}$$

where

$$\begin{aligned}\mathbf{W}_{1,1} &= [\mathbf{X}_1'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}\mathbf{M}_{X_2})^+\mathbf{X}_1]^{-1}, \\ \mathbf{W}_{1,2} &= -[\mathbf{X}_1'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}\mathbf{M}_{X_2})^+\mathbf{X}_1]^{-1}\mathbf{X}_1'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2(\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2)^{-1}, \\ \mathbf{W}_{2,1} &= \mathbf{W}_{1,2}', \\ \mathbf{W}_{2,2} &= (\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2)^{-1} + (\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_1 \times \\ &\quad \times[\mathbf{X}_1'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}\mathbf{M}_{X_2})^+\mathbf{X}_1]^{-1}\mathbf{X}_1'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2(\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2)^{-1}.\end{aligned}$$

*Proof.* Let in Lemma 2.2 (i) the matrix $\mathbf{A}$ be $(\mathbf{X}_1, \mathbf{X}_2)$. The matrix of the normal equations for the model

$$\mathbf{Y}_j \sim N_n[(\mathbf{X}_1, \mathbf{X}_2)\begin{pmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_{2,j}\end{pmatrix}, \boldsymbol{\Sigma}],$$

is of the form

$$\begin{pmatrix}\mathbf{X}_1'\boldsymbol{\Sigma}^{-1}\mathbf{X}_1, & \mathbf{X}_1'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2\\ \mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_1, & \mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2\end{pmatrix}.$$

With respect to the Rohde formula (Rao, 1965, p. 29) in the form

$$\begin{pmatrix}\mathbf{A}, & \mathbf{B}\\ \mathbf{B}', & \mathbf{C}\end{pmatrix}^{-1} =$$
$$= \begin{pmatrix}(\mathbf{A}-\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}, & -(\mathbf{A}-\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\\ -\mathbf{C}^{-1}\mathbf{B}'(\mathbf{A}-\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}, & \mathbf{C}^{-1}+\mathbf{C}^{-1}\mathbf{B}'(\mathbf{A}-\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\end{pmatrix}$$

we obtain the inverse of the matrix of the normal equations in the form

$$\begin{pmatrix}\mathbf{W}_{11}, & \mathbf{W}_{1,2}\\ \mathbf{W}_{2,1}, & \mathbf{W}_{2,2}\end{pmatrix},$$

where

$$\begin{aligned}\mathbf{W}_{1,1} &=\\ &= \left[\mathbf{X}_1'\boldsymbol{\Sigma}^{-1}\mathbf{X}_1 - X_1'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2(\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_1\right]^{-1} =\\ &= \left\{\mathbf{X}_1'\left[\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbf{X}_2(\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\right]\mathbf{X}_1\right\}^{-1} =\\ &= \left[\mathbf{X}_1'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}\mathbf{M}_{X_2})^{+}\mathbf{X}_1\right]^{-1};\end{aligned}$$

here the equality

$$(\mathbf{M}_{X_2}\boldsymbol{\Sigma}\mathbf{M}_{X_2})^{+} = \boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbf{X}_2(\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}$$

is used. (The notation $\mathbf{A}^+$ means the Moore–Penrose inverse of the matrix $\mathbf{A}$; the equalities $\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A}$, $\mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+$, $\mathbf{A}^+\mathbf{A} = (\mathbf{A}^+\mathbf{A})'$ and $\mathbf{A}\mathbf{A}^+ = (\mathbf{A}\mathbf{A}^+)'$ must be satisfied.) The first equality in our case is

$$\mathbf{M}_{X_2}\boldsymbol{\Sigma}\mathbf{M}_{X_2}[\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbf{X}_2(\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}]\mathbf{M}_{X_2}\boldsymbol{\Sigma}\mathbf{M}_{X_2} = \mathbf{M}_X\boldsymbol{\Sigma}\mathbf{M}_X,$$

since

$$[\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbf{X}_2(\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}]\mathbf{M}_{X_2} = \boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbf{X}_2(\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}$$

and $\mathbf{X}'\mathbf{M}_X = \mathbf{0}$. The further three equalities can be proved similarly.

The expressions for $\mathbf{W}_{1,2} = \mathbf{W}_{2,1}'$ and $\mathbf{W}_{2,2}$, given in the statement, can be obtained in an analogous way. □

**Remark 3.2** An obvious analogy can be written in the case $\boldsymbol{\Sigma} = \sigma^2\mathbf{V}$ and $\boldsymbol{\Sigma} = \sum_{i=1}^{p}\vartheta_i\mathbf{V}_i$ for the UBLUE of $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_{2,j}$ and $\boldsymbol{\vartheta}_0$-LBLUE of $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_{2,j}$, respectively.

