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Robust Bayesian Analysis of Lifetime Data from Maxwell Distribution

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Abstract

In this paper, we consider robust Bayesian analysis of lifetime data from the Maxwell distribution assuming an ε-contamination class of prior distributions for the parameter. We obtain robust Bayes estimates of the parameter and mean lifetime under squared error and LINEX loss functions in presence of uncensored as well as Type-I progressively hybrid censored lifetime data. A real data set is analysed for numerical illustrations.

Keywords: Maxwell distribution, *ML-II* procedure, robust Bayesian estimation, type-I progressive hybrid censoring scheme.

1. Introduction

In Bayesian analysis, the investigator is supposed to posses some subjective a priori information concerning the most probable values of the parameter. In many cases he is able to successfully present his belief about the parameter in form of a single prior density. However, when the belief of investigator cannot be adequately represented in form of a single prior density or there is a possibility of error in prior elicitation, a class of distributions may be used to successfully present the prior belief. In such cases it becomes impossible to proceed with usual Bayesian procedures to make decisions or inferences. The robust Bayesian viewpoint provides a way to deal with such problems and to make decisions that behave satisfactorily when the prior varies over a class of prior distributions. Many authors provided different methods for implementing the robust Bayesian viewpoint. For some literature review one may refer to Box and Tiao (1973), Good (1965, 1983), Dempster (1975) and Kadane and Chuang (1978).

A reasonable method for implementing uncertainties in prior elicitation is through the use of ε-contamination class of prior distributions given by

$$\Gamma = \{q : q = (1-\varepsilon)g_0 + \varepsilon g; g \in G\}, \tag{1}$$

where $\varepsilon (0 \le \varepsilon \le 1)$ is pre-assigned and represents the probability of error in the prior elicitation of the base prior g_0 and g is a distribution from the class G of all possible contaminated distributions. Many authors advocated Bayesian analysis based on the ε-contamination class of prior distributions [See Berger (1982), Berger (1983), Berger and Berliner (1986, 1984) , Sivaganesan and Berger (1987), Chaturvedi (1996) to cite a few].

Berger and Berliner (1986) provides a good review of literature and additional motivation for consideration of *ML-II* procedure of Good (1983) for selecting a prior from an ε-contamination class in a data dependent fashion. According to this procedure, one can select a prior from the considered class by maximizing the predictive density corresponding to the prior. The prior thus obtained is called type-II maximum likelihood prior or *ML-II* prior in short. The Bayes estimators obtained under *ML-II* priors are termed as *ML-II* estimators. Chaturvedi, Pati, and Tomer (2014) carried out robust Bayesian analysis of Weibull distribution by implementing *ML-II* procedure.

In many real life investigations, for example life testing and reliability, we have to deal with censored data which often arise when life testing experiments are terminated before observing lifetimes of all units on test. In this context, plenty of censoring schemes have been studied and proposed in literature during last few decades. For a general review of literature on censoring schemes one may refer to Lawless (2003) and Balakrishnan and Aggarwala (2000). In this paper we shall consider a very generalized censoring scheme termed as Type-I progressive hybrid censoring scheme(*Type-I PHCS*) [Kundu and Joarder (2006) and Childs, Chandrasekar, and Balakrishnan (2008)]. This censoring scheme is recent and quite popular in literature [see Tomer and Panwar (2015)]. *Type-I PHCS* is described as follows. Suppose in a life testing experiment, n units are put to test. The maximum duration of the experiment t_0, the integers $R_1, R_2, \cdots, R_m$ $(1 \leq m \leq n)$ are fixed before beginning of the experiment. At the time of first failure X_1, R_1 units, out of $(n-1)$ surviving units, are randomly withdrawn from the test. At the time of second failure X_2, R_2 units, out of remaining $(n - R_1 - 1)$ units, are randomly withdrawn from the test. The process continues till time $T = \min\{X_m, t_0\}$. In the case when $X_m > t_0$, we observe the sample $X_{1:m:n}, X_{2:m:n}, \cdots, X_{d:m:n}$, where $d(\leq m)$ denotes the number of failures observed before time t_0, and terminate the experiment at t_0 by withdrawing $R_d^*(= n - d - \sum_{i=1}^{d} R_i)$ units, whereas if $X_m < t_0$, we observe $X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n}$ and the experiment is terminated at X_m. The data observed under type-I *PHCS* is termed as type-I progressively hybrid censored (type-I PHC) data.

The purpose of this article is many fold. We consider robust Bayesian estimation of the parameter and mean lifetime of the Maxwell distribution(*MWD*) under ε-contamination class of prior distributions in presence of uncensored as well as censored (type-I PHC) lifetime data. Under both types of data, we provide ML-*II* estimates under symmetric(squared error) and asymmetric(LINEX) loss functions. Rest of the paper is organized as follows. In Section 2, we derive *ML-II* estimator of the parameter and mean lifetime for uncensored sample. In Section 3, we develop procedure to obtain *ML-II* estimates for *Type-I PHC* data. In Section 4, we give a numerical example based on a real data set. Finally, we conclude findings in Section 5.

2. Estimation with uncensored data

A continuous non-negative random variable(*rv*) X is said to follow *MWD* if its probability density function (*pdf*) is given by

$$f(x, \theta) = \frac{4}{\sqrt{\pi}} \frac{1}{\theta^{3/2}} x^2 e^{-x^2/\theta}; \qquad 0 \leq x < \infty, \theta > 0, \tag{2}$$

where θ is the unknown parameter. When the lifetime of a device follows the *pdf* (2), its mean lifetime is $\xi = 2\sqrt{(\theta/\pi)}$ and its reliability function $\bar{F}(t)$, at a specified mission time

$t(\geq 0)$, comes out to be

$$\bar{F}(t) = P(X \geq t) = \frac{2}{\sqrt{\pi}}\Gamma_{3/2}\Big(\frac{t^2}{\theta}\Big), \tag{3}$$

where $\Gamma_a(z) = \int_z^{\infty} u^{a-1}e^{-u}du$. Krishna and Malik (2009) has shown that MWD belongs to the class of increasing failure rate distributions. Therefore, it can be used as a lifetime model in various investigations where age of the device affects it adversely. Krishna and Malik (2012) obtained ML and Bayes estimators of the parameter and reliability function of MWD under Type-II progressive censoring scheme whereas Krishna, Vivekanand, and Kumar (2015) worked out similar problem with randomly censored data.

Suppose that the rv X denotes the lifetime of a device and follows $MWD(\theta)$. A random sample of such n independent and identically distributed lifetimes $X_1, X_2, \cdots, X_n$ (denoted by $\underline{x}$ henceforth) is observed in a certain life testing experiment. The likelihood function of θ, in the light of given sample, comes out to be

$$l(\theta|\underline{x}) = \left(\frac{4}{\sqrt{\pi}}\right)^n \frac{1}{\theta^{3n/2}} \prod_{i=1}^{n} x_i^2 \exp\left(-\frac{T}{\theta}\right), \tag{4}$$

where $T = \sum_{i=1}^{n} x_i^2$. Following our discussion in Section 1, the considered ε-contamination class of prior distributions for θ is given by

$$\Gamma = \{q(\theta) : q(\theta) = (1-\varepsilon)g_0(\theta|\mu_0) + \varepsilon g(\theta|\mu); g \in G\}. \tag{5}$$

