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Consistency of the LSE for Chirp Signal Parameters in the Models with Strongly and Weakly Dependent Noise

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Abstract

A time continuous statistical model of multiple chirp signal observed against the background of strongly-dependent stationary Gaussian noise is considered in the paper. Strong consistency of the least squares estimates for such a trigonometric regression model parameters is proved.

Keywords: multiple chirp signal, strongly dependent stationary Gaussian process, least squares estimate, uniform law of large numbers, Isserlis' theorem, Fresnel integrals, strong consistency.

1. Setting and main result

Among the various problems of nonlinear regression analysis and signal processing theory the problem of estimation amplitudes and angular frequencies of the harmonic oscillations masked by random noise should be highlighted due to its multiple applications. The problem of detecting hidden periodicities has a long own history and massive literature. We refer here only to Artis *et al.* (2004), Ivanov (2010), Quinn and Hannan (2012), Ivanov *et al.* (2015), where a lot of links to publications on this topic can be found.

Over the last decades trigonometric models on the plane, where under the signs of cosines and sines are linear forms of two variables with unknown frequencies as coefficients, have received special attention in the literature on signal and image processing due to their wide applications in the analysis of symmetrical textured surfaces. See, for example, Rao, Zhao, and Zhou (1994), Kundu and Mitra (1996), Zhang and Mandrekar (2001), Kundu and Nandi (2003), Nandi, Kundu, and Srivastava (2013), Ivanov and Malyar (2018), Ivanov and Lymar (2020).

In Brillinger (1986), Ivanov and Savych (2021), Ivanov and Dykyi (2023) some of the scalar and 2D results are extended to multivariate sinusoidal regression models. Thus, such a generalization of the detection hidden periodicities problem in statistics of stochastic processes turns it into the corresponding problem in statistics of random fields on the plane and Euclidean spaces of higher dimensions.

Another important generalization of classical trigonometric models are models of frequency-modulated sinusoidal signals observed against the background of random noises having a different nature. The problems of estimating the parameters of such signals have been studied for a long time: see, for example, Cook and Bernfeld (1967), Cramer and Leadbetter (1967). At present, the flow of literature on estimating the parameters of frequency-modulated signals is quite large and we will mention just for illustration the papers Yang *et al.* (2014), Djurovic (2017), Jiang and Wu (2021), Moradi and Mohseni (2023).

The most studied here is the case of linearly frequency-modulated signals, which we will refer to as chirp-signals. For one harmonic it can be written as

$$A\cos(\phi t+\psi t^2)+B\sin(\phi t+\psi t^2),\quad t\geq 0, \tag{1}$$

where A, B are amplitudes, ϕ is the starting frequency, ψ is the chirp rate. For discrete time t and random noise, being a linear time series, some results on consistency and asymptotic normality of LSE and some other estimates of a chirp-signal which is one harmonic or the sum of harmonics (1), have been obtained in a large numbers of works. We will point only to publications Nandi and Kundu (2004), Kundu and Nandi (2008), Lahiri (2011), Lahiri, Kundu, and Mitra (2015), Nandi and Kundu (2020), Kundu and Nandi (2021), Grover, Kundu, and Mitra (2021) and references there in.

Here we consider time continuous multiple chirp-signal observed with additive random strongly-dependent noise and prove LSE strong consistency of unknown signal parameters.

We think that the regression models with continuous time are of great importance since the discrete type models which are usually used in real life applications can be obtained from continuous time models using different methods of discretization: see, for example, Grenander (1981), pp. 249-251, or Angulo *et al.* (2008), where several discretized procedures have been proposed such as locally averages sampling, instantaneous sampling and randomized sampling. Moreover, continuous time models can be discretized in different ways in order to apply both fixed-domain asymptotic and increasing domain asymptotic, which is considered in this paper.

Suppose we observe a stochastic process

$$X(t)=g(t,\theta^0)+\varepsilon(t),\quad t\in\mathbb{R}_+, \tag{2}$$

where

$$g(t,\theta^0)=\sum_{j=1}^{N}\left(A_j^0\cos\left(\phi_j^0t+\psi_j^0t^2\right)+B_j^0\sin\left(\phi_j^0t+\psi_j^0t^2\right)\right), \tag{3}$$

$$\theta^0=\left(A_1^0,B_1^0,\phi_1^0,\psi_1^0,...,A_N^0,B_N^0,\phi_N^0,\psi_N^0\right), \tag{4}$$

$\left(A_j^0\right)^2+\left(B_j^0\right)^2>0$, $j=\overline{1,N}$; $\varepsilon=\{\varepsilon(t),t\in\mathbb{R}\}$ is a random noise defined on a probability space $(\Omega,\mathcal{F},P)$ and satisfying the following assumption.

A. ε is a sample-continuous stationary Gaussian stochastic process with zero mean and covariance function (c.f.) $B(t)=E\varepsilon(t)\varepsilon(0),t\in\mathbb{R}$, having one of the properties:

(i) $B(t)=L(|t|)|t|^{-\alpha},\alpha\in(0,1)$, with non-decreasing slowly varying at infinity function L;

(ii) $B(\cdot)\in L_1(\mathbb{R})$.

Notice that trigonometric function (3) does not satisfy for $N\geq 2$ the conditions of any general theorem on LSE consistency of nonlinear regression model parameters (see, for example, Ivanov and Leonenko (1989), Ivanov (1997)). To prove the LSE consistency of the parameters (4) we have to change the standard definition of LSE using parametric sets that allows us to distinguish the model parameters properly.

Assuming that true values of amplitudes $A_j^0, B_j^0, j = \overline{1, N}$, are different numbers and the true values of frequencies $\phi_j^0, j = \overline{1, N}$, and $\psi_j^0, j = \overline{1, N}$, are different positive numbers, we arrange the chirp rates $\psi^0 = \left(\psi_1^0, ..., \psi_N^0\right)$ in increasing order and suppose

$$\psi^0 \in \Psi(\underline{\psi}, \overline{\psi}) = \left\{\psi = (\psi_1, ..., \psi_N) \in \mathbb{R}^N : 0 \le \underline{\psi} < \psi_1 < ... < \psi_N < \overline{\psi} < +\infty\right\}.$$

In turn, we also introduce the parametric set

$$\Phi(\underline{\phi}, \overline{\phi}) = \left\{\phi = (\phi_1, ..., \phi_N) : 0 \le \underline{\phi} < \phi_j < \overline{\phi} < +\infty, j = \overline{1, N}\right\}$$

such that $\phi^0 = \left(\phi_1^0, ..., \phi_N^0\right) \in \Phi(\underline{\phi}, \overline{\phi})$.

Consider monotonically non-decreasing family of open sets $\Psi_T \subset \Psi\left(\underline{\psi}, \overline{\psi}\right), T > T_0 > 0$, containing vector ψ^0, such that $\bigcup\limits_{T > T_0} \Psi_T = \widetilde{\Psi}, \widetilde{\Psi}^c = \Psi^c\left(\underline{\psi}, \overline{\psi}\right)$, with the following properties

B. 1) $\lim\limits_{T\to\infty} \inf\limits_{\substack{1\le j\le N-1 \\ \psi\in\Psi_T}} T^2\left(\psi_{j+1} - \psi_j\right) = +\infty$;

2) $\lim\limits_{T\to\infty} \inf\limits_{\psi\in\Psi_T} T^2\psi_1 = +\infty$.