**Lemma 3.3** The expression

$$\boldsymbol{\eta}'(\mathbf{M}_A\boldsymbol{\Sigma}_0\mathbf{M}_A)^{+}\mathbf{V}_i(\mathbf{M}_A\boldsymbol{\Sigma}_0\mathbf{M}_A)^{+}\boldsymbol{\eta}$$

from Lemma 2.2 in (2) can be rewritten in the form

$$\mathbf{v}'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{v},$$

where

$$\mathbf{v} = \boldsymbol{\eta} - \mathbf{A}\hat{\boldsymbol{\Theta}},\ \hat{\boldsymbol{\Theta}} = (\mathbf{A}'\boldsymbol{\Sigma}_0^{-1}\mathbf{A})^{-1}\mathbf{A}'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\eta}.$$

*Proof.* Since

$$\begin{aligned}&(\mathbf{M}_A\boldsymbol{\Sigma}_0^{-1}\mathbf{M}_A)^{+}\boldsymbol{\eta} =\\ &= \left[\boldsymbol{\Sigma}_0^{-1} - \boldsymbol{\Sigma}_0^{-1}\mathbf{A}(\mathbf{A}'\boldsymbol{\Sigma}_0^{-1}\mathbf{A})^{-1}\mathbf{A}'\boldsymbol{\Sigma}_0^{-1}\right]\boldsymbol{\eta} =\\ &= \boldsymbol{\Sigma}_0^{-1}\left[\mathbf{I} - \mathbf{A}(\mathbf{A}'\boldsymbol{\Sigma}_0^{-1}\mathbf{A})^{-1}\mathbf{A}'\boldsymbol{\Sigma}_0^{-1}\right]\boldsymbol{\eta} = \boldsymbol{\Sigma}_0^{-1}\mathbf{v},\end{aligned}$$

the statement is obvious. □

**Lemma 3.4** (i) Let in (1) $\boldsymbol{\Sigma} = \sigma^2\mathbf{V}$. Then the best unbiased estimator of $\sigma^2$ is

$$\hat{\sigma}_j^2 = \frac{\mathbf{Y}_j'\left(\mathbf{M}_{(X_1,X_2)}\mathbf{V}\mathbf{M}_{(X_1,X_2)}\right)^{+}\mathbf{Y}_j}{n-(k_1+k_2)} \sim \sigma^2\frac{\chi^2_{n-(k_1+k_2)}(0)}{n-(k_1+k_2)}.$$

where

$$\begin{aligned}&\left(\mathbf{M}_{(X_1,X_2)}\mathbf{V}\mathbf{M}_{(X_1,X_2)}\right)^{+} =\\ &= (\mathbf{M}_{X_2}\mathbf{V}\mathbf{M}_{X_2})^{+} - (\mathbf{M}_{X_2}\mathbf{V}\mathbf{M}_{X_2})^{+}\mathbf{X}_1[\mathbf{X}_1'(\mathbf{M}_{X_2}\mathbf{V}\mathbf{M}_{X_2})^{+}\mathbf{X}_1]^{-1}(\mathbf{M}_{X_2}\mathbf{V}\mathbf{M}_{X_2})^{+}.\end{aligned}$$

*Proof.* It is a direct consequence of Lemma 2.2 (ii) and the relationship

$$\begin{aligned}&\left(\mathbf{M}_{(X_1,X_2)}\mathbf{V}\mathbf{M}_{(X_1,X_2)}\right)^{+} =\\ &= \mathbf{V}^{-1} - \mathbf{V}^{-1}(\mathbf{X}_1,\mathbf{X}_2)\begin{pmatrix}\mathbf{X}_1'\mathbf{V}^{-1}\mathbf{X}_1, & \mathbf{X}_1'\mathbf{V}^{-1}\mathbf{X}_2\\ \mathbf{X}_2'\mathbf{V}^{-1}\mathbf{X}_1, & \mathbf{X}_2'\mathbf{V}^{-1}\mathbf{X}_2\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{X}_1'\\ \mathbf{X}_2'\end{pmatrix}\mathbf{V}^{-1};\end{aligned}$$

when the Rohde formula is used, we obtain the statement. □

**Lemma 3.5** Let in (1) $\boldsymbol{\Sigma} = \sum_{i=1}^{p}\vartheta_i\mathbf{V}_i$. Then

$$\begin{aligned}\hat{\boldsymbol{\vartheta}}_j &= \mathbf{S}^{-1}_{(M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+}\begin{pmatrix}\hat{\gamma}_1\\ \vdots\\ \hat{\gamma}_p\end{pmatrix}\\ &= \mathbf{S}^{-1}_{(M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+}\begin{pmatrix}\mathbf{v}_j'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_1\boldsymbol{\Sigma}_0^{-1}\mathbf{v}_j\\ \vdots\\ \mathbf{v}_j'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_p\boldsymbol{\Sigma}_0^{-1}\mathbf{v}_j\end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}\hat{\gamma}_s &= q\mathbf{Y}_j'(\mathbf{M}_{(X_1,X_2)}\boldsymbol{\Sigma}\mathbf{M}_{(X_1,X_2)})^{+}\mathbf{V}_s(\mathbf{M}_{(X_1,X_2)}\boldsymbol{\Sigma}\mathbf{M}_{(X_1,X_2)})^{+}\mathbf{Y}_j,\ s = 1,\dots,p,\\ \mathbf{v}_j &= \mathbf{Y}_j - \mathbf{X}_1\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_j) - \mathbf{X}_2\hat{\boldsymbol{\beta}}_{2,j}(\mathbf{Y}_j),\ j = 1,\dots,m.\end{aligned}$$