Here, we take the base prior, a natural conjugate prior [see Chib and Tiwari (1991), Chaturvedi *et al.* (2014)], given by the pdf

$$g_0(\theta|\mu_0) = \frac{\mu_0^{\nu}}{\Gamma(\nu)\theta^{\nu+1}} \exp\left(-\frac{\mu_0}{\theta}\right); \qquad 0 < \theta < \infty; \mu_0, \nu > 0, \tag{6}$$

where (μ_0, ν) represents the hyper parameters. The contamination class G is the class of all natural conjugate priors with hyper parameters (μ, ν), given by

$$G = \left\{ g(\theta|\mu) = \frac{\mu^{\nu}}{\Gamma(\nu)}\theta^{\nu+1} \exp\left(-\frac{\mu}{\theta}\right); \mu \in (\mu_0, \infty)\right\}. \tag{7}$$

According to *ML-II* procedure, discussed in Section 1, we select a prior density from the class Γ by maximizing the predictive density corresponding to q. For this, we first obtain the predictive density corresponding to the base prior $g_0(\theta|\mu_0)$ as follows.

$$\begin{aligned} m(\underline{\mathrm{x}}|g_0) &= \int_0^{\infty} l(\theta|\underline{\mathrm{x}})g_0(\theta|\mu_0)d\theta \\ &= \left(\frac{4}{\sqrt{\pi}}\right)^n \frac{\mu_0^{\nu}}{\Gamma(\nu)} \prod_{i=1}^{m} x_i^2 \int_0^{\infty} \frac{1}{\theta^{3n/2+\nu+1}} \exp\left\{-\frac{1}{\theta}(T+\mu_0)\right\} \\ &= \left(\frac{4}{\sqrt{\pi}}\right)^n \frac{\mu_0^{\nu}\Gamma(3n/2+\nu)}{\Gamma(\nu)(T+\mu_0)^{(3n/2+\nu)}} \prod_{i=1}^{m} x_i^2. \end{aligned} \tag{8}$$

Similarly, the predictive density for $g(\theta|\mu)$ comes out to be

$$m(\underline{\mathrm{x}}|g) = \left(\frac{4}{\sqrt{\pi}}\right)^n \frac{\mu^{\nu}\Gamma(3n/2+\nu)}{\Gamma(\nu)(T+\mu)^{(3n/2+\nu)}} \prod_{i=1}^{m} x_i^2. \tag{9}$$

Now the predictive density corresponding to the generic prior $q \in \Gamma$ is

$$m(\underline{x}|q) = (1-\varepsilon)m(\underline{x}|g_0) + \varepsilon m(\underline{x}|g).$$

In the *ML-II* process we choose value of the unknown hyper parameter μ in a data dependent fashion by maximizing the predictive density $m(\underline{x}|q)$ over the class of all priors $q \in \Gamma$. Since g_0 is fixed, we have

$$\sup_{q\in\Gamma} m(\underline{x}|q) = (1-\varepsilon)m(\underline{x}|g_0) + \varepsilon \sup_{g\in G} m(\underline{x}|g)$$

and $m(\underline{x}|g)$ is maximized when we replace μ by its maximum likelihood estimator in $g(\theta|\mu)$ which is given by

$$\hat{\mu} = \max\left\{\mu_0, \frac{2\nu T}{3n}\right\}.$$

Then we get

$$\hat{g}(\theta|\hat{\mu}) = \begin{cases} \frac{2\nu T}{3n\theta^{\nu+1}\Gamma(\nu)} \exp\left(-\frac{2\nu T}{3n\theta}\right) = \hat{g} & \text{if} \quad \mu_0 < \frac{2\nu T}{3n} \\ g_0(\theta|\mu_0) = g_0 & \text{if} \quad \mu_0 \geq \frac{2\nu T}{3n}. \end{cases}$$

Thus, the *ML-II* prior density is given by

$$\hat{q}(\theta) = (1-\varepsilon)g_0(\theta|\mu_0) + \varepsilon\hat{g}(\theta|\hat{\mu}). \tag{10}$$

Following Berger and Berliner (1986), the *ML-II* posterior density of θ, obtained using (4) and (10), comes out to be

$$\hat{q}^*(\theta) = \hat{\lambda}g_0^*(\theta) + (1-\hat{\lambda})g^*(\theta); \qquad 0 < \theta < \infty, \tag{11}$$

where

$$\begin{aligned} g_0^*(\theta) &= \frac{l(\theta|\underline{x})g_0(\theta|\mu_0)}{\int_0^\infty l(\theta|\underline{x})g_0(\theta|\mu_0)d\theta} \\ &= \frac{(T+\mu_0)^{(3n/2+\nu)}}{\theta^{3n/2+\nu+1}\Gamma\left(3n/2+\nu\right)} \exp\left\{-\frac{1}{\theta}\left(T+\mu_0\right)\right\} \quad \text{if} \quad \mu_0 \geq \frac{2\nu T}{3n}. \end{aligned} \tag{12}$$

Similarly, we get

$$g^*(\theta) = \begin{cases} \frac{(T+\hat{\mu})^{(3n/2+\nu)}}{\theta^{3n/2+\nu+1}\Gamma(3n/2+\nu)} \exp\left\{-\frac{1}{\theta}\left(T+\hat{\mu}\right)\right\} & \text{if} \quad \mu_0 < \frac{2\nu T}{3n} \\ g_0^*(\theta) & \text{if} \quad \mu_0 \geq \frac{2\nu T}{3n}. \end{cases} \tag{13}$$

and

$$\hat{\lambda} = \frac{(1-\varepsilon)m(\underline{x}|g_0)}{(1-\varepsilon)m(\underline{x}|g_0) + \varepsilon m(\underline{x}|\hat{g})}$$

which on using (8) and (9), comes out to be

$$\hat{\lambda} = \begin{cases} \left\{1 + \frac{\varepsilon}{(1-\varepsilon)}\left(\frac{3n}{T}\right)^{3n/2}\left(\frac{2\nu}{\mu_0}\right)^{\nu}\left(\frac{T+\mu_0}{3n+2\nu}\right)^{(3n/2+\nu)}\right\}^{-1} & \text{if} \quad \mu_0 < \frac{2\nu T}{3n} \\ (1-\varepsilon) & \text{if} \quad \mu_0 \geq \frac{2\nu T}{3n}. \end{cases}$$

Remark 2.1. *In order to show the feasibility of the ML-II prior, we have*

$$\frac{\partial\hat{\lambda}}{\partial\mu_0}=\begin{cases}\dfrac{\frac{\varepsilon}{(1-\varepsilon)}\left(\frac{3n}{T}\right)^{3n/2}\frac{(2\nu)^{\nu}(T+\mu_0)^{3n/2+\nu-1}}{(3n+2\nu)^{3n/2+\nu}\mu^{\nu+1}}}{1+\frac{\varepsilon}{(1-\varepsilon)}\left(\frac{3n}{T}\right)^{3n/2}\left(\frac{2\nu}{\mu_0}\right)^{\nu}\left(\frac{T+\mu_0}{3n+2\nu}\right)^{(3n/2+\nu)}}\left(\frac{2\nu T}{3n}-\mu_0\right) & \text{if } \mu_0<\frac{2\nu T}{3n}\\ 0 & \text{if } \mu_0\geq\frac{2\nu T}{3n}.\end{cases}$$

Notice that $\frac{\partial\hat{\lambda}}{\partial\mu_0}$ *is greater than zero if* $\mu_0<\frac{2\nu T}{3n}$ *and equal to zero if* $\mu_0\geq\frac{2\nu T}{3n}$. *Thus, if the base prior is not compatible with the data,* $\hat{\lambda}$ *decreases and more weight is provided to the data based part of the ML-II posterior density* $\hat{q}^*(\theta)$ *i.e.* $\hat{g}^*(\theta)$. *As* $\mu_0\to 0, \hat{\lambda}\to 0$ *and for* $\mu_0\geq\frac{2\nu T}{3n}$, $\hat{\lambda}=1-\varepsilon$, *which is the maximum possible value of* $\hat{\lambda}$.