Definition 1. Any random vector

$$\theta_T = (A_{1T}, B_{1T}, \phi_{1T}, \psi_{1T}, ..., A_{NT}, B_{NT}, \phi_{NT}, \psi_{NT}) \tag{5}$$

such that it is a point of the functional

$$Q_T(\theta) = T^{-1} \int_0^T [X(t) - g(t, \theta)]^2 \, dt \tag{6}$$

absolute minimum on the parametric set $\Theta_T^c \subset \mathbb{R}^{4N}$, where amplitudes $A_j, B_j, j = \overline{1, N}$, can take any values and parameters (ϕ, ψ) take values in the set $\Phi^c(\underline{\phi}, \overline{\phi}) \times \Psi_T^c, T > T_0 > 0$, is called LSE of the parameter θ^0.

We study in the paper just such an estimate, namely: θ_T is defined on the parametric set Θ_T^c depending on T.

Theorem 1. *Let the conditions **A** and **B** be satisfied. Then LSE θ_T is a strongly consistent estimate of parameter θ^0 in the sense that $A_{jT} \to A_j^0$, $B_{jT} \to B_j^0$, $T\left(\phi_{jT} - \phi_j^0\right) \to 0$, $T^2\left(\psi_{jT} - \psi_j^0\right) \to 0$ a.s., as $T \to \infty, j = \overline{1, N}$.*

The proof of this theorem is postponed until the 3rd section.

2. One uniform law of large numbers

In the section an uniform strong LLN is obtained for stochastic process ε weighted by the trigonometric functions of the quadratic variable.

Theorem 2. *Under condition **A***

$$\xi_T = \sup_{\phi, \psi \in \mathbb{R}} \left| T^{-1} \int_0^T e^{-i(\phi t + \psi t^2)} \varepsilon(t) dt \right| \to 0 \text{ a.s., as } T \to \infty. \tag{7}$$

Proof. Denote by $\eta_T(\phi,\psi)$ the expression under the supremum sign in (7). Then

$$\eta_T^2(\phi,\psi) = T^{-2}\int_0^T\int_0^T \exp\left\{-i(\phi(t-s)+\psi(t^2-s^2))\right\}\varepsilon(t)\varepsilon(s)dtds$$
$$= T^{-2}\iint\limits_{t>s} + T^{-2}\iint\limits_{t<s} = I_1 + I_2.$$

Making the change of variables $t-s=u$, $s=s'$ in the integral I_1, and using again the notation s instead of s' we get

$$|I_1| = T^{-2}\left|\int_0^T e^{-i(\phi u+\psi u^2)}\int_0^{T-u} e^{-i(2\psi us)}\varepsilon(s+u)\varepsilon(s)dsdu\right|$$
$$\le T^{-2}\int_0^T\left|\int_0^{T-u} e^{-i(2\psi us)}\varepsilon(s+u)\varepsilon(s)ds\right|du. \quad (8)$$

Renaming variables t to s, and s to t in the integral I_2 we obtain similarly

$$|I_2| = T^{-2}\left|\int_0^T e^{i(\phi u+\psi u^2)}\int_0^{T-u} e^{i(2\psi us)}\varepsilon(s+u)\varepsilon(s)dsdu\right|$$
$$\le T^{-2}\int_0^T\left|\int_0^{T-u} e^{i(2\psi us)}\varepsilon(s+u)\varepsilon(s)ds\right|du. \quad (9)$$

From (8) and (9) it follows that

$$E\xi_T^2 \le 2T^{-2}\int_0^T E\sup_{\psi\in\mathbb{R}}\left|\int_0^{T-u} e^{i(2\psi us)}\varepsilon(s+u)\varepsilon(s)ds\right|du$$
$$\le 2T^{-2}\int_0^T\left(E\sup_{\psi\in\mathbb{R}}\int_0^{T-u}\int_0^{T-u} e^{-i2\psi u(t-s)}\varepsilon(t+u)\varepsilon(t)\varepsilon(s+u)\varepsilon(s)dtds\right)^{\frac{1}{2}}du. \quad (10)$$

Using the notation $\widetilde{\eta}(t)=\varepsilon(t+u)\varepsilon(t)$, we make, as above, the change of variables $t-s=v$, $s=s'$ in the double integral in the expression (10). Then

$$\int_0^{T-u}\int_0^{T-u} e^{-i(2\psi u)(t-s)}\widetilde{\eta}(t)\widetilde{\eta}(s)dtds$$
$$= 2\int_0^{T-u}\cos(2\psi uv)\int_0^{T-u-v}\widetilde{\eta}(v+s)\widetilde{\eta}(s)dsdv$$
$$\le 2\int_0^{T-u}\left|\int_0^{T-u-v}\varepsilon(u+v+s)\varepsilon(u+s)\varepsilon(v+s)\varepsilon(s)ds\right|dv,$$

and therefore

$$E\xi_T^2 \le 2\sqrt{2}T^{-2}\int_0^T\left(\int_0^{T-u} E\left|\int_0^{T-u-v}\varepsilon(u+v+s)\varepsilon(u+s)\varepsilon(v+s)\varepsilon(s)ds\right|dv\right)^{\frac{1}{2}}du$$
$$\le 2\sqrt{2}T^{-2}\int_0^T\left(\int_0^{T-u}\left(\int_0^{T-u-v}\int_0^{T-u-v} E\varepsilon(u+v+s)\varepsilon(u+s)\varepsilon(v+s)\varepsilon(s)\right.\right.$$
$$\left.\left.\times\,\varepsilon(u+v+t)\varepsilon(u+t)\varepsilon(v+t)\varepsilon(t)dtds\right)^{\frac{1}{2}}dv\right)^{\frac{1}{2}}du. \quad (11)$$

For further estimation of the right hand side of inequality (11), we use condition **A** and apply the Isserlis' theorem (see, for example, Michalovicz *et al.* (2009)) to the product of 8 values of the stationary Gaussian process ε. If such a product contains $2n$ values, then the number of terms in the Isserlis' sum is $(2n-1)!!$. In our case, $n=4$, and the number of summands is $7!!=105$. Moreover each summand is the product of some 4 values of the process ε c.f. This is the price we have to pay for getting rid of the variables ϕ and ψ in inequality (11), which is an important step in obtaining (7). The application of the Isserlis' theorem to the right hand side of (11) is given in **Appendix**.

According to the result of **Appendix**

$$E\,\overset{1}{\varepsilon}(u+v+s)\overset{2}{\varepsilon}(u+s)\overset{3}{\varepsilon}(v+s)\overset{4}{\varepsilon}(s)\overset{5}{\varepsilon}(u+v+t)\overset{6}{\varepsilon}(u+t)\overset{7}{\varepsilon}(v+t)\overset{8}{\varepsilon}(t)=\sum_{r=1}^{105} b_r(u,v,t-s), \quad (12)$$

where the terms $b_r(u,v,t-s)$ are given by formulas (A1)-(A9) and, in fact, not all of them depend on all the variables $u,v,t-s$. Therefore

$$\begin{aligned} E\xi_T^2 &\le 2\sqrt{2}T^{-2}\int_0^T\left(\int_0^{T-u}\left(\int_0^{T-u-v}\int_0^{T-u-v}\sum_{r=1}^{105} b_r(u,v,t-s)dtds\right)^{\frac12}dv\right)^{\frac12}du \\ &\le 2\sqrt{2}\sum_{r=1}^{105}T^{-2}\int_0^T\left(\int_0^{T-u}\left(\int_0^{T-u-v}\int_0^{T-u-v} b_r(u,v,t-s)dtds\right)^{\frac12}dv\right)^{\frac12}du. \quad (13)\end{aligned}$$