*Proof.* It is a direct consequence of Lemma 2.2 (iii) and Lemma 3.3. □

# 4 Estimation after $m$ Epochs

**Theorem 4.1** Let $\mathbf{\Sigma}$ be given. Then

$$\hat{\boldsymbol{\beta}}_1(\mathbf{Y}^{(m)}) = \frac{1}{m}\sum_{j=1}^{m}\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_j)$$

and

$$\left\{\hat{\boldsymbol{\beta}}_{2,(\cdot)}(\mathbf{Y}^{(m)})\right\}_j =$$
$$= \hat{\boldsymbol{\beta}}_{2,j}(\mathbf{Y}_j) + (\mathbf{X}_2'\mathbf{\Sigma}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}^{-1}\mathbf{X}_1\left[\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_j) - \frac{1}{m}\sum_{i=1}^{m}\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_i)\right];$$

further

$$\begin{pmatrix}\hat{\boldsymbol{\beta}}_1(\mathbf{Y}^{(m)})\\ \hat{\boldsymbol{\beta}}_{2,(\cdot)}(\mathbf{Y}^{(m)})\end{pmatrix} \sim$$
$$\sim N_{k_1+mk_2}\left[\begin{pmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_{2,(\cdot)}\end{pmatrix}; \begin{pmatrix}\mathrm{var}(\hat{\boldsymbol{\beta}}_1), & \mathrm{cov}(\hat{\boldsymbol{\beta}}_1, \hat{\boldsymbol{\beta}}_{2,(\cdot)})\\ \mathrm{cov}(\hat{\boldsymbol{\beta}}_{2,(\cdot)}, \hat{\boldsymbol{\beta}}_1), & \mathrm{var}(\hat{\boldsymbol{\beta}}_{2,(\cdot)})\end{pmatrix}\right],$$

where

$$\begin{aligned}
\mathrm{var}(\hat{\boldsymbol{\beta}}_1) &= \frac{1}{m}[\mathbf{X}_1'(\mathbf{M}_{X_2}\mathbf{\Sigma}\mathbf{M}_{X_2})^{+}\mathbf{X}_1]^{-1},\\
\mathrm{cov}(\hat{\boldsymbol{\beta}}_1, \hat{\boldsymbol{\beta}}_{2,(\cdot)}) &= -\frac{\mathbf{1}}{m}\otimes[\mathbf{X}_1'(\mathbf{M}_{X_2}\mathbf{\Sigma}\mathbf{M}_{X_2})^{+}\mathbf{X}_1]^{-1}\times\\
&\quad\times\mathbf{X}_1'\mathbf{\Sigma}^{-1}\mathbf{X}_2(\mathbf{X}_2'\mathbf{\Sigma}^{-1}\mathbf{X}_2)^{-1},\\
\mathrm{cov}(\hat{\boldsymbol{\beta}}_{2,(\cdot)}, \hat{\boldsymbol{\beta}}_1) &= \mathrm{cov}(\hat{\boldsymbol{\beta}}_1, \hat{\boldsymbol{\beta}}_{2,(\cdot)})',\\
\mathrm{var}(\hat{\boldsymbol{\beta}}_{2,(\cdot)}) &= \mathbf{I}\otimes(\mathbf{X}_2'\mathbf{\Sigma}^{-1}\mathbf{X}_2)^{-1} +\\
&\quad+\mathbf{P}_m\otimes(\mathbf{X}_2'\mathbf{\Sigma}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}^{-1}\mathbf{X}_1\times\\
&\quad\times[\mathbf{X}_1'(\mathbf{M}_{X_2}\mathbf{\Sigma}\mathbf{M}_{X_2})^{+}\mathbf{X}_1]^{-1}\mathbf{X}_1'\mathbf{\Sigma}^{-1}\mathbf{X}_2(\mathbf{X}_2'\mathbf{\Sigma}^{-1}\mathbf{X}_2)^{-1}.
\end{aligned}$$

*Proof.* The expression (cf. Lemma 2.2 (i))

$$\left[\begin{pmatrix}\mathbf{1}'\otimes\mathbf{X}_1'\\ \mathbf{I}\otimes\mathbf{X}_2'\end{pmatrix}(\mathbf{I}\otimes\mathbf{\Sigma}^{-1})(\mathbf{1}\otimes\mathbf{X}_1, \mathbf{I}\otimes\mathbf{X}_2)\right]^{-1}\begin{pmatrix}\mathbf{1}'\otimes\mathbf{X}_1'\\ \mathbf{I}\otimes\mathbf{X}_2'\end{pmatrix}(\mathbf{I}\otimes\mathbf{\Sigma}^{-1})\mathbf{Y}^{(m)}$$

can be rewritten using the Rohde formula as $\begin{pmatrix}\mathbf{U}\\ \mathbf{W}\end{pmatrix}$, where