2.1. Estimation under SELF

We derive *ML-II* estimators of the parameter θ and mean lifetime ξ under squared error loss function (*SELF*) along with their posterior variances in the following theorems.

Theorem 1. *The ML-II posterior mean and variance of* θ *are given, respectively, by*

$$\hat{\theta}=\begin{cases}\frac{1}{3n/2+\nu-1}\left\{\left(1+\frac{2\nu}{3n}\right)T+\hat{\lambda}\left(\mu_0-\frac{2\nu T}{3n}\right)\right\} & \text{if } \mu_0<\frac{2\nu T}{3n}\\ \frac{1}{3n/2+\nu-1}\left(\mu_0+T\right) & \text{if } \mu_0\geq\frac{2\nu T}{3n}.\end{cases} \tag{14}$$

and

$$V_{q^*}(\theta)=\begin{cases}\frac{1}{(3n/2+\nu-1)^2}\left[\frac{1}{3n/2+\nu-2}\left\{\hat{\lambda}\left(T+\mu_0\right)^2+(1-\hat{\lambda})T^2\left(1+\frac{2\nu}{3n}\right)^2\right\}\right] & \\ +\frac{\hat{\lambda}(1-\hat{\lambda})}{(3n/2+\nu-1)^2}\left(\mu_0-\frac{2\nu T}{3n}\right) & \text{if } \mu_0<\frac{2\nu T}{3n}\\ \frac{1}{(3n/2+\nu-1)^2(3n/2+\nu-2)}\left(T+\mu_0\right)^2 & \text{if } \mu_0\geq\frac{2\nu T}{3n}.\end{cases} \tag{15}$$

Proof. See Appendix. □

Theorem 2. *The ML-II posterior mean and variance of* ξ *are given, respectively, by*

$$\hat{\xi}=\begin{cases}\frac{2\Gamma\left(3n/2+\nu-\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(3n/2+\nu)}\left[\hat{\lambda}\left(T+\mu_0\right)^{1/2}+(1-\hat{\lambda})\sqrt{T}\left(1+\frac{2\nu}{3n}\right)^{\frac{1}{2}}\right] & \text{if } \mu_0<\frac{2\nu T}{3n}\\ \frac{2\Gamma(3n/2+\nu-1/2)}{\sqrt{\pi}\Gamma(3n/2+\nu)}\left(T+\mu_0\right)^{1/2} & \text{if } \mu_0\geq\frac{2\nu T}{3n}.\end{cases}$$

and

$$V_{q^*}(\xi)=\begin{cases}\frac{4}{\pi(\Gamma(3n/2+\nu))^2}\left[\left\{\Gamma\left(3n/2+\nu\right)\Gamma\left(3n/2+\nu-1\right)-\left(\Gamma\left(3n/2+\nu-(1/2)\right)\right)^2\right\}\right. & \\ \left.\left\{\left(1+\frac{2\nu}{3n}\right)T+\hat{\lambda}\left(\mu_0-\frac{2\nu T}{3n}\right)\right\}\right]+\frac{4\hat{\lambda}(1-\hat{\lambda})(\Gamma(3n/2+\nu-1/2))^2}{\pi(\Gamma(3n/2+\nu))^2} & \\ .\left\{\left(T+\mu_0\right)^{\frac{1}{2}}-\sqrt{T}\left(1+\frac{2\nu}{3n}\right)^{\frac{1}{2}}\right\}^2 & \text{if } \mu_0<\frac{2\nu T}{3n}\\ \frac{4}{\pi(3n/2+\nu-1)}\left(T+\mu_0\right). & \text{if } \mu_0\geq\frac{2\nu T}{3n}.\end{cases}$$

Proof. The *ML-II* posterior mean of ξ is

$$\begin{aligned}\hat{\xi}=&\frac{2}{\sqrt{\pi}}E_{q^*}(\sqrt{\theta})\\ =&\hat{\lambda}E_{g_0^*}(\sqrt{\theta})+(1-\hat{\lambda})E_{g^*}(\sqrt{\theta})\end{aligned}$$

and rest part of the Proof is similar to that of Theorem 1. □

2.2. Estimation under LINEX loss function

In previous section, we used a symmetric loss function *SELF* for estimation of the unknown parameter θ. This loss function is appropriate for the inferential problems when underestimation and overestimation of the parameter are of equal consequences. However, there may be circumstances when it does not happen. For example, overestimation of average lifetime or reliability of a component of an aircraft may be more serious than its underestimation. In such cases asymmetric loss functions are preferred. Among several asymmetric loss functions [see Calabria and Pulcini (1994)], LINEX loss function introduced by Varian (1975) is quite popular in literature. Zellner (1986) used LINEX loss function for Bayesian estimation of scale parameter. Kim, Jung, and Chung (2011), Doostparast, Ahmadi, and Ahmadi (2013) and Panwar, Tomer, and Kumar (2015) used it for different problems of estimation in presence of censored lifetime data. The expression of the LINEX loss function while estimating the parameter θ by its estimator $\hat{\theta}$ is

$$L(\Delta) = \exp(a\Delta) - a\Delta - 1, \quad a \neq 0 \tag{16}$$

where $\Delta = \hat{\theta} - \theta$.
Under the LINEX loss function (16), the *ML-II* estimator of θ is given by

$$\begin{aligned}\hat{\theta}_L &= -\frac{1}{a}\ln E_{q^*}[\exp(-a\theta)],\\ &= \begin{cases} -\frac{1}{a}\ln\left(\hat{\lambda}E_{g_0^*}[\exp(-a\theta)] + (1-\hat{\lambda})E_{g^*}[\exp(-a\theta)]\right) & \text{if} \quad \mu_0 < \frac{2\nu T}{3n} \\ -\frac{1}{a}\ln\left(E_{g_0^*}[\exp(-a\theta)]\right) & \text{if} \quad \mu_0 \geq \frac{2\nu T}{3n}. \end{cases}\end{aligned}$$

Here, on using (13), we obtain for $\mu_0 \geq \frac{2\nu T}{3n}$ that

$$\begin{aligned}E_{g_0^*}[\exp(-a\theta)] =& \frac{(T+\mu_0)^{(3n/2+\nu)}}{\Gamma(3n/2+\nu)}\int_0^\infty \frac{1}{\theta^{3n/2+\nu+1}}\exp\left\{-a\theta - \frac{1}{\theta}(T+\mu_0)\right\}d\theta\\ =& \frac{2\{a(T+\mu_0)\}^{(3n/2+\nu)/2}}{\Gamma(3n/2+\nu)}H_{-(3n/2+\nu)}\left(2\sqrt{a(T+\mu_0)}\right),\end{aligned}$$

where $H_\nu(z)$ is a modified Bessel function of third kind of order ν,[Gradshteyn and Ryzhik (1965), pp.340].

