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Tests Based on Kurtosis for Multivariate Normality

Danush K. Wijekularathna
Troy university

Huijun Yi
Troy university

Adhiraj Roka
Troy university

Abstract

In this paper, we first transform a multivariate normal random vector into a random vector with elements that are approximately independent standard normal random variables. Then we propose the multivariate version generalized from the univariate normality test based on kurtosis from the literature. Power is investigated through the Monte Carlo Simulation with different significance level, dimension, and sample size. To assess the validity and accuracy of the new tests, we carry a comparative study with several other existing tests by selecting certain types of symmetric and asymmetric alternative distributions.

Keywords: transformation, multivariate normal, Pearson kurtosis, Monte Carlo simulation.

1. Introduction

Testing multivariate normality is a key premise in modern statistical inference. This is due to the fact that parametric statistical techniques have been developed based on normal distribution theory. As a result, testing the normality assumption is more important before we start any analysis. A considerable number of tests procedures can be found in the literature. Some references are Kim (2015), Kim (2016), Enomoto (2013), Koizumi, Okamoto, and and (2009), Székely and Rizzo (2005), Thode (2002), Doornik and Hansen (1994), Royston (1983), Mardia (1970) and Olive (2017).

Even though numerous multivariate tests have been proposed, there is not a single test that can be used for all the situations. As an example, some tests are better for long tailed distributions but not for short tailed distributions. Some references for comparative power studies are Joenssen and Vogel (2014), Romeu and Ozturk (1993), Mecklin and Mundfrom (2005) and Alpu and Yuksek (2016).

Most of multivariate normality tests are extensions of univariate normality tests. Kim (2016) recently proposed a robustified Jarque-Bera test for multivariate and univariate normality. He investigates the multivariate versions of the Jarque-Bera test and its modifications using orthogonalization or an empirical standardization of data. Alva and Estrada (2009) has proposed a goodness-of-fit test for multivariate normality which is based on Shapiro-Wilk's statistics, one of the best omnibus tests for the univariate normality. Both these articles proposed a multivariate version of univariate tests.

Hanusz and Tarasinska (2014) proposed two new tests for multivariate normality based on

Mardia's and Srivastava's more accurate moments of multivariate sample skewness and kurtosis. The proposed two tests have an asymptotic Student's t-distribution with $\frac{1}{6}p(p+1)(p+2)$ and p degrees of freedom, respectively. In their simulation studies, sample significance level and power against chosen alternative distributions of both tests were calculated. Their proposed tests were compared with the two improved Jarque-Bera's tests and the Henze-Zirkler test. The proposed tests do not recognize the mixture of two multivariate normal variates with different means and covariance matrices.

Looney (1995) presented how to use tests for univariate normality to assess multivariate normality. He described several techniques for assessing multivariate normality based on well-known tests for univariate normality and offered suggestions for their practical application. In his simulation study, one of the examples is shown that simply testing each of the marginal distributions for univariate normality can lead to a mistaken conclusion. Shapiro and Wilk's test is a powerful procedure for detecting departures from univariate normality. Royston (1983) the application of Shapiro-Wilk-W to testing multivariate normality.

Koizumi, Sumikawa, and Pavlenko (2014) proposed new definitions for multivariate skewness and kurtosis as natural extensions of Mardia's measures when the covariance matrix has a block diagonal structure. Expectations and the variances for new multivariate sample measures of skewness and kurtosis were presented. They also derived asymptotic distributions of the statistics under multivariate normality. In their simulation study, they investigated accuracies of upper percentage points of the proposed statistics based on new multivariate skewness and kurtosis.

In this paper, we propose some multivariate tests that are an extension of univariate tests proposed by Bonett and Seier (2002) based on transformation proposed by Anscombe and Glynn (1983).

The article is organized as follows: in section 2 the univariate tests proposed by Anscombe and Glynn (1983), and Bonett and Seier (2002) are reviewed. The proposed multivariate tests are presented in section 3. In section 4, we studied a Monte Carlo comparative study of the power comparison of the proposed tests against the Mardia (1970) tests Skewness (M_S) and Kurtosis (M_K), Henze and Zirkler (1990) test (H_Z), and Shapiro-Wilk's test (S_W) Alva and Estrada (2009) and two modified tests denoted by (C^*_β) and (C^{**}_β) based on Kim (2016).