$$\begin{aligned}
\mathbf{U} &= \left\{\frac{\mathbf{1}'}{m}\otimes[\mathbf{X}_1'(\mathbf{M}_{X_2}\mathbf{\Sigma}\mathbf{M}_{X_2})^{+}\mathbf{X}_1]^{-1}\mathbf{X}_1'(\mathbf{M}_{X_2}\mathbf{\Sigma}\mathbf{M}_{X_2})^{+}\right\}\mathbf{Y}^{(m)},\\
\mathbf{W} &= \mathbf{I}\otimes[(\mathbf{X}_2'\mathbf{\Sigma}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}^{-1}]\mathbf{Y}^{(m)} -\\
&\quad-\mathbf{P}_m\otimes(\mathbf{X}_2'\mathbf{\Sigma}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}^{-1}\mathbf{X}_1\times\\
&\quad\times[\mathbf{X}_1'(\mathbf{M}_{X_2}\mathbf{\Sigma}\mathbf{M}_{X_2})^{+}\mathbf{X}_1]^{-1}\mathbf{X}_1'(\mathbf{M}_{X_2}\mathbf{\Sigma}\mathbf{M}_{X_2})^{+}\mathbf{Y}^{(m)}.
\end{aligned}$$

Since (cf. Theorem 3.1)

$$\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_j) = [\mathbf{X}_1'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}\mathbf{M}_{X_2})^+\mathbf{X}_1]^{-1}\mathbf{X}_1'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}\mathbf{M}_{X_2})^+\mathbf{Y}_j$$

and

$$\hat{\boldsymbol{\beta}}_{2,j}(\mathbf{Y}_j) = (\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}[\mathbf{Y}_j - \mathbf{X}_1\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_j)],$$

we have

$$\left\{\frac{\mathbf{1}'}{m}\otimes[\mathbf{X}_1'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}\mathbf{M}_{X_2})^+\mathbf{X}_1]^{-1}\mathbf{X}_1'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}\mathbf{M}_{X_2})^+\right\}\mathbf{Y}^{(m)} = \frac{1}{m}\sum_{i=1}^{m}\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_i)$$

and

$$\begin{aligned}
&\mathbf{I}\otimes[(\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}]\mathbf{Y}^{(m)} - \mathbf{P}_m\otimes(\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_1\times\\
&\times[\mathbf{X}_1'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}\mathbf{M}_{X_2})^+\mathbf{X}_1]^{-1}\mathbf{X}_1'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}\mathbf{M}_{X_2})^+\mathbf{Y}^{(m)} =\\
&= \begin{pmatrix}(\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{Y}_1\\ \vdots\\ (\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{Y}_m\end{pmatrix} -\\
&-\mathbf{1}\otimes(\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_1\frac{1}{m}\sum_{i=1}^{m}\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_i) \Rightarrow\\
&\Rightarrow \left\{\hat{\boldsymbol{\beta}}_{2,(\cdot)}(\mathbf{Y}^{(m)})\right\}_j = (\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\left[\mathbf{Y}_j - \mathbf{X}_1\frac{1}{m}\sum_{i=1}^{m}\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_i)\right] =\\
&= (\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\Big\{\mathbf{Y}_j - \mathbf{X}_1\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_j) +\\
&+\left[\mathbf{X}_1\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_j) - \mathbf{X}_1\frac{1}{m}\sum_{i=1}^{m}\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_i)\right]\Big\} =\\
&= \hat{\boldsymbol{\beta}}_{2,j}(\mathbf{Y}_j) + (\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_1\left[\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_j) - \frac{1}{m}\sum_{i=1}^{m}\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_i)\right].
\end{aligned}$$

□

**Corollary 4.2** The model assumption on the stableness of the points with the parameters $\boldsymbol{\beta}_1$ can be verified by testing the hypothesis

$$H_{0,1}: E[\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_j)] = E[\hat{\boldsymbol{\beta}}_1(\mathbf{Y}^{(m)})], \quad m = 2, 3, \ldots$$

Regarding Theorem 3.1 and 4.1, the test statistic can be chosen as

$$\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_j) - \hat{\boldsymbol{\beta}}_1(\mathbf{Y}^{(m)}) \sim N_{k_1}\left(\mathbf{0}, \frac{m-1}{m}\mathrm{var}[\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_j)]\right).$$

It is a consequence of the following facts. With respect to the mentioned theorems it is valid

$$\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_j) = [\mathbf{X}_1'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}\mathbf{M}_{X_2})^+\mathbf{X}_1]^{-1}\mathbf{X}_1'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}\mathbf{M}_{X_2})^+\mathbf{Y}_j,$$

$$\begin{aligned}
\hat{\boldsymbol{\beta}}_1(\mathbf{Y}^{(m)}) &= \frac{1}{m}\sum_{j-1}^{m}\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_j),\\
\text{cov}\left(\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_j), \hat{\boldsymbol{\beta}}_1(\mathbf{Y}^{(m)})\right) &= \frac{1}{m}\text{var}\left(\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_j)\right),\\
\text{var}\left(\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_j) - \hat{\boldsymbol{\beta}}_1(\mathbf{Y}^{(m)})\right) &= \text{var}\left(\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_j)\right) - \frac{2}{m}\text{var}\left(\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_j)\right) +\\
&\quad + \frac{1}{m}\text{var}\left(\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_j)\right)\\
&= \frac{m-1}{m}\text{var}[\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_j)].
\end{aligned}$$