Obviously, for $r=\overline{1,105}$, making consecutive changes of variables $u\to Tu$, $v\to Tv$, and then $t\to tT$, $s\to Ts$, we get

$$\begin{aligned} &T^{-2}\int_0^T\left(\int_0^{T-u}\left(\int_0^{T-u-v}\int_0^{T-u-v} b_r(u,v,t-s)dtds\right)^{\frac12}dv\right)^{\frac12}du \\ &\quad = T^{-1}\int_0^T\left(T^{-1}\int_0^{T-u}\left(T^{-2}\int_0^{T-u-v}\int_0^{T-u-v} b_r(u,v,t-s)dtds\right)^{\frac12}dv\right)^{\frac12}du \\ &\quad = \int_0^1\left(\int_0^{1-u}\left(\int_0^{1-u-v}\int_0^{1-u-v} b_r(Tu,Tv,T(t-s))dtds\right)^{\frac12}dv\right)^{\frac12}du \\ &\quad \le \int_0^1\left(\int_0^1\left(\int_0^1\int_0^1 b_r(Tu,Tv,T(t-s))dtds\right)^{\frac12}dv\right)^{\frac12}du = J_r. \quad (14)\end{aligned}$$

Consider in (A1)-(A9) the c.f. B values covering all the combinations of variables under their signs, allowing integrals (14) to be estimated properly under condition $\mathbf{A}(\boldsymbol{i})$. In **Appendix**, for convenience in the lists (A1)-(A9) of products of the c.f. B values, constituting the terms b_r, we underlined those values from which the integral are further estimated. The rest of the factors we bound by $B(0)$. This coarsens the inequalities, but makes it easier to write down the answer.

Firstly, in the integrals of b_r we consider the factors

$$\begin{aligned} &B\left(T(t-s)\right), \quad B\left(T(t-s+u)\right)B\left(T(t-s-u)\right), \\ &\quad B\left(T(t-s+v)\right)B\left(T(t-s-v)\right), \quad B\left(T(t-s+u+v)\right)B\left(T(t-s-u-v)\right). \quad (15)\end{aligned}$$

Let $a(t)\ge 0$ be an even function and $\sup_t a(t)=a_0<+\infty$. Then

$$\int_0^1\int_0^1 a(t-s)dtds=\int_{-1}^1(1-|t|)a(t)dt\le\int_{-1}^1 a(t)dt. \quad (16)$$

Applying (16) to the even functions

$$a(t) = B(Tt), \quad B(T(t+w))B(T(t-w)), \quad w = u, v, u+v > 0, \tag{17}$$

we get

$$\int_0^1 \int_0^1 B(T(t-s))dtds \le 2\int_0^1 B(Tt)dt; \tag{18}$$

$$\int_0^1 \int_0^1 B(T(t-s+w))B(T(t-s-w))dtds$$
$$\le B(0)\left(\int_0^1 B(T(t+w))dt + \int_{-1}^0 B(T(t-w))dt\right) = 2B(0)\int_0^1 B(T(t+w))dt. \tag{19}$$

Secondly, in the integrals of b_r we have to look at the factors

$$B(T(t-s+u))B(T(t-s-u-v)), \quad B(T(t-s+v))B(T(t-s-u-v)),$$
$$B(T(t-s+u+v))B(T(t-s-u)), \quad B(T(t-s+u+v))B(T(t-s-v)),$$
$$B(T(t-s+v))B(T(t-s-u)), \quad B(T(t-s+u))B(T(t-s-v)). \tag{20}$$

They all can be written as generalization of (17) in the form

$$B(T(t-s+w_1))B(T(t-s-w_2)), \quad w_1, w_2 > 0. \tag{21}$$

We get successively

$$I = \int_0^1 \int_0^1 a(t-s+w_1)a(t-s-w_2)dtds = \iint_{t>s} + \iint_{t<s},$$

$$\iint_{t>s} = \int_0^1 \int_0^{1-u} a(u+w_1)a(u-w_2)dsdu = \int_0^1 (1-u)a(u+w_1)a(u-w_2)du$$
$$\le a_0 \int_0^1 a(u+w_1)du,$$

$$\iint_{t<s} = \int_0^1 \int_0^{1-u} a(-u+w_1)a(-u-w_2)dsdu = \int_0^1 (1-u)a(-u+w_1)a(-u-w_2)du$$
$$\le a_0 \int_0^1 a(u+w_2)du,$$

that is

$$\int_0^1 \int_0^1 B(T(t-s+w_1))B(T(t-s-w_2))dsdt$$
$$\le B(0)\left(\int_0^1 B(T(t+w_1))dt + \int_0^1 B(T(t+w_2))dt\right). \tag{22}$$

Thirdly, it remains to consider cases where the terms b_r can be bounded by the c.f. values $B(Tu)$, $B(Tv)$, $B(T(u+v))$. Then, respectively,

$$J_r \le B^{\frac{3}{4}}(0)\int_0^1 B^{\frac{1}{4}}(Tu)du, \quad J_r \le B^{\frac{3}{4}}(0)\left(\int_0^1 B^{\frac{1}{2}}(Tv)dv\right)^{\frac{1}{2}},$$
$$J_r \le B^{\frac{3}{4}}(0)\int_0^1 \left(\int_0^1 B^{\frac{1}{2}}(T(u+v))dv\right)^{\frac{1}{2}} du. \tag{23}$$

Let's estimate from above the values $B(T(t+w))$, where $t \in [0,1]$ and w can be equal to 0, u, v, $u+v$ with $u,v \in (0,1]$. By condition $\mathbf{A(}\boldsymbol{i}\mathbf{)}$ for any $\delta > 0$ and $T > T_0 = T_0(\delta)$ $L(T(t+w)) \le L(3T) \le (1+\delta)L(T)$. Besides, $(T(t+w))^{-\alpha} \le T^{-\alpha}t^{-\alpha}$, and for $T > T_0$

$$\int_0^1 B(T(t+w))dt \le \frac{1+\delta}{1-\alpha}B(T). \tag{24}$$

We note that in the lists (A1)-(A9) the underlined values of c.f. B corresponding to formulas (15) and (20) are in 75 terms of sum (13). Denoting these terms by $J_{r_i}, i = \overline{1,75}$, and taking into account the inequalities (18), (19), (22), and (24), we obtain for $T > T_0$

$$2\sqrt{2}\sum_{i=1}^{75} J_{r_i} \le 75(2^{1+\frac{1}{2}+\frac{1}{4}})B^{\frac{3}{4}}(0)\left(\frac{1+\delta}{1-\alpha}\right)^{\frac{1}{4}} B^{\frac{1}{4}}(T). \tag{25}$$

It remains to estimate the integrals in (23). The 2nd and the 3rd integrals are bounded identically. For the 2nd integral for $T > T_0$ we get immediately

$$\left(\int_0^1 B^{\frac{1}{2}}(Tv)dv\right)^{\frac{1}{2}} \le \frac{(1+\delta)^{\frac{1}{4}}}{(1-\frac{\alpha}{2})^{\frac{1}{2}}}B^{\frac{1}{4}}(T). \tag{26}$$

In the lists (A1)-(A9) there are 25 underlined values of c.f B corresponding to the 2nd and 3rd integrals in (23). Denoting the suitable terms (14) in the sum (13) by $J_{r_i}, i = \overline{76,100}$, we obtain from (26)

$$2\sqrt{2}\sum_{i=76}^{100} J_{r_i} \le 25(2^{1+\frac{1}{2}})B^{\frac{3}{4}}(0)\frac{(1+\delta)^{\frac{1}{4}}}{(1-\frac{\alpha}{2})^{\frac{1}{2}}}B^{\frac{1}{4}}(T). \tag{27}$$

Finally, for the last 5 terms $J_{r_i}, i = \overline{101,105}$, corresponding to the 1st integral in (23), we derive the obvious inequality

$$2\sqrt{2}\sum_{i=101}^{105} J_{r_i} \le 5(2^{1+\frac{1}{2}})B^{\frac{3}{4}}(0)\frac{(1+\delta)^{\frac{1}{4}}}{1-\frac{\alpha}{4}}B^{\frac{1}{4}}(T). \tag{28}$$