Similarly, for $\mu_0 < \frac{2\nu T}{3n}$, we get

$$E_{g^*}[\exp(-a\theta)] = \frac{2\{a(1+2\nu/3n)T\}^{(3n/2+\nu)/2}}{\Gamma(3n/2+\nu)}H_{-(3n/2+\nu)}\left(2\sqrt{a(1+2\nu/3n)T}\right).$$

The expectation of the LINEX loss function for $\hat{\theta}_L$ with respect to *ML-II* posterior distribution of θ comes out to be

$$\begin{aligned}aE_{\hat{q}^*}[\theta - \hat{\theta}_L] =& a(E_{\hat{q}^*}[\theta] - E[\hat{\theta}_L]\\ =& \begin{cases} a\left[\frac{1}{\frac{3n}{2}+\nu-1}\left\{\left(1+\frac{2\nu}{3n}\right)T + \hat{\lambda}\left(\mu_0 - \frac{2\nu T}{3n}\right)\right\} - \hat{\theta}_L\right] & \text{if} \quad \mu_0 < \frac{2\nu T}{3n} \\ a\left[\frac{1}{\frac{3n}{2}+\nu-1}(\mu_0+T) - \hat{\theta}_L\right] & \text{if} \quad \mu_0 \geq \frac{2\nu T}{3n} \end{cases}\end{aligned}$$

Under the LINEX loss function (16), the *ML-II* estimator of ξ when $A = 2a/\sqrt{\pi}$ is

$$\begin{aligned}\hat{\xi}_L &= -\frac{1}{a}\ln E_{q^*}[\exp(-A\theta^{1/2})],\\ &= \begin{cases} -\frac{1}{a}\ln(\hat{\lambda}\hat{\xi}_{L_0} + (1-\hat{\lambda})\hat{\xi}_{L^*}) & \text{if} \quad \mu_0 < \frac{2\nu T}{3n} \\ -\frac{1}{a}\ln(\hat{\xi}_{L_0}) & \text{if} \quad \mu_0 \geq \frac{2\nu T}{3n}, \end{cases}\end{aligned}$$

where $\hat{\xi}_{L_0} = E_{g_0^*}[\exp(-A\theta^{1/2})]$ and $\hat{\xi}_{L^*} = E_{g_0^*}[\exp(-A\theta^{1/2})]$. The expressions for these are derived in Appendix. The expectation of the *LINEX* loss function for $\hat{\xi}_L$ is

$$aE_{\hat{q}^*}[\xi - \hat{\xi}_L] = \begin{cases} a\left[\frac{2\Gamma\left(\frac{3n}{2}+\nu-\frac{1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{3n}{2}+\nu\right)}\left[\hat{\lambda}\,(T+\mu_0)^{\frac{1}{2}} + (1-\hat{\lambda})\sqrt{T}\left(1+\frac{2\nu}{3n}\right)^{\frac{1}{2}}\right] - \hat{\xi}_L\right] & \\ & \text{if} \quad \mu_0 < \frac{2\nu T}{3n} \\ a\left[\frac{2\Gamma\left(\frac{3n}{2}+\nu-\frac{1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{3n}{2}+\nu\right)}\,(T+\mu_0)^{\frac{1}{2}} - \hat{\xi}_L\right] & \text{if} \quad \mu_0 \geq \frac{2\nu T}{3n}. \end{cases}$$

3. Estimation under type-I PHCS

Suppose that a type-I *PHC* sample $x_{1:m:n}, x_{2:m:n}, \cdots, x_{m:m:n}$ (denoted by $\underset{\sim}{x}$ henceforth) is obtained by placing n units on a lifetest and following type-I *PHCS*, described in Section 1. Henceforth, we use notation x_i instead of $x_{i:m:n}$, for brevity. The likelihood function of given observations $\underset{\sim}{x}$ [see Tomer and Panwar (2015)] can be written as follows

$$L(\theta|\underline{x}) = C_d \prod_{i=1}^{d} f(x_i)\{\bar{F}(x_i)\}^{R_i}\{\bar{F}(t_0)\}^{R_d^*}, \tag{17}$$

where $C_d = n(n-R_1-1)(n-R_1-R_2-2)\cdots(n-R_1-\cdots-R_{d-1}-d+1)$.

Remark 3.1. *Note that for the case $X_m \leq t_0$ we get $d = m$ and $R_m^* = n - m - \sum_{i=1}^{m} R_i = 0$. Therefore, (17) reduces to*

$$L(\theta|\underline{x}) = C_m \prod_{i=1}^{m} f(x_i)\{\bar{F}(x_i)\}^{R_i}, \quad x_m < t_0.$$

We proceed with the general case (17). Using (2) and (3), the likelihood (17) becomes

$$L(\theta|\underline{x}) = C_d\Big(\frac{4}{\sqrt{\pi}}\Big)^n \frac{2^d}{\theta^{3d/2}} exp\Big(-\frac{1}{\theta}\sum_{i=1}^{d} x_i^2\Big)\Big\{\Gamma_{3/2}\Big(\frac{t_0^2}{\theta}\Big)\Big\}^{R_d^*} \prod_{i=1}^{d} x_i^2 \Big\{\Gamma_{3/2}\Big(\frac{x_i^2}{\theta}\Big)\Big\}^{R_i}. \tag{18}$$

The predictive density corresponding to the prior $g(\theta|\mu)$ on using (18) comes out to be

$$\begin{aligned} m(\underset{\sim}{x}|g) =& C_d \frac{2^{n+2d-m}}{(\sqrt{\pi})^{n+d-m}} \frac{\mu^\nu}{\Gamma(\nu)} \prod_{i=1}^{d} x_i^2 \int_0^\infty \frac{1}{\theta^{3d/2+\nu+1}} \exp\left\{-\frac{1}{\theta}\,(T_d+\mu)\right\} \\ & \prod_{i=1}^{d}\left\{\Gamma_{3/2}\left(x_i^2/\theta\right)\right\}^{R_i}\left\{\Gamma_{3/2}\left(t_0^2/\theta\right)\right\}^{R_d^*} d\theta. \\ =& C_d \frac{2^{n+2d-m}}{(\sqrt{\pi})^{n+d-m}} \frac{\mu^\nu}{\Gamma(\nu)} \prod_{i=1}^{d} x_i^2 \int_0^\infty I(\theta,\mu) d\theta \end{aligned} \tag{19}$$

and the predictive density corresponding to the base prior $g(\theta|\mu_0)$ can be obtained from (19) when $\mu = \mu_0$.

Here, the value of μ which maximizes the predictive density $m(\underline{x}|g)$ is

$$\tilde{\mu} = \max\{\mu_0,\ \hat{\mu}_d\},$$

where $\hat{\mu}_d$ is the solution of

$$\frac{\hat{\mu}_d}{\nu} + \frac{\int_0^\infty \frac{1}{\theta} I(\theta, \hat{\mu}_d) d\theta}{\int_0^\infty I(\theta, \hat{\mu}_d) d\theta} = 0.$$

Then we have

$$g(\theta|\tilde{\mu}) = \begin{cases} \frac{\hat{\mu}_d^\nu}{\Gamma(\nu)\theta^{\nu+1}} \exp\left(-\frac{\hat{\mu}_d}{\theta}\right) = \tilde{g}, & \text{if} \quad \mu_0 < \hat{\mu}_d \\ g_0(\theta|\mu_0) = g_0, & \text{if} \quad \mu_0 \geq \hat{\mu}_d. \end{cases} \tag{20}$$

We write the *ML-II* prior density for this case as follows.

$$\tilde{q}(\theta) = (1-\varepsilon) g_0(\theta|\mu_0) + \varepsilon \tilde{g}(\theta|\tilde{\mu}). \tag{21}$$