If the dimension of the vector $\boldsymbol{\beta}_{2,j}$ and the number of the epochs $m$ are large, then the rounding errors can destroy the agreement between $\hat{\boldsymbol{\beta}}_{2,j}(\mathbf{Y}_j)$ and $\hat{\boldsymbol{\beta}}_{2,j}(\mathbf{Y}^{(m)})$. It can be checked by testing the hypothesis

$$H_{0,2} : E[\hat{\boldsymbol{\beta}}_{2,j}(\mathbf{Y}_j)] = E[\hat{\boldsymbol{\beta}}_{2,j}(\mathbf{Y}^{(m)})].$$

On the basis of Theorem 3.1 and Theorem 4.1, the test statistic can be chosen in the form

$$\hat{\boldsymbol{\beta}}_{2,j}(\mathbf{Y}_j) - \hat{\boldsymbol{\beta}}_{2,j}(\mathbf{Y}^{(m)}) \sim N_{k_2}\left(\mathbf{0}, \text{var}[\hat{\boldsymbol{\beta}}_{2,j}(\mathbf{Y}_j)] - (\mathbf{X}_2'\boldsymbol{\Sigma}^{-1}\mathbf{X}_2)^{-1}\right).$$

**Remark 4.3** The estimators of the position parameters after $m$ epochs can be expressed by estimators in separate epochs. When estimating the parameter $\sigma^2$ in the case $\boldsymbol{\Sigma} = \sigma^{\mathbf{2}}\mathbf{V}$, the situation is not so simple.

**Theorem 4.4** Let $\boldsymbol{\Sigma} = \sigma^2\mathbf{V}$ in the model (1). Then the uniformly best estimator of $\sigma^2$ after $m$ epochs is

$$\begin{aligned}
\hat{\sigma}_*^2 &= \left\{\text{Tr}\left[\mathbf{M}_{X_2}\mathbf{V}\mathbf{M}_{X_2}\sum_{i=1}^{m}(\mathbf{Y}_i - \overline{\mathbf{Y}})(\mathbf{Y}_i - \overline{\mathbf{Y}})'\right] + \right.\\
&\quad \left. + m\overline{\mathbf{Y}}'(\mathbf{M}_{(X_1,X_2)}\mathbf{V}\mathbf{M}_{(X_1,X_2)})^+\overline{\mathbf{Y}}\right\} \Big/ (mn - k_1 - mk_2)\\
&\sim \frac{\sigma^2\chi^2_{mn-k_1-mk_2}(0)}{mn - k_1 - mk_2};
\end{aligned}$$

here $\overline{\mathbf{Y}} = (1/m)\sum_{j=1}^{m}\mathbf{Y}_j$.

*Proof.* Regarding Lemma 2.2 (ii), we can write

$$\hat{\sigma}^2 = (\mathbf{Y}^{(m)})'[\mathbf{M}_{(1\otimes X_1, I\otimes X_2)}(\mathbf{I}\otimes\mathbf{V})\mathbf{M}_{(1\otimes X_1, I\otimes X_2)}]^+\mathbf{Y}^{(m)}/(mn - k_1 - mk_2).$$

Further

$$\begin{aligned}
&[\mathbf{M}_{(1\otimes X_1, I\otimes X_2)}(\mathbf{I}\otimes\mathbf{V})\mathbf{M}_{(1\otimes X_1, I\otimes X_2)}]^+ =\\
&= (\mathbf{I}\otimes\mathbf{V}^{-1}) - (\mathbf{I}\otimes\mathbf{V}^{-1})(\mathbf{1}\otimes\mathbf{X}_1, \mathbf{I}\otimes\mathbf{X}_2)\times\\
&\quad \times\left[\begin{pmatrix}\mathbf{1}'\otimes\mathbf{X}_1'\\ \mathbf{I}\otimes\mathbf{X}_2'\end{pmatrix}(\mathbf{I}\otimes\mathbf{V}^{-1})(\mathbf{1}\otimes\mathbf{X}_1, \mathbf{I}\otimes\mathbf{X}_2)\right]^{-1}\begin{pmatrix}\mathbf{1}'\otimes\mathbf{X}_1'\\ \mathbf{I}\otimes\mathbf{X}_2'\end{pmatrix}(\mathbf{I}\otimes\mathbf{V}^{-1}) =\\
&= \mathbf{P}_m\otimes[\mathbf{M}_{(X_1,X_2)}\mathbf{V}\mathbf{M}_{(X_1,X_2)}]^+ + \mathbf{M}_m\otimes(\mathbf{M}_{X_2}\mathbf{V}\mathbf{M}_{X_2})^+,
\end{aligned}$$