Combining the bounds (25), (27), and (28) for any $\delta > 0$ and $T > T_0(\delta)$, we arrive at the inequality

$$E\xi_T^2 \le 2\sqrt{2}\left(75\sqrt[4]{2}(1-\alpha)^{-\frac{1}{4}} + 25(1-\alpha/2)^{-\frac{1}{2}} + 5(1-\alpha/4)^{-1}\right)(1+\delta)^{\frac{1}{4}}B^{\frac{3}{4}}(0)B^{\frac{1}{4}}(T), \tag{29}$$

that is

$$E\xi_T^2 = O\left(B^{\frac{1}{4}}(T)\right), \quad as \quad T \to \infty. \tag{30}$$

Now let condition $\mathbf{A(}\boldsymbol{ii}\mathbf{)}$ be satisfied. Then instead of (13) we have to write the inequality

$$E\xi_T^2 \le 2\sqrt{2}\sum_{r=1}^{105} T^{-2}\int_0^T\left(\int_0^{T-u}\left(\int_0^{T-u-v}\int_0^{T-u-v}|b_r(u,v,t-s)|dtds\right)^{\frac{1}{2}}dv\right)^{\frac{1}{2}}du, \tag{31}$$

in which the sum contains $24+72 = 96$ summands, where in the products of 4 values of c.f. B the difference $t-s$ is present in the arguments 4 or 2 times. In these summands we majorize the double integral by

$$B^3(0)\|B\|_1(T-u-v), \quad \|B\|_1 = \int_{-\infty}^{\infty}|B(t)|dt, \quad \text{and get}$$

$$2\sqrt{2}T^{-2}\int_0^T\left(\int_0^{T-u}\left(B^3(0)\|B\|_1(T-u-v)\right)^{\frac{1}{2}}dv\right)^{\frac{1}{2}}du$$
$$= \frac{16}{7\sqrt{3}}B^{\frac{3}{4}}(0)\|B\|_1^{1/4}T^{-\frac{1}{4}}. \tag{32}$$

In the remaining 9 terms of the sum (31), which include the products of c.f. B values from the list (A9), the double integral, for example, can be bounded 5 times by $B^3(0)(T-u-v)^2|B(v)|$, 3 times by $B^3(0)(T-u-v)^2|B(u+v)|$, and once (see formula 5 in A(9)) by $(T-u-v)^2B^4(u)$. So,

$$2\sqrt{2}B^{\frac{3}{4}}(0)T^{-2}\int_0^T\left(\int_0^{T-u}(T-u-v)|B(v)|^{\frac{1}{2}}dv\right)^{\frac{1}{2}}du \le \frac{8\sqrt{2}}{7\sqrt{3}}B^{\frac{3}{4}}(0)\|B\|_1^{1/4}T^{-\frac{1}{4}}. \tag{33}$$

For integrals with $|B(u+v)|$ instead of $|B(v)|$ we get the same bound. And finally,

$$2\sqrt{2}T^{-2}\int_0^T|B(u)|\left(\int_0^{T-u}(T-u-v)dv\right)^{\frac{1}{2}}du \le 2\|B\|_1T^{-1}. \tag{34}$$

It follows from (31)-(34) that under condition $\mathbf{A}(\boldsymbol{ii})$

$$E\xi_T^2 \le \left(\frac{1536+64\sqrt{2}}{7\sqrt{3}}\right)B^{\frac{3}{4}}(0)\|B\|_1^{1/4}T^{-\frac{1}{4}} + 2\|B\|_1T^{-1}, \tag{35}$$

that is

$$E\xi_T^2 = O\left(T^{-\frac{1}{4}}\right), \quad \text{as} \quad T\to\infty. \tag{36}$$

Returning to relation (30) let us take $T_n = n^\beta$ with number $\beta > \frac{4}{\alpha}$. Then $\sum\limits_{n=1}^{\infty} E\xi_{T_n}^2 < +\infty$, and $\xi_{T_n}\to 0$ a.s., as $n\to\infty$. Consider the sequence of random variables

$$\begin{aligned}
\zeta_n &= \sup_{T_n\le T<T_{n+1}}|\xi_T-\xi_{T_n}| \\
&= \sup_{T_n\le T<T_{n+1}}\left|\sup_{\phi,\psi\in\mathbb{R}}\left|T^{-1}\int_0^T e^{-i(\phi t+\psi t^2)}\varepsilon(t)dt\right| - \sup_{\phi,\psi\in\mathbb{R}}\left|T_n^{-1}\int_0^{T_n}e^{-i(\phi t+\psi t^2)}\varepsilon(t)dt\right|\right| \\
&\le \sup_{T_n\le T<T_{n+1}}\sup_{\phi,\psi\in\mathbb{R}}\left|T^{-1}\int_0^T e^{-i(\phi t+\psi t^2)}\varepsilon(t)dt - T_n^{-1}\int_0^{T_n}e^{-i(\phi t+\psi t^2)}\varepsilon(t)dt\right| \\
&\le \sup_{T_n\le T<T_{n+1}}\left[\sup_{\phi,\psi\in\mathbb{R}}\left|(T^{-1}-T_n^{-1})\int_0^{T_n}e^{-i(\phi t+\psi t^2)}\varepsilon(t)dt\right|\right. \\
&\left. + \sup_{\phi,\psi\in\mathbb{R}}\left|T^{-1}\int_{T_n}^T e^{-i(\phi t+\psi t^2)}\varepsilon(t)dt\right|\right] \le \frac{T_{n+1}-T_n}{T_n}\xi_{T_n} + T_n^{-1}\int_{T_n}^{T_{n+1}}|\varepsilon(t)|dt = \zeta_{n1}+\zeta_{n2}.
\end{aligned}$$

Obviously, $\zeta_{n1}\to 0$ a.s., as $n\to\infty$. On the other hand,

$$E\zeta_{n2}^2 = T_n^{-2}\int_{T_n}^{T_{n+1}}\int_{T_n}^{T_{n+1}}E|\varepsilon(t)\varepsilon(s)|dtds \le B(0)\left(\frac{T_{n+1}-T_n}{T_n}\right)^2 = O(n^{-2}),$$

$\sum\limits_{n=1}^{\infty}E\zeta_{n2}^2 < +\infty$. So, $\zeta_{n2}\to 0$ a.s, as $n\to\infty$, and under condition $\mathbf{A}(\boldsymbol{i})$ the relation (7) is true. Under condition $\mathbf{A}(\boldsymbol{ii})$ the same convergence follows from (36). □

3. Proof of Theorem 1

Consider a system of linear equations for A_{kT}, B_{kT}, $k=\overline{1,N}$, which is a subsystem of the system of normal equations for LSE θ_T:

$$\frac{\partial Q_T(\theta_T)}{\partial A_p} = \frac{\partial Q_T(\theta_T)}{\partial B_p} = 0, \quad p=\overline{1,N},$$

and rewrite it in more detail in the form

$$\begin{cases} \sum\limits_{j=1}^{N} a_{jp}^{(1)}(T)A_{jT} + \sum\limits_{j=1}^{N} b_{jp}^{(1)}(T)B_{jT} = c_p^{(1)}, \quad p = \overline{1,N}; \\ \sum\limits_{j=1}^{N} a_{jp}^{(2)}(T)A_{jT} + \sum\limits_{j=1}^{N} b_{jp}^{(2)}(T)B_{jT} = c_p^{(2)}, \quad p = \overline{1,N}. \end{cases} \tag{37}$$