2. Review of the univariate tests

Bonett and Seier (2002) used the modified Geary measure and the Pearson measure to define a joint test of kurtosis that has high uniform power across a very wide range of symmetric non-normal distributions.

The population value of Pearson's measure of kurtosis can be defined as

$$\beta_2 = \frac{E(X-\mu)^4}{\{E(X-\mu)^2\}^2}.$$

The estimator of β_2 is

$$\hat{\beta}_2 = \frac{n\sum(X_i-\hat{\mu})^4}{\{\sum(X_i-\hat{\mu})^2\}^2} \tag{1}$$

where $\hat{\mu} = \sum X_i/n$ and n is the sample size.

Anscombe and Glynn (1983) proposed the following transformation of $\hat{\beta}_2$,

$$Z_\beta = \left[(1-\frac{2}{9A}) - [\frac{1-2/A}{1+C\sqrt{2/(A-4)}}]^{1/3}\right]\frac{1}{\sqrt{2/(9A)}} \tag{2}$$

where

$$C = \frac{\hat{\beta}_2 - E(\hat{\beta}_{2)}}{\sqrt{Var(\hat{\beta}_2)}}$$

and

$$A = 6 + \frac{8}{\sqrt{\beta_1(\hat{\beta}_2)}}\left[\frac{2}{\sqrt{\beta_1(\hat{\beta}_2)}} + \sqrt{1 + \frac{4}{\beta_1(\hat{\beta}_2)}}\right]$$

with

$$\sqrt{\beta_1(\hat{\beta_2})} = \frac{E\{\hat{\beta}_2 - E(\hat{\beta}_2)\}^3}{\{var(\hat{\beta}_2)\}^{3/2}} = \frac{6(n^2-5n+2)}{(n+7)(n+9)}\sqrt{\frac{6(n+3)(n+5)}{n(n-2)(n-3)}}.$$

It is known that

$$E(\hat{\beta}_2) = \frac{3(n-1)}{n+1}$$

and

$$Var(\hat{\beta}_2) = \frac{24n(n-2)(n-3)}{(n+1)^2(n+3)(n+5)}.$$

Z_β in equation (2) has an approximate standard normal distribution.

The next univariate test statistic we consider is the test proposed by Bonett and Seier (2002). It is denoted by Z_w,

$$Z_w = \frac{(n+2)^{1/2}(\hat{w}-3)}{3.54} \tag{3}$$

where, $\hat{w} = 13.29(ln(\hat{\sigma}) - ln(\hat{\tau}))$, $\hat{\tau} = \frac{\sum|X_i - \hat{\mu}|}{n}$, $\hat{\sigma}^2 = \frac{\sum(X_i-\hat{\mu})^2}{n}$ and $\hat{\mu} = n^{-1}\sum_{i=1}^{n} X_i$.

The test Z_w in equation (3) has an approximate standard normal distribution under the null hypothesis of normality.

3. Proposed tests for multivariate normality

Let $\mathbf{X_1}, \mathbf{X_2}, ..., \mathbf{X_n}$ be independent identically distributed (i.i.d) p-dimensional random vectors with sample mean and covariance matrix $\bar{\mathbf{X}} = n^{-1}\sum_{i=1}^{n}\mathbf{X_i}$ and $\mathbf{S} = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{X}_i - \bar{\mathbf{X}})(\mathbf{X}_i - \bar{\mathbf{X}})'$ respectively and let $N_p(\mu, \mathbf{\Sigma})$ be the $p-$variate multivariate normal distribution with mean μ and covariance matrix $\mathbf{\Sigma}$.