$$(\mathbf{Y}^{(m)})'[\mathbf{M} \otimes (\mathbf{M}_{X_2}\mathbf{V}\mathbf{M}_{X_2})^{+}\mathbf{Y}^{(m)} = \mathrm{Tr}\left[(\mathbf{M}_{X_2}\mathbf{V}\mathbf{M}_{X_2})^{+}\sum_{i=1}^{m}(\mathbf{Y}_i - \overline{\mathbf{Y}})(\mathbf{Y}_i - \overline{\mathbf{Y}})'\right]$$

and

$$(\mathbf{Y}^{(m)})'\{\mathbf{P}_m \otimes \mathbf{M}_{(X_1,X_2)}\mathbf{V}\mathbf{M}_{(X_1,X_2)}]^{+}\}\mathbf{Y}^{(m)} = m\overline{\mathbf{Y}}'[\mathbf{M}_{(X_1,X_2)}\mathbf{V}\mathbf{M}_{(X_1,X_2)}]^{+}\overline{\mathbf{Y}}.$$

Further

$$\begin{aligned}\xi_1 &= (\mathbf{Y}^{(m)})'[\mathbf{M}_m \otimes (\mathbf{M}_{X_2}\mathbf{V}\mathbf{M}_{X_2})^{+}]\mathbf{Y}^{(m)} = \\ &= [(\mathbf{M}_m \otimes \mathbf{M}_{X_2})\mathbf{Y}^{(m)}]'[\mathbf{M}_m \otimes (\mathbf{M}_{X_2}\mathbf{V}\mathbf{M}_{X_2})^{+}][(\mathbf{M}_m \otimes \mathbf{M}_{X_2})\mathbf{Y}^{(m)}]\end{aligned}$$

(here the relationships $\mathbf{M}_m^2 = \mathbf{M}_m$ and $(\mathbf{M}_{X_2}\mathbf{V}\mathbf{M}_{X_2})^{+} = \mathbf{M}_{X_2}(\mathbf{M}_{X_2}\mathbf{V}\mathbf{M}_{X_2})^{+}$ $= (\mathbf{M}_{X_2}\mathbf{V}\mathbf{M}_{X_2})^{+}\mathbf{M}_{X_2}$ were used).

Since $(\mathbf{M}_m \otimes \mathbf{M}_{X_2})\mathbf{Y}^{(m)} \sim N_{mn}(\mathbf{0}, \mathbf{M}_m \otimes \sigma^2\mathbf{M}_{X_2}\mathbf{V}\mathbf{M}_{X_2})$ the random variable $\xi_1$ is $\sigma^2\chi^2_{(m-1)(n-k_2)}(0)$ distributed (it is to be remarked that $\mathbf{M}_m^{+} = \mathbf{M}_m$). Analogously it can be proved

$$\xi_2 = (\mathbf{Y}^{(m)})'[\mathbf{P}_m \otimes (\mathbf{M}_{(X_1,X_2)}\mathbf{V}\mathbf{M}_{(X_1,X_2)})^{+}]\mathbf{Y}^{(m)} \sim \sigma^2\chi^2_{n-(k_1+k_2)}$$

Since $\xi_1 + \xi_2 = \sigma^2\chi_{mn-k_1-mk_2}$, the statement is proved. □

**Remark 4.5** It can be seen that estimators

$$\hat{\sigma}_j^2 = \frac{1}{n-k_1-k_2}\mathbf{Y}_j'[\mathbf{M}_{(X_1,X_2)}\mathbf{V}\mathbf{M}_{(X_1,X_2)}]^{+}\mathbf{Y}_j,\ j = 1,\dots,m,$$

from separate epochs cannot be used for the final estimator. Also the arithmetic mean $\frac{1}{m}\sum_{i=1}^{m}\hat{\sigma}^2$ is significantly worse (cf. further lemma) than the estimator from Theorem 4.4.

**Lemma 4.6** Under our assumptions

(i)

$$\frac{1}{m}\sum_{j=1}^{m}\hat{\sigma}_j^2 \sim_1 \left(\sigma^2, \frac{2\sigma^4}{m(n-k_1-k_2)}\right),$$

(ii)

$$\begin{aligned}&\frac{(\mathbf{Y}^{(m)})'[\mathbf{M}_m \otimes (\mathbf{M}_{X_2}\mathbf{V}\mathbf{M}_{X_2})^{+}]\mathbf{Y}^{(m)}}{(m-1)(n-k_2)} = \\ &= \frac{\mathrm{Tr}[(\mathbf{M}_{X_2}\mathbf{V}\mathbf{M}_{X_2})\sum_{i=1}^{m}(\mathbf{Y}_i - \overline{\mathbf{Y}})(\mathbf{Y}_i - \overline{\mathbf{Y}})']}{(m-1)(n-k_2)} \sim_1 \left(\sigma^2, \frac{2\sigma^4}{(m-1)(n-k_2)}\right),\end{aligned}$$

and

(iii)