On using (18) and (21), the *ML-II* posterior density of θ comes out to be

$$\tilde{q}^*(\theta) = \tilde{\lambda} g_0'^*(\theta) + (1-\tilde{\lambda}) \tilde{g}^*(\theta), \qquad 0 < \theta < \infty, \tag{22}$$

where

$$g_0'^*(\theta) = \frac{I(\theta, \mu_0)}{\int_0^\infty I(\theta, \mu_0) d\theta} \quad \text{if} \quad \mu_0 \geq \hat{\mu}_d, \tag{23}$$

$$\tilde{g}^*(\theta) = \begin{cases} \frac{I(\theta, \tilde{\mu})}{\int_0^\infty I(\theta, \tilde{\mu}) d\theta} & \text{if} \quad \mu_0 \leq \hat{\mu}_d \\ g_0'^*(\theta) & \text{if} \quad \mu_0 \geq \hat{\mu}_d \end{cases} \tag{24}$$

and

$$\tilde{\lambda} = \begin{cases} \left[1 + \frac{\varepsilon}{(1-\varepsilon)} \frac{\tilde{\mu}^\nu}{\mu_0^\nu} \frac{\int_0^\infty I(\theta, \tilde{\mu}) d\theta}{\int_0^\infty I(\theta, \mu_0) d\theta}\right]^{-1} & \text{if} \quad \mu_0 \leq \hat{\mu}_d \\ (1-\varepsilon) & \text{if} \quad \mu_0 \geq \hat{\mu}_d. \end{cases} \tag{25}$$

3.1. Estimation under SELF

In presence of type-I *PHC* data we obtain the *ML-II* estimator of θ under SELF, on using (22), as follows

$$\begin{aligned} \tilde{\theta} &= E_{q^*}(\theta) \\ &= \tilde{\lambda} E_{g_0'^*}(\theta) + (1-\tilde{\lambda}) E_{\tilde{g}^*}(\theta). \end{aligned} \tag{26}$$

Since the posterior densities $g_0'^*(\theta)$ and $\tilde{g}^*(\theta)$ given by expressions (23) and (24), respectively, do not follow standard distributions, we use *M-H* algorithm [Metropolis and Ulam (1949)] to evaluate the posterior expectations $E_{g_0'^*}(\theta)$ and $E_{\tilde{g}^*}(\theta)$. Similarly, by using *M-H* algorithm *ML-II* estimate of mean lifetime can be obtained as follows.

$$\begin{aligned} \tilde{\xi} &= \frac{2}{\sqrt{\pi}} E_{q^*}(\sqrt{\theta}) \\ &= \frac{2}{\sqrt{\pi}} \left[\tilde{\lambda} E_{g_0'^*}(\sqrt{\theta}) + (1-\tilde{\lambda}) E_{\tilde{g}^*}(\sqrt{\theta})\right] \end{aligned} \tag{27}$$

The posterior variances of θ can be obtained from (31) of Appendix on replacing g_0^* by $g_0^{'*}$ and g^* by $\tilde{g}^*$ and implementing *M-H* algorithm. Similarly, we evaluate the posterior variance of ξ.

3.2. Estimation under LINEX loss function

The expressions for the *ML-II* estimators of θ and ξ under LINEX loss function can be obtained on using (22) as

$$\begin{aligned}\tilde{\theta}_L &= -\frac{1}{a}\ln E_{\tilde{q}^*}[\exp(-a\theta)]\\ &= \begin{cases} -\frac{1}{a}\ln\left(\tilde{\lambda}E_{g_0^{'*}}[\exp(-a\theta)] + (1-\tilde{\lambda})E_{\tilde{g}^*}[\exp(-a\theta)]\right) & \text{if} \quad \mu_0 \leq \hat{\mu}_d \\ -\frac{1}{a}\ln\left(E_{g_0^{'*}}[\exp(-a\theta)]\right) & \text{if} \quad \mu_0 \geq \hat{\mu}_d. \end{cases}\end{aligned}$$

and

$$\begin{aligned}\tilde{\xi}_L &= -\frac{1}{a}\ln E_{\tilde{q}^*}[\exp(-A\theta^{1/2})]\\ &= \begin{cases} -\frac{1}{a}\ln\left[\tilde{\lambda}E_{g_0^{'*}}[\exp(-A\theta^{1/2})] + (1-\tilde{\lambda})E_{\tilde{g}^*}[\exp(-A\theta^{1/2})]\right] & \text{if} \quad \mu_0 \leq \hat{\mu}_d \\ -\frac{1}{a}\ln\left[E_{g_0^{'*}}[\exp(-A\theta^{1/2})]\right] & \text{if} \quad \mu_0 \geq \hat{\mu}_d. \end{cases}\end{aligned}$$

The expectations of LINEX loss function for $\tilde{\theta}_L$ and $\tilde{\xi}_L$ are respectively given by $a(E_{\tilde{q}^*}[\theta]-\tilde{\theta}_L)$ and $a(E_{\tilde{q}^*}[\xi]-\tilde{\xi}_L)$. Like in Section 3.1, the *ML-II* estimates and their posterior risks can be obtained using *M-H* algorithm.

4. Real data analysis

Here, we consider a real data set of 23 ball bearings from Lawless (2003). The data presents the number of revolutions (in millions) completed by any ball bearing before its failure. Tomer and Panwar (2015) have shown that *MWD* is a suitable model for this data. The data is given below.

17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.40, 51.84,
51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12,
93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.

In order to illustrate the *ML-II* procedure discussed in Section 2, we consider two different base priors $IG_1(2000, 2)$ and $IG_2(7000, 2)$ for θ. Then we obtain *ML-II* estimates of θ as well ξ assuming different values of ε that ranges from 0 to 1. The values of these estimates along with their posterior standard deviations (SDs) under *SELF* and *LINEX* loss functions are presented in Table 2. For each loss function, we observe from Table 2 that when $\varepsilon = 0$, the *ML-II* estimates corresponding to the considered base priors differ significantly but as $\varepsilon \to 1$, the estimates under two prior come closer and almost coincide at $\varepsilon = 1$.

Further, to study the behaviour of *ML-II* estimators in presence of *Type-I PHC* data, we use expressions that are obtained in Section 3. We consider three *Type-I PHC* samples which are generated from the original data. These samples are presented in Table 1. With these samples, we obtained the *ML-II* estimates of θ and mean lifetime ξ under *SELF* and *LINEX* loss functions with the same values of hyper-parameters of base priors i.e. $IG_1(2000, 2)$ and $IG_2(7000, 2)$. The findings are presented in Tables 3-5 which exhibit same behaviour as we observed in complete sample study.

Table 1: Samples obtained under three different *Type-I PHCS* from the ball bearings data.

Scheme	Sample observations								
$S_{18:23}$= ({1,0,0}*6), $t_0 = 120$	17.88	28.92	33.00	41.52	42.12	45.60	51.84	51.96	54.12
	55.56	67.80	68.64	68.64	68.88	84.12	93.12	98.64	105.12
$S_{15:23}$= (1,{0,1}*7), $t_0 = 110$	17.88	28.92	33.00	41.52	42.12	45.60	48.48	51.84	54.12
	55.56	67.80	68.64	68.64	68.88	93.12			
$S_{12:23}$= ({1}*5,1,0,{1}*5), $t_0 = 100$	17.88	28.92	33.00	41.52	42.12	45.60	48.48	51.84	51.96
	84.12	93.12	98.64						

Note: $a * b$=(a, a, a, ..., (b times))

5. Conclusion

We considered robust Bayesian estimation of the parameter and mean lifetime in the presence of uncensored as well as *Type-I PHC* lifetime data. In this study, we have shown that ε-contamination class of prior distributions can give robust results when the prior belief of the investigator cannot be represented in form of a single prior density or there is a possibility of error in the prior elicitation of unique prior for the parameter. We have illustrated with the help of a real data set that ε-contamination class is a sensible class of priors which may be thought to promote objective thinking by removing the judgment error in prior elicitation process.

Table 2: *ML-II* estimate of θ and ξ, along with their posterior *SD*s(in parentheses), under two different base priors for uncensored data.