We want to test the null hypothesis

$$H_o : \mathbf{X_1}, ...\mathbf{X_n} \text{ is a sample from } N_p(\boldsymbol{\mu}, \mathbf{\Sigma}) \text{ for some } \boldsymbol{\mu} \text{ and } \mathbf{\Sigma}.$$

First we use the transformation proposed by Doornik and Hansen (1994) to transform $\mathbf{X_1}, ...\mathbf{X_n}$ as

$$\mathbf{Z}_i = \mathbf{H}\Lambda^{-1/2}\mathbf{H}'\mathbf{V}^{-1/2}(\mathbf{X}_i - \bar{\mathbf{X}}) = \mathbf{A}(\mathbf{X}_i - \bar{\mathbf{X}}) \tag{4}$$

where $\mathbf{A} = \mathbf{H}\Lambda^{-1/2}\mathbf{H}'\mathbf{V}^{-1/2}$ "$\prime$" denotes a transpose, $\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_p)$, is the matrix with the eigenvalues of $\mathbf{C} = \mathbf{V}^{-1/2}\mathbf{S}\mathbf{V}^{-1/2}$, on the diagonal, $\mathbf{V}^{-1/2}$ is a matrix with the reciprocals of the standard deviation on the diagonal, $\mathbf{V}^{-1/2} = diag(S_{11}^{-1/2}, S_{22}^{-1/2}, ..., S_{pp}^{-1/2})$, the columns of $\mathbf{H}$ are the corresponding eigenvectors, such that $\mathbf{H}'\mathbf{H} = \mathbf{I}_p$, identity matrix of order $p \times p$, and $\mathbf{\Lambda} = \mathbf{H}'\mathbf{C}\mathbf{H}$. If $\mathbf{X_1}, \mathbf{X_2}, ..., \mathbf{X_n}$ are a sample from the $p-$variate multivariate normal $N_p(\boldsymbol{\mu}, \mathbf{\Sigma})$ with mean vector $\boldsymbol{\mu}$ and covariance matrix $\mathbf{\Sigma}$, then $\mathbf{Z} = (\mathbf{Z}_{1i}, \mathbf{Z}_{2i}, ..., \mathbf{Z}_{pi})$ is standard normals.

Now we transform $\mathbf{X_1}, \mathbf{X_2}, ..., \mathbf{X_n}$ using the equation (3) as discussed above. Now, we can think of test statistics that calculate $\mathbf{Z}_w^*$ by the equation (2) for each coordinate of $\mathbf{Z}$ such as $\mathbf{Z}_w^* = (Z_{w1}, Z_{w2}, ..., Z_{wp})'$, that has an approximate standard normal distribution under the null hypothesis of normality. Then the proposed test statistic can be stated as

$$D_w^* = \mathbf{Z}_w^{*\,'}\mathbf{Z}_w^* \tag{5}$$

has an asymptotic χ^2 distribution with degrees of freedom p.

The following test statistic can also be used to check normality.

$$D_w^{**} = \max_{1\le k\le p} Z_{wk}^2. \tag{6}$$

Under the null hypothesis of normality, Z_{wk} is approximately standard normal distribution. Therefore the limit distribution of D_w^{**} has the becomes

$$D_w^{**} \xrightarrow{d} \max_{1\le k\le p} U_k$$

where $U_1, U_2, ..., U_p$ follow i.i.d. chi-squared distribution with 1 degree of freedom, χ_1^2, and we have

$$\lim_{n\to\infty} p(D_w^{**} \le x) = (P(U_1 \le \chi_1^2))^p$$

Now, we can also calculate the $\mathbf{Z}_\beta = (Z_{\beta_1}, Z_{\beta_2}, ..., Z_{\beta_p})'$ by (1) from each coordinate, which yields the following two statistics that can be used to check the multivariate normality:

$$D_\beta^* = \mathbf{Z}_\beta \mathbf{Z}_\beta' \sim \chi_p^2 \tag{7}$$

and

$$D_\beta^{**} = \max_{1\le k\le p} Z_{\beta_k}^2 \tag{8}$$

Similarly the test based on D_β^* rejects H_0 at a test size α, if $D_\beta^* > \chi^2_{p,1-\alpha}$, where $P(X \le \chi^2_{p,1-\alpha}) = 1-\alpha$ if $X \sim \chi_p^2$ and D_β^{**} has $\max_{1\le k\le p} U_k$.