In (37) we use the following notation. Let

$$\cos(\phi_{jT}t + \psi_{jT}t^2) = \cos_j(t), \quad \sin(\phi_{jT}t + \psi_{jT}t^2) = \sin_j(t),$$
$$\cos(\phi_j^0 t + \psi_j^0 t^2) = \cos_j^0(t), \quad \sin(\phi_j^0 t + \psi_j^0 t^2) = \sin_j^0(t).$$

Then the coefficients of system (37) can be written as

$$a_{jp}^{(1)}(T) = T^{-1}\int_0^T \cos_j(t)\cos_p(t)dt, \quad a_{jp}^{(2)}(T) = T^{-1}\int_0^T \cos_j(t)\sin_p(t)dt,$$
$$b_{jp}^{(1)}(T) = T^{-1}\int_0^T \sin_j(t)\cos_p(t)dt, \quad b_{jp}^{(2)}(T) = T^{-1}\int_0^T \sin_j(t)\sin_p(t)dt,$$
$$c_p^{(1)}(T) = T^{-1}\int_0^T X(t)\cos_p(t)dt, \quad c_p^{(2)}(T) = T^{-1}\int_0^T X(t)\sin_p(t)dt.$$

Let also denote by $o_T(1), T > 0$, possibly different stochastic processes such that $o_T(1) \to 0$ a.s., as $T \to \infty$. Using condition **B**, we will show that

$$a_{jp}^{(1)}(T) = o_T(1), \quad j \neq p, \quad a_{pp}^{(1)}(T) = \frac{1}{2} + o_T(1); \quad a_{jp}^{(2)}(T) = o_T(1), \quad j,p = \overline{1,N};$$
$$b_{jp}^{(1)}(T) = o_T(1); \quad b_{jp}^{(2)}(T) = o_T(1), \quad j \neq p, \quad b_{pp}^{(2)}(T) = \frac{1}{2} + o_T(1), \quad j,p = \overline{1,N}. \tag{38}$$

The following simple statement will help to do this.

Lemma 1. *Let α_T, β_T, $T > 0$, be some functions, and $\beta_T \to +\infty$, as $T \to \infty$. Then*

$$\int_0^1 \cos(\alpha_T t + \beta_T t^2)dt \to 0, \quad \int_0^1 \sin(\alpha_T t + \beta_T t^2)dt \to 0, \text{ as } T \to \infty. \tag{39}$$

Proof. Obviously,

$$\begin{aligned} \int_0^1 \begin{matrix}\cos\\ \sin\end{matrix} (\alpha_T t + \beta_T t^2)dt &= \int_0^1 \begin{matrix}\cos\\ \sin\end{matrix} \left(\beta_T\left(t + \frac{\alpha_T}{2\beta_T}\right)^2 - \frac{\alpha_T^2}{4\beta_T}\right)dt \\ &= \cos\left(\frac{\alpha_T^2}{4\beta_T}\right)\left(\frac{1}{\sqrt{\beta_T}} \int_{\alpha_T/2\sqrt{\beta_T}}^{\sqrt{\beta_T}+\alpha_T/2\sqrt{\beta_T}} \begin{matrix}\cos\\ \sin\end{matrix} (t^2)dt\right) \\ &\pm \sin\left(\frac{\alpha_T^2}{4\beta_T}\right)\left(\frac{1}{\sqrt{\beta_T}} \int_{\alpha_T/2\sqrt{\beta_T}}^{\sqrt{\beta_T}+\alpha_T/2\sqrt{\beta_T}} \begin{matrix}\sin\\ \cos\end{matrix} (t^2)dt\right). \end{aligned} \tag{40}$$

Since the Fresnel integrals

$$C(x) = \int_0^x \cos(t^2)dt, \quad S(x) = \int_0^x \sin(t^2)dt, \quad x > 0,$$

taking positive values, have the property $\max\limits_x C(x), \max\limits_x S(x) < 1$, then from (40) it follows that

$$\left|\int_0^1 \cos(\alpha_T t + \beta_T t^2)dt\right|, \left|\int_0^1 \sin(\alpha_T t + \beta_T t^2)dt\right| < 4\beta_T^{-\frac{1}{2}}. \tag{41}$$

□

Assume $j > p$. Then by condition **B** and **Lemma 1**

$$a_{jp}^{(1)}(T) = \frac{1}{2}\int_0^1 \cos(T(\phi_{jT} + \phi_{pT})t + T^2(\psi_{jT} + \psi_{pT})t^2)dt$$
$$+\frac{1}{2}\int_0^1 \cos(T(\phi_{jT} - \phi_{pT})t + T^2(\psi_{jT} - \psi_{pT})t^2)dt = c_{jp}^{+} + c_{jp}^{-} = o_T(1).$$

Case $j < p$ is considered similarly. Moreover, $b_{jp}^{(2)}(T) = c_{jp}^{-} - c_{jp}^{+} = o_T(1), j \neq p$. On the other hand,

$$a_{pp}^{(1)}(T) = \frac{1}{2} + \frac{1}{2}\int_0^1 \cos((2\phi_{pT}T)t + (2\psi_{pT}T^2)t^2)dt$$
$$= \frac{1}{2} + \frac{1}{2}c_{pp}^{+} = \frac{1}{2} + o_T(1), \quad b_{pp}^{(2)}(T) = \frac{1}{2} - \frac{1}{2}c_{pp}^{+} = \frac{1}{2} + o_T(1).$$

We get further for $j > p$

$$b_{jp}^{(1)}(T) = \frac{1}{2}\int_0^1 \sin(T(\phi_{jT} + \phi_{pT})t + T^2(\psi_{jT} + \psi_{pT})t^2)dt$$
$$+\frac{1}{2}\int_0^1 \sin(T(\phi_{jT} - \phi_{pT})t + T^2(\psi_{jT} - \psi_{pT})t^2)dt = s_{jp}^{+} + s_{jp}^{-} = o_T(1),$$

$a_{jp}^{(2)}(T) = s_{jp}^{+} - s_{jp}^{-} = o_T(1)$. Case $j < p$ is considered similarly.
Note also that

$$a_{pp}^{(2)}(T) = b_{pp}^{(1)} = \frac{1}{2}s_{pp}^{+} = o_T(1),$$

and relations (38) are valid.

Besides,

$$c_p^{(1)}(T) = T^{-1}\int_0^T \varepsilon(t)\cos_p(t)dt + T^{-1}\int_0^T g(t,\theta^0)\cos_p(t)dt, \tag{42}$$

and the 1st term of the sum (42) is $o_T(1)$, $p = \overline{1, N}$, due to **Theorem 2**. On the other hand, for any p

$$T^{-1}\int_0^T g(t,\theta^0)\cos_p(t)dt = \sum_{j=1}^{N} A_j^0 T^{-1}\int_0^T \cos_j^0(t)\cos_p(t)dt$$
$$+\sum_{j=1}^{N} B_j^0 T^{-1}\int_0^T \sin_j^0(t)\cos_p(t)dt$$
$$= A_p^0(2T)^{-1}\int_0^T \cos((\phi_{pT} - \phi_p^0)t + (\psi_{pT} - \psi_p^0)t^2)dt$$
$$- B_p^0(2T)^{-1}\int_0^T \sin((\phi_{pT} - \phi_p^0)t + (\psi_{pT} - \psi_p^0)t^2)dt + o_T(1), \tag{43}$$

by condition **B** and **Lemma 1**. Using notation

$$x_{pT} = T^{-1}\int_0^T \cos((\phi_{pT} - \phi_p^0)t + (\psi_{pT} - \psi_p^0)t^2)dt, \tag{44}$$

$$y_{pT} = T^{-1}\int_0^T \sin((\phi_{pT} - \phi_p^0)t + (\psi_{pT} - \psi_p^0)t^2)dt, \tag{45}$$

from (42) and (43) we get

$$c_p^{(1)}(T) = \frac{1}{2}[A_p^0 x_{pT} - B_p^0 y_{pT}] + o_T(1), \quad p = \overline{1, N}. \tag{46}$$