	SELF				LINEX a=-.01				LINEX a=.01			
ε	$\hat{\theta}$		$\hat{\xi}$		$\hat{\theta}_L$		$\hat{\xi}_L$		$\hat{\theta}_L$		$\hat{\xi}_L$	
	$IG_1(2000,2)$	$IG_2(7000,2)$	$IG_1(2000,2)$	$IG_2(7000,2)$	$IG_1(2000,2)$	$IG_2(7000,2)$	$IG_1(2000,2)$	$IG_2(7000,2)$	$IG_1(2000,2)$	$IG_2(7000,2)$	$IG_1(2000,2)$	$IG_2(7000,2)$
0	4306.90	4447.75	73.7921	74.9915	5007.17	5377.19	79.8456	82.7433	3480.25	3593.20	66.5672	67.6388
	(233.26)	(252.23)	(6.2045)	(6.3036)	(206.26)	(221.23)	(5.7284)	(5.8612)	(218.26)	(242.23)	(5.7461)	(5.8726)
0.2	4401.40	4457.94	74.5978	75.0726	4996.59	5213.19	79.7612	81.4717	3571.26	3642.26	67.4320	68.0990
	(255.52)	(259.24)	(6.3296)	(6.3135)	(209.52)	(212.24)	(6.0432)	(6.0102)	(214.52)	(235.89)	(6.0351)	(6.0697)
0.4	4444.71	4467.96	74.9685	75.1695	4985.47	5108.87	79.6724	80.6245	3625.23	3661.75	67.9396	68.2809
	(261.60)	(261.02)	(6.3437)	(6.3277)	(203.60)	(212.04)	(5.6904)	(6.0356)	(214.60)	(221.87)	(6.0146)	(5.9614)
0.6	4469.56	4477.80	75.1795	75.2471	4974.35	5026.23	79.5835	79.9979	3665.52	3676.24	68.3161	68.4159
	(263.96)	(262.24)	(6.3414)	(6.3363)	(199.96)	(203.24)	(5.8871)	(5.9341)	(212.96)	(215.23)	(5.9083)	(6.0053)
0.8	4485.67	4487.47	75.3142	75.3264	4963.23	5009.89	79.4945	79.8673	3679.89	3684.49	68.4499	68.4926
	(265.05)	(264.25)	(6.3401)	(6.3323)	(203.05)	(205.25)	(5.9176)	(5.8843)	(209.05)	(210.25)	(5.8598)	(5.8865)
1.0	4496.97	4496.97	75.4013	75.4040	4952.34	4952.34	79.4000	79.4072	3688.25	3688.26	68.5276	68.5277
	(265.62)	(265.62)	(6.3476)	(6.3405)	(198.62)	(200.62)	(5.8500)	(5.8684)	(205.62)	(205.62)	(5.8847)	(5.8991)

Table 3: *ML-II* estimate of θ and ξ, along with their posterior *SD*s(in parentheses), under two different base priors for $S_{18:23}$ censoring Scheme.

	SELF				LINEX a=-.01				LINEX a=.01			
ε	$\tilde{\theta}$		$\tilde{\xi}$		$\tilde{\theta}_L$		$\tilde{\xi}_L$		$\tilde{\theta}_L$		$\tilde{\xi}_L$	
	$IG_1(2000,2)$	$IG_2(7000,2)$	$IG_1(2000,2)$	$IG_2(7000,2)$	$IG\_1(2000,2)$	$IG_2(7000,2)$	$IG_1(2000,2)$	$IG_2(7000,2)$	$IG_1(2000,2)$	$IG_2(7000,2)$	$IG_1(2000,2)$	$IG_2(7000,2)$
0	4968.09	4931.49	79.5334	79.1595	5776.17	5805.32	85.4542	85.6329	4148.20	4205.40	72.9913	73.3418
	(259.88)	(239.62)	(8.9181)	(8.8971)	(232.88)	(209.62)	(8.4358)	(8.4601)	(224.88)	(226.62)	(8.4735)	(8.4604)
0.2	4959.94	4952.42	79.4682	79.1912	5687.59	5716.75	85.1962	85.5244	4218.41	4236.41	73.5389	73.5945
	(252.50)	(238.87)	(8.9411)	(8.8989)	(207.50)	(185.87)	(8.5795)	(8.7027)	(202.50)	(205.87)	(8.6871)	(8.6534)
0.4	4952.23	4927.70	79.3477	79.2095	5631.47	5642.49	84.6481	84.7458	4253.38	4259.38	73.6606	73.6346
	(245.05)	(238.40)	(8.9077)	(8.8999)	(191.05)	(189.40)	(8.5091)	(8.5615)	(196.05)	(195.40)	(8.5312)	(8.5729)
0.6	4944.91	4929.19	79.3477	79.2215	5595.35	5614.50	84.4062	84.2996	4266.67	4269.67	73.7757	73.6844
	(237.53)	(238.08)	(8.9077)	(8.9006)	(180.53)	(183.08)	(8.5051)	(8.4941)	(182.53)	(182.86)	(8.4915)	(8.5325)
0.8	4937.96	4930.25	79.2919	79.2229	5549.23	5551.38	84.1182	84.3000	4273.04	4274.04	73.8150	73.8001
	(229.94)	(237.86)	(8.9046)	(8.9011)	(163.94)	(176.86)	(8.4723)	(8.4246)	(174.94)	(183.86)	(8.7520)	(8.5070)
1.0	4931.34	4931.03	79.2387	79.2362	5533.34	5533.49	84.0887	84.0887	4277.4	4277.4	73.8522	73.8522
	(222.29)	(237.68)	(8.9016)	(8.9014)	(155.29)	(174.68)	(8.4285)	(8.4363)	(159.29)	(178.68)	(8.4233)	(8.4693)

Table 4: *ML-II* estimate of θ and ξ, along with their posterior *SD*s(in parentheses), under two different base priors for $S_{15:23}$ censoring Scheme.

	SELF				LINEX a=-.01				LINEX a=.01			
ε	$\tilde{\theta}$		$\tilde{\xi}$		$\tilde{\theta}_L$		$\tilde{\xi}_L$		$\tilde{\theta}_L$		$\tilde{\xi}_L$	
	$IG_1(2000,2)$	$IG_2(7000,2)$	$IG_1(2000,2)$	$IG_2(7000,2)$	$IG_1(2000,2)$	$IG_2(7000,2)$	$IG_1(2000,2)$	$IG_2(7000,2)$	$IG_1(2000,2)$	$IG_2(7000,2)$	$IG_1(2000,2)$	$IG_2(7000,2)$
0	4830.09	4820.21	78.4210	78.3408	5646.82	5785.32	85.4542	86.6329	4148.20	4225.40	72.9913	73.3018
	(235.74)	(280.24)	(6.3095)	(7.7825)	(220.74)	(242.24)	(5.8347)	(7.3384)	(253.24)	(247.24)	(5.8640)	(7.3506)
0.2	48225.76	4819.01	78.3859	78.3310	5581.24	5716.75	85.1962	85.5244	4188.41	4236.41	73.0189	73.1945
	(235.99)	(278.21)	(6.3830)	(7.7800)	(181.99)	(233.21)	(5.9942)	(7.4802)	(243.21)	(245.21)	(6.2600)	(7.4546)
0.4	4821.21	4817.40	78.3489	78.3489	5530.12	5642.49	84.6481	84.7458	4233.38	4245.38	73.4606	73.2346
	(236.17)	(275.45)	(6.3246)	(7.2994)	(185.17)	(218.45)	(5.9790)	(6.9767)	(220.45)	(232.45)	(6.0320)	(6.9340)
0.6	4816.42	4815.11	78.0399	78.2994	5506.00	5514.50	84.4062	84.2996	4246.67	4252.67	73.5757	73.3844
	(236.27)	(271.46)	(6.3278)	(7.5080)	(176.27)	(209.46)	(5.9590)	(7.2730)	(211.46)	(222.46)	(5.9690)	(7.0890)
0.8	4812.12	4811.60	78.2601	78.2496	5464.88	5481.38	84.1182	84.3000	4243.04	4256.04	73.4750	73.5010
	(236.26)	(265.20)	(6.3277)	(7.2987)	(174.27)	(207.20)	(5.8390)	(6.8790)	(209.20)	(207.20)	(5.9288)	(6.8479)
1.0	4806.05	4805.55	78.2256	78.2215	5433.99	5433.49	84.0887	84.0887	4257.43	4257.41	73.5922	73.5922
	(236.14)	(253.91)	(6.3235)	(6.9266)	(171.14)	(190.91)	(5.8550)	(6.4540)	(193.91)	(194.91)	(5.8690)	(6.4980)

Table 5: *ML-II* estimate of θ and ξ, along with their posterior SDs(in parentheses), under two different base priors for $S_{12:23}$ censoring Scheme.