Kim (2015) proposed two tests for multivariate normality based on the the transformation

$$\mathbf{Z_i} = \mathbf{S}^{*'}(\mathbf{X}_i - \bar{\mathbf{X}}) \tag{3.7}$$

where $\mathbf{S}^*$ is defined by $\mathbf{S}^{*'}\mathbf{S}\mathbf{S}^* = \mathbf{I}$. In this paper, according to Kim (2015)'s transformation above, we calculate the $\mathbf{Z}_\beta = (Z_{\beta_1}, Z_{\beta_2}, ..., Z_{\beta_p})'$ by (2) from each coordinate, which yields the following similar multivariate normality statistics given by

$$C_\beta^* = \mathbf{Z}_\beta \mathbf{Z}_\beta' \sim \chi_p^2 \quad \text{and} \quad C_\beta^{**} = \max_{1\le k\le p} Z_{\beta_k}^2.$$

4. Simulation study

4.1. Application

We consider the famous multivariate data set first introduced by Rao (1948) that consists of weights of cork borings from the north (N), east (E), west (W), and south (S) (in centigrams) for 28 trees. He considered the following three constraints

$$y_1 = N - E - W + S, \qquad y_2 = S - W, \qquad y_3 = N - S.$$

After applying the transformation $\hat{\beta}$ in (1), values of Pearson's measure of kurtosis are

$$(\hat{\beta}_2(1), \hat{\beta}_2(2), \hat{\beta}_2(3)) = (3.1474, 2.6182, 1.9266).$$

Then applying the transformations Z_β in (2) yields

$$(Z_\beta(1), Z_\beta(2), Z_\beta(3)) = (0.7057, -0.0579, 1.6755).$$

And applying the transformations Z_w in (2.4) yields

$$(Z_w(1), Z_w(2), Z_w(3)) = (0.1304584, -0.6599033, -1.6101341).$$

Table 1 provides the values of the proposed test statistics in (5), (6), (7), (8) and p values of them. According to our result, the null hypothesis, constraints coming from multivariate normal distribution could not be rejected, that confirms Rao's test for contrasts as valid.

Table 1: Statistics and p-values for the Rao's bark deposit data

Tests	D_w^*	D_w^{**}	D_β^*	D_β^{**}
Statistic	3.0450	2.5925	3.309	2.8072
p-value	0.3847	0.2888	0.3465	0.2559

4.2. Simulation

A simulation was performed to compute the power of the proposed new test statistics denoted by D_w^*, D_w^{**} and D_β^*, D_β^{**} respectively according to different measure of kurtosis. We also compute the power of C_β^*, C_β^{**} according to Kim (2015)'s transformation and four other existing tests: Mardia's Skewness (M_S), Mardia's Kurtosis (M_K), Henze-Zirkler's (H_Z) and Shapiro-Wilk (S_w). The choices of simulation parameters are: $p = 2,\ 5$; $n = 20,\ 50$ and significance level $\alpha = 0.05, 0.1$. The alternatives included in the comparison study are standard normal ($N(0, 1)$), t_k distribution with k degrees of freedom (t_2, t_5), Cauchy, logistic and Laplace as opposed to asymmetric distributions χ_k^2 distribution with k degrees of freedom (χ_2^2, χ_5^2), Gamma distribution ($\Gamma(2, 0.5)$) and $exp(1)$. In the simulation, 10,000 Monte Carlo replications are carried out. The results of power computation are displayed in Tables 2 to 9.