Similarly,

$$c_p^{(2)}(T) = \frac{1}{2}[A_p^0 y_{pT} + B_p^0 x_{pT}] + o_T(1), \quad p = \overline{1, N}. \tag{47}$$

Applying relations (38), (46), (47) to the system of equations (37) we obtain the following expressions for LSE A_{jT}, B_{jT}:

$$A_{jT} = A_j^0 x_{jT} - B_j^0 y_{iT} + o_T(1), \quad B_{jT} = A_j^0 y_{jT} + B_j^0 x_{iT} + o_T(1), \quad j = \overline{1, N}. \tag{48}$$

Obviously, $|x_{jT}|, |y_{jT}| \le 1$, then as it follows from (48)

$$|A_{jT}| \le |A_j^0| + |B_j^0| + o_T(1), \quad |B_{jT}| \le |A_j^0| + |B_j^0| + o_T(1), \quad j = \overline{1, N}. \tag{49}$$

Let

$$G_T(\theta_1, \theta_2) = T^{-1} \int_0^T (g(t, \theta_1) - g(t, \theta_2))^2 dt, \quad \theta_1, \theta_2 \in \Theta_T^c.$$

From LSE θ_T definition we get

$$0 \ge Q_T(\theta_T) - Q_T(\theta^0) = G_T(\theta_T, \theta^0) + 2T^{-1} \int_0^T \varepsilon(t)(g(t, \theta^0) - g(t, \theta_T))dt. \tag{50}$$

By **Theorem 2** and (49)

$$T^{-1} \int_0^T \varepsilon(t)(g(t, \theta^0) - g(t, \theta_T))dt \to 0 \text{ a.s., as } T \to \infty,$$

and therefore due to (50)

$$G_T(\theta_T, \theta^0) \to 0 \text{ a.s., as } T \to \infty. \tag{51}$$

Using notation

$$g_{jT}(t) = A_{jT} \cos_j(t) + B_{jT} \sin_j(t) - A_j^0 \cos_j^0(t) - B_j^0 \sin_j^0(t),$$

we get

$$G_T(\theta_T, \theta^0) = \sum_{j=1}^N T^{-1} \int_0^T g_{jT}^2(t)dt + 2 \sum_{j<p} T^{-1} \int_0^T g_{jT}(t) g_{pT}(t)dt. \tag{52}$$

From **Lemma 1**, condition **B**, inequalities (49) and previous arguments, we find that the 2nd sum in (52) is $o_T(1)$. On the other hand,

$$\begin{aligned}
T^{-1} \int_0^T g_{jT}^2(t)dt &= A_{jT}^2 T^{-1} \int_0^T \cos_j^2(t)dt + B_{jT}^2 T^{-1} \int_0^T \sin_j^2(t)dt \\
&\quad + (A_j^0)^2 T^{-1} \int_0^T (\cos_j^0(t))^2 dt + (B_j^0)^2 T^{-1} \int_0^T (\sin_j^0(t))^2 dt \\
&\quad + 2A_{jT} B_{jT} T^{-1} \int_0^T \cos_j(t) \sin_j(t)dt + 2A_j^0 B_j^0 T^{-1} \int_0^T \cos_j^0(t) \sin_j^0(t)dt \\
&\quad - 2A_{jT} A_j^0 T^{-1} \int_0^T \cos_j(t) \cos_j^0(t)dt - 2B_{jT} B_j^0 T^{-1} \int_0^T \sin_j(t) \sin_j^0(t)dt \\
&\quad - 2A_{jT} B_j^0 T^{-1} \int_0^T \cos_j(t) \sin_j^0(t)dt - 2B_{jT} A_j^0 T^{-1} \int_0^T \sin_j(t) \cos_j^0(t)dt \\
&= \frac{1}{2} \left[A_{jT}^2 + B_{jT}^2 + (A_j^0)^2 + (B_j^0)^2\right] - \left(A_{jT} A_j^0 + B_{jT} B_j^0\right) x_{jT} \\
&\quad + \left(A_{jT} B_j^0 - A_j^0 B_{jT}\right) y_{jT} + o_T(1), \quad j = \overline{1, N}.
\end{aligned} \tag{53}$$

Substituting equalities (48) into expression (53), we derive

$$\begin{aligned} T^{-1}\int_0^T g_{jT}^2(t)dt &= \frac{1}{2}\left[(A_j^0 x_{jT} - B_j^0 y_{iT})^2 + (A_j^0 y_{jT} + B_j^0 x_{iT})^2 + (A_j^0)^2 + (B_j^0)^2\right] \\ &\quad - \left[(A_j^0 x_{jT} - B_j^0 y_{iT})A_j^0 + (A_j^0 y_{jT} + B_j^0 x_{iT})B_j^0\right] x_{jT} \\ &\quad + \left[(A_j^0 x_{jT} - B_j^0 y_{iT})B_j^0 - (A_j^0 y_{jT} + B_j^0 x_{iT})A_j^0\right] y_{jT} + o_T(1) \\ &= \frac{1}{2}\left((A_j^0)^2 + (B_j^0)^2\right)\left(1 - x_{jT}^2 - y_{jT}^2\right) + o_T(1), \quad j = \overline{1, N}. \end{aligned} \tag{54}$$

Taking into account (51), (52), and (54), we arrive at the relation

$$\sum_{j=1}^N \frac{1}{2}\left((A_j^0)^2 + (B_j^0)^2\right)\left(1 - x_{jT}^2 - y_{jT}^2\right) \to 0 \text{ a.s., as } T \to \infty. \tag{55}$$

From (55) it follows that

$$x_{jT}^2 + y_{jT}^2 \to 1 \text{ a.s., as } T \to \infty, \quad j = \overline{1, N}. \tag{56}$$

Introducing the notation $\lambda_{jT} = T(\phi_{jT} - \phi_j^0)$, $\mu_{jT} = T^2(\psi_{jT} - \psi_j^0)$, we can write

$$x_{jT} = \int_0^1 \cos(\lambda_{jT} t + \mu_{jT} t^2)dt, \quad y_{jT} = \int_0^1 \sin(\lambda_{jT} t + \mu_{jT} t^2)dt, \quad j = \overline{1, N}. \tag{57}$$

Let $\Omega_0 \subset \Omega$, $P(\Omega_0) = 1$, be a random event for which (56) is valid. If for any elementary event $\omega \in \Omega_0$

$$\lambda_{jT}, \mu_{jT} \to 0, \text{ as } T \to \infty, \quad j = \overline{1, N}, \tag{58}$$

then (56) is true by Lebesgue majorized convergence theorem.

Assume that (58) is not true for some $\omega_0 \in \Omega_0$ and consider all possible cases of λ_{jT}, μ_{jT} behavior.