ε	SELF				LINEX a=-.01				LINEX a=.01			
	$\tilde{\theta}$		$\tilde{\xi}$		$\tilde{\theta}_L$		$\tilde{\xi}_L$		$\tilde{\theta}_L$		$\tilde{\xi}_L$	
	$IG_1(2000,2)$	$IG_2(7000,2)$	$IG_1(2000,2)$	$IG_2(7000,2)$	$IG_1(2000,2)$	$IG_2(7000,2)$	$IG_1(2000,2)$	$IG_2(7000,2)$	$IG_1(2000,2)$	$IG_2(7000,2)$	$IG_1(2000,2)$	$IG_2(7000,2)$
0	4896.71	4913.05	78.9600	79.0917	5686.82	5859.82	85.0922	86.3768	4314.40	4461.35	74.1166	75.5187
	(280.76)	(229.70)	(7.7990)	(6.0995)	(252.76)	(206.70)	(7.3167)	(5.6657)	(260.76)	(207.70)	(7.3543)	(5.6604)
0.2	4897.88	4912.28	78.9695	79.0854	5621.24	5724.82	84.6002	85.6271	4365.41	4414.41	74.5534	75.1167
	(277.49)	(231.44)	(7.6950)	(6.6030)	(222.49)	(178.44)	(7.3616)	(5.9530)	(237.49)	(187.44)	(7.4265)	(5.8684)
0.4	4899.11	4911.23	79.0770	79.0770	5564.12	5580.50	84.1692	84.5344	4326.38	4380.9	74.2193	74.8282
	(274.01)	(233.76)	(6.2400)	(6.2400)	(223.01)	(181.76)	(5.9076)	(5.9105)	(220.01)	(189.76)	(7.2245)	(5.8764)
0.6	4900.41	4909.76	78.9898	79.0650	5530.00	5584.86	83.9108	83.8023	4344.67	4394.39	74.3761	74.9444
	(270.29)	(237.01)	(7.4638)	(6.3533)	(212.29)	(184.01)	(7.0142)	(5.9881)	(210.29)	(187.01)	(7.0561)	(5.9879)
0.8	4901.77	4907.49	79.0008	79.0456	5500.88	5531.52	83.6895	83.6912	4348.04	4356.64	74.4050	74.5052
	(266.30)	(241.87)	(7.3346)	(6.5204)	(202.30)	(175.87)	(6.8507)	(6.0932)	(208.30)	(181.87)	(6.9098)	(6.1118)
1.0	4903.21	4903.87	79.0240	79.0124	5472.99	5472.97	83.4771	83.4597	4337.35	4337.41	74.3139	74.3254
	(262.02)	(249.99)	(7.9470)	(6.7953)	(193.02)	(184.99)	(6.7082)	(6.3458)	(198.02)	(187.99)	(6.7223)	(6.3502)

References

Balakrishnan N, Aggarwala R (2000). *Progressive Censoring: Theory, Methods and Applications*. Birkhäuser, Boston.

Berger J (1982). "Bayesian Robustness and the Stein Effect." *Journal of the American Statistical Association*, **77**, 358–368.

Berger J (1983). "The Robust Bayesian Viewpoint (with Discussion): Robustness of Bayesian Statistics." In J Kadane (ed.), *Robustness of Bayesian Analysis*. North-Holland, Amsterdam, Oxford.

Berger J, Berliner (1984). "Bayesian Input in Stein Estimation and a New Minimax Empirical Bayes Estimator." *Journal of Econometrics*, **25**, 87–108.

Berger J, Berliner LM (1986). "Robust Bayes and Empirical Bayes Analysis with $\varepsilon-$ contaminated Priors." *The Annals of Statistics*, **14**, 461–486.

Box GEP, Tiao GC (1973). *Bayesian Inference in Statistical Analysis*. Wiley, New York.

Calabria R, Pulcini G (1994). "Bayesian 2-sample Prediction for the Inverse Weibull Distribution." *Communications in Statistics- Theory and Methods*, **23**, 1811–1821.

Chaturvedi A (1996). "Robust Bayesian Analysis of the Linear Regression Model." *Journal of Statistical Planning and Inference*, **50**(2), 175–186.

Chaturvedi A, Pati M, Tomer SK (2014). "Robust Bayesian Analysis of Weibull Failure Model." *Metron*, **72**(1), 77–95.

Chib S, Tiwari RC (1991). "Robust Bayes Analysis in Normal Linear Regression with an Improper Mixture Prior." *Communications in Statistics-Theory and Methods*, **20**(3), 807–829.

Childs A, Chandrasekar B, Balakrishnan N (2008). *Exact Likelihood Inference for an Exponential Parameter Under Progressive Hybrid Censoring Schemes*, pp. 319–330. Birkhäuser, Boston.

Dempster AP (1975). "A Subjectivist Look at Robustness." *Bull. Internat. Statist. Inst*, **46**, 349–374.

Doostparast M, Ahmadi MV, Ahmadi J (2013). "Bayes Estimation Based on Joint Progressive Type-II Censored Data under LINEX Loss Function." *Communications in Statistics - Simulation and Computation*, **42**, 1865–1886.

Good IJ (1965). *The Estimation of Probabilities: An Essay on Modern Bayesian Methods*, volume 258. MIT press Cambridge, MA.

Good IJ (1983). *Good Thinking: The Foundations of Probability and Its Applications*. University of Minnesota Press.

Gradshteyn IS, Ryzhik IM (1965). *Tables of Integrals, Series and Products*. Acadmic Press, N. Y.

Kadane JB, Chuang DT (1978). "Stable Decision Problems." *The Annals of Statistics*, **6**, 1095–1110.

Kim C, Jung J, Chung Y (2011). "Bayesian Estimation for the Exponentiated Weibull Model under Type II Progressive Censoring." *Statistical Papers*, **52**(1), 53–70.

Krishna H, Malik M (2009). "Reliability Estimation of Maxwell Distribution with Type-II Censored Data." *International Journal of Quality & Reliability Management*, **26**(2), 184–195.

Krishna H, Malik M (2012). "Reliability Estimation in Maxwell Distribution with Progressively Type-II Censored Data." *Journal of Statistical Computation and Simulation*, **82**(4), 623–641.

Krishna H, Vivekanand, Kumar K (2015). "Estimation in Maxwell Sistribution with Randomly Censored Data." *Journal of Statistical Computation and Simulation*, **85**(17), 3560–3578.

Kundu D, Joarder A (2006). "Analysis of Type-II progressively hybrid censored data." *Computational Statistics & Data Analysis*, **50**(10), 2509–2528.