Table 2: Power comparison of test statistics for $\alpha = 0.05, n = 20, p = 2$ under several alternatives

Tests	D_w^*	D_w^{**}	D_β^*	D_β^{**}	C_β^*	C_β^{**}	M_S	M_K	H_Z	S_W
$N(0, 1)$	0.0487	0.0469	0.0421	0.0443	0.0441	0.0441	0.0261	0.0052	0.0443	0.0484
t_2	0.6021	0.5964	0.5992	0.5750	0.6102	0.5930	0.6605	0.5931	0.5931	0.6182
t_5	0.2062	0.2070	0.2180	0.2113	0.2221	0.2214	0.2539	0.1580	0.1580	0.2205
Logistic	0.1079	0.1031	0.1518	0.1493	0.1831	0.1819	0.2566	0.1080	0.1080	0.2852
Laplace	0.2310	0.2276	0.2045	0.1981	0.2089	0.1985	0.2348	0.1467	0.1467	0.2203
Cauchy	0.9168	0.9076	0.9005	0.8799	0.9069	0.8909	0.9186	0.9154	0.9154	0.9259
χ_2^2	0.2923	0.2870	0.5048	0.4879	0.4954	0.4785	0.6935	0.2960	0.2960	0.9535
χ_5^2	0.1492	0.1419	0.2706	0.2600	0.2663	0.2554	0.3390	0.1058	0.1058	0.6036
$\Gamma(2, 0.5)$	0.1741	0.1688	0.3191	0.3071	0.3122	0.2976	0.4239	0.1395	0.5211	0.7126
$exp(1)$	0.2967	0.2911	0.5122	0.4906	0.5039	0.4812	0.7018	0.3042	0.8538	0.9563

4.3. Simulation results

The first line of each table shows Type 1 error of the considered tests for different p, n and nominal α values. It is expected that the empirical rejection rates generated from multivariate normal distributions are close to the significance level α. However, it is observed that the tests M_S, M_K shows severe deviations from α when $n = 20$, especially the test M_K.

Tables 2-5 report the mean power at significance level $\alpha = 0.05$ when $n = 20,\ 50$ and $p = 2,\ 5$. Certain reasonable patterns emerge from these tables. It is observed that all four proposed tests D_w^*, D_w^{**}, D_β^* and D_β^{**} are sensitive to the sample size. The powers are seen

Table 3: Power comparison of test statistics for $\alpha = 0.05, n = 50, p = 2$ under several alternatives

$Tests$	D^*_w	D^{**}_w	D^*_β	D^{**}_β	C^*_β	C^{**}_β	M_S	M_K	H_Z	S_W
$N(0,1)$	0.0465	0.0447	0.0474	0.0479	0.0477	0.0477	0.0419	0.0153	0.0447	0.0496
t_2	0.9645	0.9586	0.9541	0.9398	0.9536	0.9378	0.9203	0.9756	0.9686	0.9537
t_5	0.4834	0.4749	0.4930	0.4738	0.4941	0.4760	0.4987	0.5705	0.4146	0.1894
Cauchy	1.0000	1.0000	1.0000	1.000	1.0000	0.9999	1.0000	1.0000	1.0000	1.0000
Logistic	0.2083	0.2059	0.3479	0.3348	0.3824	0.3697	0.7368	0.4250	0.6678	0.7265
Laplace	0.5714	0.5512	0.4685	0.4328	0.4675	0.4300	0.4464	0.5795	0.5409	0.4695
χ^2_2	0.5531	0.5440	0.8452	0.8180	0.8422	0.8160	0.9984	0.7937	0.9996	1.0000
χ^2_5	0.2601	0.2542	0.5206	0.4951	0.5130	0.4905	0.9065	0.4328	0.8630	0.9869
$\Gamma(2, 0.5)$	0.3103	0.3045	0.5976	0.5760	0.5965	0.5734	0.9515	0.5202	0.9382	0.9964
$exp(1)$	0.5612	0.5560	0.8539	0.8278	0.8514	0.8252	0.9988	0.8065	0.9993	1.0000

Table 4: Power comparison of test statistics for $\alpha = 0.05, n = 20, p = 5$ under several alternatives