Let for some $j \in \{1, ..., N\}$ and $\omega_0 \in \Omega_0$ $\mu_{jT} \not\to 0$, as $T \to \infty$. Then there exist $\varepsilon_0 > 0$ and sequence $\{T_n, n \geq 1\}$, $T_n \to \infty$, as $n \to \infty$, such that $|\mu_{jT_n}| \geq \varepsilon_0$, $n \geq 1$. Let set of the values $\{\mu_{jT_n}, n \geq 1\}$ be bounded, that is there exists a subsequence $\{T_{n_k}, k \geq 1\}$ such that $\mu_{jT_{n_k}} \to \mu_j \neq 0$, as $k \to \infty$. Then for some subsequence $\{T_{n_{k_m}}, m \geq 1\}$ $\lambda_{jT_{n_{k_m}}} \to \lambda_j \neq \pm\infty$, or $\lambda_{jT_{n_{k_m}}} \to +\infty$ or $-\infty$, as $m \to \infty$. If set of the values $\{\mu_{jT_n}, n \geq 1\}$ is unbounded, then for some subsequence $\{T_{n_k}, k \geq 1\}$ $\mu_{jT_{n_k}} \to +\infty$ or $-\infty$, as $k \to \infty$. At the same time, subsequences $\lambda_{jT_{n_{k_m}}}$ can have the properties described above. Interchanging the values λ_{jT}, μ_{jT} we arrive at the symmetric properties of their subsequences.

Noting $\lambda_{jT_{n_{k_m}}} = \lambda_{jm}$, $\mu_{jT_{n_{k_m}}} = \mu_{jm}$ and considering previous remarks, indicate the possible options of λ_{jm}, μ_{jm} convergence, as $m \to \infty$.

(i) $\lambda_{jm} \to +\infty$ or $-\infty$, $\mu_{jm} \to \mu_j \neq 0$; *(ii)* $\lambda_{jm} \to +\infty$ or $-\infty$, $\mu_{jm} \to 0$;

(iii) $\lambda_{jm} \to \lambda_j$, $\mu_{jm} \to \mu_j$, $\lambda_j^2 + \mu_j^2 > 0$; *(iv)* $\lambda_{jm} \to +\infty$ or $-\infty$, $\mu_{jm} \to +\infty$ or $-\infty$;

(v) $\lambda_{jm} \to \lambda_j$, $\mu_{jm} \to +\infty$ or $-\infty$.

Set also $x_{jT_{n_{k_m}}} = x_{jm}$, $y_{jT_{n_{k_m}}} = y_{jm}$ and show that for the options *(i)*-*(v)*

$$x_{jm}^2 + y_{jm}^2 \not\to 1, \text{ as } m \to \infty. \tag{59}$$

Let $\mu_{jm} \not\to 0$, as $m \to \infty$, and $\mu_{jm} > 0$ for sufficiently large m. Using equality (40) we express $x_{jm}^2 + y_{jm}^2$ in terms of Fresnel integrals. Designate $\gamma_{jm} = \lambda_{jm}/2\sqrt{\mu_{jm}}$, then

$$\begin{aligned} x_{im}^2 + y_{im}^2 = {} & \frac{\cos^2(\gamma_{jm}^2)}{\mu_{jm}} \left(\int\limits_{\gamma_{jm}}^{\sqrt{\mu_{jm}}+\gamma_{jm}} \cos(t^2)dt \right)^2 + \frac{\sin^2(\gamma_{jm}^2)}{\mu_{jm}} \left(\int\limits_{\gamma_{jm}}^{\sqrt{\mu_{jm}}+\gamma_{jm}} \sin(t^2)dt \right)^2 \\ & + \frac{\sin(2\gamma_{jm}^2)}{\mu_{jm}} \left(\int\limits_{\gamma_{jm}}^{\sqrt{\mu_{jm}}+\gamma_{jm}} \cos(t^2)dt \right) \left(\int\limits_{\gamma_{jm}}^{\sqrt{\mu_{jm}}+\gamma_{jm}} \sin(t^2)dt \right) \\ & + \frac{\cos^2(\gamma_{jm}^2)}{\mu_{jm}} \left(\int\limits_{\gamma_{jm}}^{\sqrt{\mu_{jm}}+\gamma_{jm}} \sin(t^2)dt \right)^2 + \frac{\sin^2(\gamma_{jm}^2)}{\mu_{jm}} \left(\int\limits_{\gamma_{jm}}^{\sqrt{\mu_{jm}}+\gamma_{jm}} \cos(t^2)dt \right)^2 \\ & - \frac{\sin(2\gamma_{jm}^2)}{\mu_{jm}} \left(\int\limits_{\gamma_{jm}}^{\sqrt{\mu_{jm}}+\gamma_{jm}} \cos(t^2)dt \right) \left(\int\limits_{\gamma_{jm}}^{\sqrt{\mu_{jm}}+\gamma_{jm}} \sin(t^2)dt \right) \\ = {} & \frac{1}{\mu_{jm}} \left(\int\limits_{\gamma_{jm}}^{\sqrt{\mu_{jm}}+\gamma_{jm}} \cos(t^2)dt \right)^2 + \frac{1}{\mu_{jm}} \left(\int\limits_{\gamma_{jm}}^{\sqrt{\mu_{jm}}+\gamma_{jm}} \sin(t^2)dt \right)^2 . \end{aligned} \tag{60}$$

In the case $\mu_{jm} < 0$ for sufficiently large m and $\gamma'_{jm} = \lambda_{jm}/2\sqrt{|\mu_{jm}|}$ we get similarly

$$x_{im}^2 + y_{im}^2 = \frac{1}{|\mu_{jm}|} \left(\int\limits_{-\gamma'_{jm}}^{\sqrt{|\mu_{jm}|}-\gamma'_{jm}} \cos(t^2)dt \right)^2 + \frac{1}{|\mu_{jm}|} \left(\int\limits_{-\gamma'_{jm}}^{\sqrt{|\mu_{jm}|}-\gamma'_{jm}} \sin(t^2)dt \right)^2 . \tag{61}$$

Using formulas (60), (61) and properties of Fresnel integrals we obtain for options *(i)*, *(iv)* and *(v)*

$$x_{im}^2 + y_{im}^2 \to 0, \text{ as } m \to \infty. \tag{62}$$

Going over to the option *(ii)* we derive

$$\begin{aligned} x_{im}^2 + y_{im}^2 = {} & \int_0^1 \int_0^1 \cos(\lambda_{jm}(t-s) + \mu_{jm}(t^2 - s^2))dtds \\ & \le \left| \int_0^1 \int_0^1 \cos(\lambda_{jm}(t-s)) \cos(\mu_{jm}(t^2 - s^2))dtds \right| \\ & + \left| \int_0^1 \int_0^1 \sin(\lambda_{jm}(t-s)) \sin(\mu_{jm}(t^2 - s^2))dtds \right| \\ \le {} & \left(\int_0^1 \int_0^1 \cos^2(\lambda_{jm}(t-s))dtds \right)^{\frac{1}{2}} \left(\int_0^1 \int_0^1 \cos^2(\mu_{jm}(t^2 - s^2))dtds \right)^{\frac{1}{2}} \\ & + \left(\int_0^1 \int_0^1 \sin^2(\lambda_{jm}(t-s))dtds \right)^{\frac{1}{2}} \left(\int_0^1 \int_0^1 \sin^2(\mu_{jm}(t^2 - s^2))dtds \right)^{\frac{1}{2}} \\ \le {} & \left(\int_0^1 \int_0^1 \cos^2(\lambda_{jm}(t-s))dtds \right)^{\frac{1}{2}} + \left(\int_0^1 \int_0^1 \sin^2(\mu_{jm}(t^2 - s^2))dtds \right)^{\frac{1}{2}} . \end{aligned} \tag{63}$$