$$\hat{\sigma}_*^2 \sim_1 \left(\sigma^2, \frac{2\sigma^4}{mn-k_1-mk_2}\right)$$

*Proof.* It is a consequence of the relationships

$$\hat{\sigma}^2 \sim \sigma^2 \frac{\chi_f^2}{f},$$

$E(\chi_f^2) = f$ and $\mathrm{var}(\chi_f^2) = 2f$. □

For comparison let us calculate the ratio of dispersions of the following estimators:

$$\frac{\mathrm{var}[(1/m)\sum_{j=1}^m \hat{\sigma}_j^2]}{\mathrm{var}(\hat{\sigma}_*^2)} = \frac{2\sigma^4[m(n-k_2)-k_1]}{2\sigma^4 m(n-k_1-k_2)},$$

which for $m$ tending to infinity tends to

$$\frac{n-k_2}{n-k_1-k_2}.$$

For example if $n = 40,\ k_2 = 10$ and $k_1 = 10$, then

$$lim_{m\to\infty}\frac{\mathrm{var}[(1/m)\sum_{j=1}^m \hat{\sigma}_j^2]}{\mathrm{var}(\hat{\sigma}_*^2)} = \frac{3}{2}.$$

**Remark 4.7** From Lemma 4.6 it can be seen that the estimator from (iii) is the best estimator and the worst is the estimator from (i). If $m$ tends to infinity, then the difference between estimators (ii) and (iii) can be neglected. Thus (ii) is an essential part of the estimator $\hat{\sigma}_*^2$, however it cannot be established by the help of the estimators of separate epochs.

Let us try to use another approach in order to use in a better way the epoch results for the final estimator.

Since also

$$\hat{\sigma}_*^2 = [\mathbf{v}_1'(\mathbf{Y}^{(m)}),\ldots,\mathbf{v}_m'(\mathbf{Y}^{(m)})](\mathbf{I}\otimes\mathbf{V}^{-1})\begin{pmatrix}\mathbf{v}_1(\mathbf{Y}^{(m)})\\ \vdots \\ \mathbf{v}_m(\mathbf{Y}^{(m)})\end{pmatrix}\Big/(mn-k_1-mk_2)$$

and

$$\mathbf{v}_j(\mathbf{Y}^{(m)}) = \mathbf{Y}_j - \mathbf{X}_1\hat{\boldsymbol{\beta}}_1(\mathbf{Y}^{(m)}),$$

we can write

$$\begin{aligned}\mathbf{v}_j(\mathbf{Y}^{(m)}) &= \mathbf{v}_j(\mathbf{Y}_j) + \mathbf{X}_1\left[\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_j) - \frac{1}{m}\sum_{i=1}^m \hat{\boldsymbol{\beta}}_1(\mathbf{Y}_i)\right] - \\ &\quad -\mathbf{X}_2(\mathbf{X}_2\mathbf{V}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2\mathbf{V}^{-1}\mathbf{X}_1\left[\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_j) - \frac{1}{m}\sum_{i=1}^m \hat{\boldsymbol{\beta}}_1(\mathbf{Y}_i)\right] = \\ &= \mathbf{v}_j(\mathbf{Y}_j) + \mathbf{M}_{X_2}^{V^{-1}}\mathbf{X}_1\left[\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_j) - \frac{1}{m}\sum_{i=1}^m \hat{\boldsymbol{\beta}}_1(\mathbf{Y}_i)\right],\ j = 1,\ldots,m,\end{aligned}$$

and thus the estimate $\hat{\sigma}_*^2$ can be calculated on the basis of the corrected residuals $\mathbf{v}_j(\mathbf{Y}^{(m)}),\ j = 1,\dots,m$.

This approach seems to be suitable also for calculating the final estimator $\hat{\boldsymbol{\vartheta}}(\mathbf{Y}^{(m)})$ in the case $\boldsymbol{\Sigma} = \sum_{i=1}^m \vartheta_i \mathbf{V}_i$.

**Theorem 4.8** Let in model (1) $\boldsymbol{\Sigma} = \sum_{i=1}^m \vartheta_i \mathbf{V}_i$. Then the $\boldsymbol{\vartheta}_0$-LBQUIE (locally best quadratic unbiased invariant estimator) of $\boldsymbol{\vartheta}$ is

$$\hat{\boldsymbol{\vartheta}}(\mathbf{Y}^{(m)}) = \mathbf{S}_{[M_{(1\otimes X_1, I\otimes X_2)}(I\otimes\Sigma_0)(M_{(1\otimes X_1, I\otimes X_2)}]^+}\hat{\boldsymbol{\gamma}},$$

$$\hat{\boldsymbol{\gamma}} = (\hat{\gamma}_1,\dots,\hat{\gamma}_p)'$$

$$\hat{\gamma}_i = (\mathbf{v}_1'(\mathbf{Y}^{(m)}),\dots \mathbf{v}_m'(\mathbf{Y}^{(m)}))(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1}\mathbf{V}_i\boldsymbol{\Sigma}^{-1})\begin{pmatrix}\mathbf{v}_1(\mathbf{Y}^{(m)})\\ \vdots \\ \mathbf{v}_m(\mathbf{Y}^{(m)})\end{pmatrix} =$$

$$= \mathrm{Tr}\left[\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\sum_{s=1}^m \mathbf{v}_s(\mathbf{Y}^{(m)})\mathbf{v}_s'(\mathbf{Y}^{(m)})\right],\ i = 1,\dots,p,$$

where

$$\mathbf{S}_{[M_{(1\otimes X_1, I\otimes X_2)}(I\otimes\Sigma_0)(M_{(1\otimes X_1, I\otimes X_2)}]^+} = $$
$$= (m-1)\mathbf{S}_{(M_{X_2}\Sigma_0 M_{X_2})^+} + \mathbf{S}_{[M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)}]^+}.$$