$Tests$	D^*_w	D^{**}_w	D^*_β	D^{**}_β	C^*_β	C^{**}_β	M_S	M_K	H_Z	S_W
$N(0,1)$	0.0525	0.0459	0.0383	0.0400	0.0387	0.0368	0.0088	0.0064	0.0456	0.0467
t_2	0.4792	0.4579	0.4326	0.3606	0.6381	0.5893	0.8637	0.6426	0.8312	0.5449
t_5	0.1687	0.1694	0.1637	0.1498	0.2154	0.2130	0.3460	0.0769	0.2776	0.1894
Cauchy	0.8352	0.7976	0.7388	0.5946	0.9304	0.8886	0.9906	0.9673	0.9931	0.8958
Logistic	0.1525	0.1472	0.2141	0.2056	0.2319	0.2252	0.1798	0.0127	0.2170	0.4150
Laplace	0.1208	0.1198	0.1016	0.0962	0.1132	0.1056	0.1508	0.0171	0.1594	0.1152
χ^2_2	0.4785	0.4577	0.7206	0.6441	0.6801	0.6073	0.6222	0.1180	0.8024	0.9967
χ^2_5	0.2247	0.2128	0.3839	0.3423	0.3507	0.3106	0.1992	0.0122	0.3360	0.8197
$\Gamma(2, 0.5)$	0.2741	0.2600	0.4704	0.4102	0.4279	0.3787	0.2827	0.0228	0.4287	0.9072
$exp(1)$	0.4811	0.4508	0.7185	0.6422	0.6815	0.6071	0.6244	0.1198	0.8072	0.9962

Table 5: Power comparison of test statistics for $\alpha = 0.05, n = 50, p = 5$ under several alternatives

$Tests$	D^*_w	D^{**}_w	D^*_β	D^{**}_β	C^*_β	C^{**}_β	M_S	M_K	H_Z	S_W
$N(0,1)$	0.0496	0.0502	0.0555	0.0577	0.0523	0.0595	0.0391	0.0262	0.0559	0.0504
t_2	0.9938	0.9872	0.9904	0.9714	0.9919	0.9757	0.9991	0.9999	0.9990	0.9899
t_5	0.6062	0.5688	0.6288	0.5457	0.6428	0.5663	0.8673	0.8610	0.7107	0.6020
Cauchy	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	1.0000	1.0000	1.0000	1.0000
Logistic	0.3047	0.2965	0.5214	0.4732	0.5395	0.4918	0.8687	.4612	0.6341	0.9349
Laplace	0.3270	0.3086	0.2948	0.2447	0.3011	0.2498	0.5745	0.5342	0.4511	0.2918
χ^2_2	0.8230	0.7918	0.9859	0.9558	0.9812	0.9480	1.000	0.8959	1.0000	1.0000
χ^2_5	0.4302	0.4111	0.7699	0.6908	0.7565	0.6737	0.9476	0.4449	0.8715	0.9999
$\Gamma(2, 0.5)$	0.5124	0.4772	0.8520	0.7703	0.8359	0.7521	0.9836	0.5554	0.9486	1.0000
$exp(1)$	0.8270	0.7943	0.9868	0.9586	0.9842	0.9514	1.0000	0.8969	0.9999	1.0000

to rise as the sample size increases. Due to different kurtosis measure, it is noticed that the pair of tests D^*_β and D^{**}_β has better power than the pair of tests D^*_w and D^{**}_w in most cases. In addition, we can see that D^*_w is slightly superior to D^{**}_w. This pattern occurs in the pairs of D^*_β, D^{**}_β and C^*_β, C^{**}_β. Considering same Pearson kurtosis measure $\hat{\beta}_2$, Tests D^*_β and D^{**}_β perform similarly as C^*_β and C^{**}_β. This fact indicates that the multivariate standard normal transformation stated in (4) is competitive to the transformation proposed by Kim (2015).

For multivariate t(2) distribution and Cauchy distribution, when $n = 50$, all tests display high power. When $n = 20$, these two distributions still have better power than other distributions. Among the skewed alternative distributions, exp(1) has higher power, but is sensitive to sample size, and the power decreases when $n = 20$. When $n = 20$, the test M_k shows lowest power, which is related to its Type 1 error's deviation from the significance level α. When samples are from symmetric distributions $(t_2, t_5, logistic, Laplace)$, all four proposed tests

Table 6: Power comparison of test statistics for $\alpha = 0.1, n = 20, p = 2$ under several alternatives