Lawless JF (2003). *Statistical Models and Methods for Lifetime Data*. 2nd edition. John Wiley and Sons Inc, New York.

Metropolis N, Ulam S (1949). "The Monte Carlo Method." *Journal of the American Statistical Association*, **44**(247), 335–341.

Panwar MS, Tomer SK, Kumar J (2015). "Competing Risk Analysis of Failure Censored Data from Maxwell Distribution." *International Journal of Agricultural and Statistical Sciences*, **11**(1), 29–34.

Prudinikov AP, Brychkov YUA, Marichev OI (1986). *Integrals and Series*, volume 1. Gordon and Breach Science Publishers, NewYork.

Sivaganesan S, Berger J (1987). "Ranges of Posterior Measures Priors with Unimodal Contaminations." *Technical Report No. 86-41, Department of Statistics, Purdue University, W. Lafayette.*

Tomer SK, Panwar MS (2015). "Estimation Procedures for Maxwell Distribution under Type-I Progressively Hybrid Censoring Scheme." *Journal of Statistical Computation and Simulation*, **85**, 339–356.

Varian HR (1975). "A Bayesian Approach to Real Estate Assessment." *North Holand, Amsterdam*, pp. 195–208.

Zellner A (1986). "On Assessing Prior Distributions and Bayesian Regression Analysis with g-prior Distributions." In PK Goel, A Zellne (eds.), *Bayesian Inference and Decision Techniques: Essays in Honor of Bruno de Finetti*, pp. 233–243. Elsevier Science Publishers, Inc., New York.

Appendix

Proof of Theorem 1. We have

$$E_{q^*}(\theta) = \hat{\lambda} E_{g_0^*}(\theta) + (1-\hat{\lambda}) E_{g^*}(\theta). \tag{28}$$

For $\mu_0 \geq \frac{2\nu T}{3n}$, we obtain using (12) that

$$\begin{aligned} E_{g_0^*}(\theta) =& \frac{(T+\mu_0)^{(3n/2+\nu)}}{\Gamma\left(3n/2+\nu\right)} \int\limits_0^\infty \frac{1}{\theta^{3n/2+\nu}} \exp\left\{-\frac{1}{\theta}\left(T+\mu_0\right)\right\} d\theta \\ =& \frac{\Gamma(3n/2+\nu-1)}{\Gamma(3n/2+\nu)} \left(T+\mu_0\right). \end{aligned} \tag{29}$$

Similarly, for $\mu_0 < \frac{2\nu T}{3n}$, it immediately follows from (13) that

$$E_{g^*}(\theta) = \frac{\Gamma(3n/2+\nu-1)}{\Gamma(3n/2+\nu)}\left(1+\frac{2\nu}{3n}\right)T. \tag{30}$$

The first part of the theorem follows on substituting the expressions of $E_{g_0^*}(\theta)$ and $E_{g^*}(\theta)$ in (28).
In order to prove second result of the theorem, we use following expression from Berger and Berliner (1986)

$$V_{q^*}(\theta) = \hat{\lambda}V_{g_0^*}(\theta) + (1-\hat{\lambda})V_{g^*}(\theta) + \hat{\lambda}(1-\hat{\lambda})\left\{E_{g_0^*}(\theta) - E_{g^*}(\theta)\right\}^2. \tag{31}$$

and the result follows from (31) by using (29), (30) with

$$\begin{aligned}V_{g_0^*}(\theta) =& E_{g_0^*}(\theta^2) - (E_{g_0^*}(\theta))^2\\ =& \frac{(T+\mu_0)^2}{\Gamma(3n/2+\nu-2)(\Gamma(3n/2+\nu-1)^2)} \qquad \mu_0 \geq \frac{2\nu T}{3n}.\\ &\text{and}\\ V_{g^*}(\theta) =& \frac{\left(1+\frac{2\nu}{3n}\right)^2 T^2}{\Gamma(3n/2+\nu-2)(\Gamma(3n/2+\nu-1)^2)} \qquad \mu_0 < \frac{2\nu T}{3n}.\end{aligned}$$

Expressions of $\hat{\theta}_{L_0} = E_{g_0^*}[\exp(-A\theta^{1/2})]$ and $\hat{\theta}_{L^*} = E_{g_0^*}[\exp(-A\theta^{1/2})]$

For $\mu_0 \geq \frac{2\nu T}{3n}$, we obtain

$$\begin{aligned}E_{g_0^*}[\exp(-A\theta^{1/2})] =& \frac{(T+\mu_0)^{(3n/2+\nu)}}{\Gamma(3n/2+\nu)}\int_0^\infty \frac{1}{\theta^{3n/2+\nu+1}}\exp\left\{-A\theta^{1/2} - \frac{1}{\theta}(T+\mu_0)\right\}d\theta\\ =& \frac{(T+\mu_0)^{(3n/2+\nu)}}{\Gamma(3n/2+\nu)}\int_0^\infty y^{3n/2+\nu-1}\exp\left\{-Ay^{-1/2} - y(T+\mu_0)\right\}dy\\ =& \frac{(T+\mu_0)^{(3n/2+\nu)}}{\Gamma(3n/2+\nu)}\left[\sum_{j=0}^{J-1}\frac{(-A)^j}{j!}\Gamma\left(\frac{3n}{2}+\nu-\frac{j}{2}\right)(T+\mu_0)^{j/2-3n/2-\nu}\right.\\ & {}_1F_{p+J}(1;\Delta(p,1-3n/2-\nu+j/2),\Delta(J,1+j);z)\\ & +\sum_{h=0}^{p-1}\frac{(-1)^h}{h!}2\Gamma(-3n-2\nu-2h)A^{3n+2\nu+2h}(T+\mu_0)^h\\ & {}_1F_{p+J}(1;\Delta(J,1+3n+2\nu+2h),\Delta(p,1+h);z)]\end{aligned} \tag{32}$$

where,

$$z = (-1)^{p+J}\left(\frac{T+\mu_0}{p}\right)^p\left(\frac{A}{J}\right)^J$$

The last expression in (32) is obtained by utilizing Prudinikov, Brychkov, and Marichev

(1986),(formula 14, p. 322). Similarly, for $\mu_0 < \frac{2\nu T}{3n}$, it follows that

$$E_{g^*}[\exp(-A\theta^{1/2})] = \frac{[T(1+2\nu/3n)]^{(3n/2+\nu)}}{\Gamma(3n/2+\nu)} \int_0^\infty y^{3n/2+\nu-1} \exp\left\{-Ay^{-1/2} - y\,[T(1+2\nu/3n)]\right\} dy$$

$$= \frac{[T(1+2\nu/3n)]^{(3n/2+\nu)}}{\Gamma(3n/2+\nu)} \left[\sum_{j=0}^{J-1} \frac{(-A)^j}{j!} \Gamma\left(\frac{3n}{2}+\nu-\frac{j}{2}\right) [T(1+2\nu/3n)]^{j/2-3n/2-\nu}\right.$$

$${}_1F_{p+J}(1; \Delta(p, 1-3n/2-\nu+j/2), \Delta(J, 1+j); z^*)$$

$$+\sum_{h=0}^{p-1} \frac{(-1)^h}{h!} 2\Gamma(-3n-2\nu-2h) A^{3n+2\nu+2h} [T(1+2\nu/3n)]^h$$

$$\left. {}_1F_{p+J}(1; \Delta(J, 1+3n+2\nu+2h), \Delta(p, 1+h); z^*)\right]$$

where,

$$z^* = (-1)^{p+J} \left[\frac{T(1+2\nu/3n)}{p}\right]^p \left(\frac{A}{J}\right)^J$$

□

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