*Proof.* It suffices to prove the last equality only. With respect to the definition of the matrix $\mathbf{S}_{[M_{(1\otimes X_1, I\otimes X_2)}(I\otimes\Sigma_0)(M_{(1\otimes X_1, I\otimes X_2)}]^+}$ it can be written (cf. the proof of Theorem 4.4)

$$\{\mathbf{S}_{[M_{(1\otimes X_1, I\otimes X_2)}(I\otimes\Sigma_0)(M_{(1\otimes X_2, I\otimes X_2)}]^+}\}_{i,j} =$$
$$= \mathrm{Tr}\{[\mathbf{M}_m\otimes(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_0\mathbf{M}_{X_2})^+\mathbf{P}_m\otimes(\mathbf{M}_{(X_1,X_2)}\boldsymbol{\Sigma}_0\mathbf{M}_{(X_1,X_2)})^+](\mathbf{I}\otimes\mathbf{V}_i)\times$$
$$\times[\mathbf{M}_m\otimes(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_0\mathbf{M}_{X_2})^+\mathbf{P}_m\otimes(\mathbf{M}_{(X_1,X_2)}\boldsymbol{\Sigma}_0\mathbf{M}_{(X_1,X_2)})^+](\mathbf{I}\otimes\mathbf{V}_j)\} =$$
$$= \mathrm{Tr}\{\mathbf{M}_m\otimes[(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_0\mathbf{M}_{X_2})^+\mathbf{V}_i(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_0\mathbf{M}_{X_2})^+\mathbf{V}_j]\} +$$
$$+\mathrm{Tr}\{\mathbf{P}_m\otimes[(\mathbf{M}_{(X_1,X_2)}\boldsymbol{\Sigma}_0\mathbf{M}_{(X_1,X_2)})^+\mathbf{V}_i(\mathbf{M}_{(X_1,X_2)}\boldsymbol{\Sigma}_0\mathbf{M}_{(X_1,X_2)}) + \mathbf{V}_j]\} =$$
$$= (m-1)\{\mathbf{S}_{(M_{X_2}\Sigma_0 M_{X_2})^+}\}_{i,j} + \{\mathbf{S}_{[M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)}]^+}\}_{i,j},$$

$j = 1,\dots,p$. □

# 5 Conclusions

From Section 4 it can be seen how the partial results, obtained after each epoch, can be used in a calculation of results after several epochs.

This possibility has two practical advantages; it is simple to calculate estimators after several epochs and in addition to it the partial results create a sample which can be statistically analyzed from several points of view. For example hypotheses as $E(\hat{\boldsymbol{\beta}}_1(\mathbf{Y}_1)) = \dots = E(\hat{\boldsymbol{\beta}}_1(Y_m))$, $E(\hat{\boldsymbol{\beta}}_{2,j}(\mathbf{Y}_j)) = E(\hat{\boldsymbol{\beta}}_{2,j}(\mathbf{Y}^{(m)}))$, $E(\hat{\sigma}^2(\mathbf{Y}_j)) = E(\hat{\sigma}^2(\mathbf{Y}^{(m)}))$, $j = 1,\dots,m$ (a check whether an accuracy of measurement is homogeneous), etc. can be tested.

## References

L. Kubáček. Statistical modelling of deformation measurements. *Acta Math. Univ. Comenianae*, Vol. LXIII, 1, 77–106, 1994.

L. Kubáček, L. Kubáčková, and J. Volaufová. *Statistical Models with Linear Structures*. Veda, Bratislava, 1995.

L. Kubáčková. Locally best estimators of the second order parameters in fundamental replicated structures with nuisance parameters. Technical Report No. 12, Institute of Geodesy, University of Stuttgart, 1–67, 1990.

L. Kubáčková. Optimum elimination of systematic effects in deformation models. *Folia Fac. Sci. Nat. Univ. Masarykianae Brunensis, Mathematica 7*, 67–72, 1998.

C.R. Rao. *Linear Statistical Inference and Its Applications*. Wiley, New York, 1965.

C.R. Rao and J. Kleffe. *Estimation of Variance Components and Applications*. North-Holland, Amsterdam, 1988.

C.R. Rao and S.K. Mitra. *Generalized Inverse of Matrices and Its Applications*. Wiley, New York, 1971.

Authors' address:

Lubomír Kubáček and Ludmila Kubáčková
Department of Mathematical Analysis and Applications of Mathematics
Faculty of Science, Palacký University
Tomkova 40, CZ-779 00 Olomouc

Email: kubacekl@risc.upol.cz