$Tests$	D_w^*	D_w^{**}	D_β^*	D_β^{**}	C_β^*	C_β^{**}	M_S	M_K	H_Z	S_W
$N(0,1)$	0.0972	0.0971	0.0887	0.0878	0.0860	0.0897	0.0512	0.0126	0.1016	0.0959
t_2	0.6686	0.6626	0.6814	0.6633	0.6917	0.6753	0.7196	0.6438	0.7558	0.6902
t_5	0.2756	0.2707	0.3047	0.2963	0.3083	0.3068	0.3277	0.2010	0.3069	0.3060
Cauchy	0.9369	0.9319	0.9298	0.9195	0.9346	0.9255	0.9391	0.9329	0.9736	0.9458
Logistic	0.1647	0.1597	0.2288	0.2269	0.2629	0.2604	0.3460	0.1401	0.4312	0.3903
Laplace	0.4267	0.4268	0.3964	0.3865	0.3979	0.3886	0.4278	0.3072	0.5031	0.4172
χ_2^2	0.3684	0.3601	0.5988	0.5876	0.5906	0.5790	0.7862	0.3517	0.9048	0.9777
χ_5^2	0.2208	0.2121	0.3659	0.3542	0.3575	0.3472	0.4453	0.1415	0.5612	0.7195
$\Gamma(2, 0.5)$	0.2474	0.2420	0.4186	0.4060	0.4087	0.3980	0.5333	0.1802	0.6576	0.8107
$exp(1)$	0.3797	0.3665	0.6037	0.5897	0.5971	0.5812	0.7915	0.3616	0.9159	0.9787

Table 7: Power comparison of test statistics for $\alpha = 0.1, n = 20, p = 5$ under several alternatives

$Tests$	D_w^*	D_w^{**}	D_β^*	D_β^{**}	C_β^*	C_β^{**}	M_S	M_K	H_Z	S_W
$N(0,1)$	0.1014	0.0845	0.0837	0.823	0.0826	0.0843	0.0207	0.0619	0.0994	0.0950
t_2	0.5634	0.5414	0.5425	0.4826	0.7198	0.6854	0.9032	0.7163	0.8786	.6361
t_5	0.2389	0.2396	0.2529	0.2405	0.3101	0.3024	0.4357	0.1234	0.3814	0.2732
Cauchy	0.8762	0.8515	0.8198	0.7213	0.9553	0.9346	0.9946	0.9776	0.9951	0.9268
Logistic	0.2197	0.2116	0.3007	0.2926	0.3239	0.3111	0.2585	0.0343	0.3325	0.5326
Laplace	0.4478	0.4366	0.3897	0.3438	0.4367	0.3938	0.7120	0.4038	0.7828	0.4428
χ_2^2	0.5575	0.5352	0.7952	0.7422	0.7606	0.7105	0.7173	0.1722	0.8824	0.9985
χ_5^2	0.3060	0.2849	0.4936	0.4576	0.4521	0.4218	0.2883	0.0350	0.4755	0.8864
$\Gamma(2, 0.5)$	0.3628	0.3364	0.5764	0.5312	0.5338	0.4910	0.3797	0.0482	0.5681	0.9455
$exp(1)$	0.5555	0.5315	0.7985	0.7443	0.7649	0.7115	0.7219	0.1732	0.8838	0.9989

Table 8: Power comparison of test statistics for $\alpha = 0.1, n = 50, p = 2$ under several alternatives

$Tests$	D_w^*	D_w^{**}	D_β^*	D_β^{**}	C_β^*	C_β^{**}	M_S	M_K	H_Z	S_W
$N(0,1)$	0.0923	0.0929	0.0936	0.0958	0.0960	0.0989	0.0823	0.0405	0.1008	0.1034
t_2	0.9740	0.9711	0.9702	0.9632	0.9697	0.9621	0.9380	0.9823	0.9790	0.9677
t_5	0.5603	0.5534	0.5854	0.5715	0.5872	0.5725	0.5732	0.6250	0.5171	0.5548
Cauchy	1.0000	0.9999	0.9997	0.9992	0.9997	0.9994	0.9970	1.0000	1.0000	0.9998
Logistic	0.2820	0.2778	0.4370	0.4284	0.4716	0.4610	0.8154	0.4769	0.7664	0.8084
Laplace	0.8621	0.8448	0.7612	0.7203	0.7640	0.7275	0.6733	0.8363	0.8758	0.7788
χ_2^2	0.6285	0.6214	0.8926	0.8796	0.8910	0.8781	0.9995	0.8312	1.0000	1.0000
χ_5^2	0.3378	0.3329	0.6107	0.5950	0.6107	0.5930	0.9475	0.4891	0.9222	0.9946
$\Gamma(2, 0.5)$	0.3951	0.3884	0.6865	0.6656	0.6816	0.6658	0.9753	0.5783	0.9707	0.9984
$exp(1)$	0.6316	0.6236	0.9003	0.8857	0.8983	0.8844	0.9997	0.8438	0.9999	1.0000

display similar power results as other existing tests under the this study.

Tables 6-9 report the mean power at significance level $\alpha = 0.1$ when $n = 20,\ 50$ and $p = 2,\ 5$. Noticeably the tests are more powerful compared with significance level $\alpha = 0.05$. The detailed power analysis is paralleled to the above comparison for $\alpha = 0.05$, which has confirmed the validity of the proposed test statistics.

5. Concluding remarks

In general, all four proposed tests (D_w^*, D_w^{**}, D_β^* and D_β^{**}) had power close to α when the data distribution was multivariate normal, and the tests D_β^* and D_β^{**} had power competitive with some of the existing tests when the data distribution was not multivariate normal. When

Table 9: Power comparison of test statistics for $\alpha = 0.1, n = 50, p = 5$ under several alternatives

Tests	D_w^*	D_w^{**}	D_β^*	D_β^{**}	C_β^*	C_β^{**}	M_S	M_K	H_Z	S_W
$N(0,1)$	0.0960	0.0920	0.0978	0.1028	0.1001	0.1053	0.0675	0.0810	0.0998	0.0969
t_2	0.9958	0.9927	0.9943	0.9869	0.9945	0.9875	0.9991	1.0000	0.9995	0.9940
t_5	0.6881	0.6547	0.7162	0.6555	0.7283	0.6770	0.9045	0.8953	0.7790	0.6833
Cauchy	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Logistic	0.3928	0.3791	0.6195	0.5798	0.6376	0.5930	0.9168	0.5316	0.7339	0.9614
Laplace	0.9607	0.9333	0.9030	0.8104	0.9094	0.8216	0.9825	0.9961	0.9979	0.9101
χ_2^2	0.8669	0.8468	0.9934	0.9797	0.9894	0.9729	1.0000	0.9250	1.0000	1.0000
χ_5^2	0.5162	0.4919	0.8392	0.7821	0.8216	0.7709	0.9708	0.5217	0.9232	1.0000
$\Gamma(2, 0.5)$	0.5933	0.5677	0.9019	0.8531	0.8886	0.8386	0.9930	0.6224	0.9717	1.0000
$exp(1)$	0.8701	0.8513	0.9934	0.9812	0.9911	0.9779	1.0000	0.9221	1.0000	1.0000

sample size $n = 20$, noticeably D_w^*, D_w^{**}, D_β^* and D_β^{**} perform better than the test M_k for most cases. In the study, there is no situation where one test reaches the best power under all combinations of n, p, and α. But overall tests H_Z and S_W are well known as the powerful tests. For the skewed alternative distributions, when sample size tends to moderately large, the new tests D_β^* and D_β^{**} usually had better power than some of the existing tests. But the new proposed tests D_w^* and D_w^{**} particularly possesses inferior performance. It is of a future problem, when we form a new $p-$dimensional multivariate test by combining chi-squared distributions based on kurtosis measures, it may be noted that we investigate a weighted chi-squared distribution test. For further numerical summaries, we also recommend the graphical methods as aids to detect the departures from multivariate normality.

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Affiliation:

Danush K. Wijekularathna
Department of Mathematics
Troy Univeristy
Troy, AL, USA
E-mail: dwijekularathna@troy.edu

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