By the Lebesgue theorem mentioned above the 2nd term in (63) tends to zero, as $m \to \infty$. The 1st term

$$\left(\int_0^1 \int_0^1 \cos^2(\lambda_{jm}(t-s))dtds \right)^{\frac{1}{2}} = \left(\frac{1}{2} + \frac{1}{2} \int_0^1 \int_0^1 \cos(2\lambda_{jm}(t-s))dtds \right)^{\frac{1}{2}}$$

$$= \left(\frac{1}{2} + \frac{1}{2} \left(\frac{\sin \lambda_{jm}}{\lambda_{jm}} \right)^2 \right)^{\frac{1}{2}},$$

that is for the option *(ii)*

$$\limsup_{m\to\infty} (x_{jm}^2 + y_{jm}^2) \leq \frac{1}{\sqrt{2}}. \tag{64}$$

In the option *(iii)* by the Lebesgue theorem and Cauchy-Schwartz inequality

$$\lim_{m\to\infty} (x_{jm}^2 + y_{jm}^2) = \left(\int_0^1 \cos(\lambda_j t + \mu_j t^2) dt \right)^2 + \left(\int_0^1 \sin(\lambda_j t + \mu_j t^2) dt \right)^2$$
$$\leq \int_0^1 \cos^2(\lambda_j t + \mu_j t^2) dt + \int_0^1 \sin^2(\lambda_j t + \mu_j t^2) dt = 1, \tag{65}$$

and equality is achieved if and only if for some constants k_1, k_2

$$\cos(\lambda_j t + \mu_j t^2) = k_1, \quad \sin(\lambda_j t + \mu_j t^2) = k_2, \quad t \in [0, 1].$$

However it is impossible, and therefore

$$\lim_{m\to\infty} (x_{jm}^2 + y_{jm}^2) < 1. \tag{66}$$

Together with (55), this means that for $j = \overline{1, N}$

$$\lambda_{jT} = T(\phi_{jT} - \phi_j^0), \quad \mu_{jT} = T^2(\psi_{jT} - \psi_j^0) \to 0 \text{ a.s., as } T \to \infty. \tag{67}$$

From (48), (57), and (67) it follows also

$$A_{jT} \to A_j^0, \quad B_{jT} \to B_j^0 \text{ a.s., as } T \to \infty, \quad j = \overline{1, N}. \tag{68}$$

Appendix

We number the factors under the expectation sign in (12) from the left to the right with numbers 1, 2,..., 8 and separate all the 105 terms from the Isserlis' theorem into 3 sets.

The 1st set consists of terms in which all 4 values of the c.f. contain $t - s$ in the argument. There are 24 such terms and they can be found, for example, as follows. Let us denote by (ab) the expectation of the product of factors with numbers a and b in (12). Then we get

1. $(15)(26)(37)(48) = \underline{B(t-s)}B(t-s)B(t-s)B(t-s);$
2. $(15)(26)(38)(47) = \underline{B(t-s)}B(t-s)B(t-s-v)B(t-s+v);$
3. $(15)(27)(36)(48) = \underline{B(t-s)}B(t-s+v-u)B(t-s+u-v)B(t-s);$
4. $(15)(27)(38)(46) = \underline{B(t-s)}B(t-s+v-u)B(t-s-v)B(t-s+u);$
5. $(15)(28)(36)(47) = \underline{B(t-s)}B(t-s-u)B(t-s+u-v)B(t-s+v);$
6. $(15)(28)(37)(46) = \underline{B(t-s)}B(t-s-u)B(t-s)B(t-s+u);$ (A1)
7. $(16)(25)(37)(48) = B(t-s-v)B(t-s+v)B(t-s)\underline{B(t-s)};$
8. $(16)(25)(38)(47) = \underline{B(t-s-v)B(t-s+v)}B(t-s-v)B(t-s+v);$
9. $(16)(27)(35)(48) = B(t-s-v)B(t-s+v-u)B(t-s+u)\underline{B(t-s)};$

10. $(16)(27)(38)(45) = B(t-s-v)B(t-s+v-u)\underline{B(t-s-v)B(t-s+u+v)};$
11. $(16)(28)(35)(47) = \underline{B(t-s-v)}B(t-s-u)B(t-s+u)\underline{B(t-s+v)};$
12. $(16)(28)(37)(45) = B(t-s-v)B(t-s-u)\underline{B(t-s)}B(t-s+u+v);$

13. $(17)(25)(36)(48) = B(t-s-u)B(t-s+v)B(t-s+u-v)\underline{B(t-s)};$
14. $(17)(25)(38)(46) = B(t-s-u)\underline{B(t-s+v)B(t-s-v)}B(t-s+u);$
15. $(17)(26)(35)(48) = B(t-s-u)\underline{B(t-s)}B(t-s+u)B(t-s);$
16. $(17)(26)(38)(45) = B(t-s-u)\underline{B(t-s)}B(t-s-v)B(t-s+u+v);$
17. $(17)(28)(35)(46) = \underline{B(t-s-u)}B(t-s-u)\underline{B(t-s+u)}B(t-s+u);$
18. $(17)(28)(36)(45) = \underline{B(t-s-u)}B(t-s-u)B(t-s+u-v)\underline{B(t-s+u+v)};$ (A2)
19. $(18)(25)(36)(47) = \underline{B(t-s-u-v)B(t-s+v)}B(t-s+u-v)B(t-s+v);$
20. $(18)(25)(37)(46) = B(t-s-u-v)B(t-s+v)\underline{B(t-s)}B(t-s+u);$
21. $(18)(26)(35)(47) = B(t-s-u-v)\underline{B(t-s)}B(t-s+u)B(t-s+v);$
22. $(18)(26)(37)(45) = B(t-s-u-v)\underline{B(t-s)}B(t-s)B(t-s+u+v);$
23. $(18)(27)(35)(46) = \underline{B(t-s-u-v)}B(t-s+v-u)B(t-s+u)\underline{B(t-s+u)};$
24. $(18)(27)(36)(45) = \underline{B(t-s-u-v)}B(t-s+v-u)B(t-s+u-v)\underline{B(t-s+u+v)}.$

The 2nd set consists of the terms in which 2 values of the c.f. contain $t-s$ in the argument. There are 72 terms of such a kind and they can be determined, for example, using the following reasoning. There exists 6 expectations made out of the first 4 multipliers in (12):

(i) (12), (13), (14), (23), (24), (34),

and correspondently 6 expectations made out of the second 4 multipliers in (12):

(ii) (56), (57), (58), (67), (68), (78).

Each element of the 2nd set is the product of one value of the c.f. from ***(i)*** and one from ***(ii)***, as well as two values of c.f. obtained by taking expectation from the product of process ε values with the remaining numbers from the 1st quadruplet by the values of ε with the remaining numbers from the 2nd quadruplet.

The 1st dozen: (12)×... .

1. $(12)(56)(37)(48) = B(v)B(v)B(t-s)\underline{B(t-s)};$
2. $(12)(56)(38)(47) = B(v)B(v)\underline{B(t-s-v)B(t-s+v)};$
3. $(12)(57)(36)(48) = B(v)B(u)B(t-s+u-v)\underline{B(t-s)};$
4. $(12)(57)(38)(46) = B(v)B(u)\underline{B(t-s-v)B(t-s+u)};$
5. $(12)(58)(36)(47) = \underline{B(v)}B(u+v)B(t-s+u-v)B(t-s+v);$
6. $(12)(58)(37)(46) = B(v)B(u+v)\underline{B(t-s)}B(t-s+u);$ (A3)
7. $(12)(67)(35)(48) = B(v)B(u-v)B(t-s+u)\underline{B(t-s)};$

8. $(12)(67)(38)(45) = B(v)B(u-v)\underline{B(t-s-v)B(t-s+u+v)};$
9. $(12)(68)(35)(47) = \underline{B(v)}B(u)B(t-s+u)B(t-s+v);$
10. $(12)(68)(37)(45) = B(v)B(u)\underline{B(t-s)}B(t-s+u+v);$
11. $(12)(78)(35)(46) = \underline{B(v)}B(v)B(t-s+u)B(t-s+u);$
12. $(12)(78)(36)(45) = \underline{B(v)}B(v)B(t-s+u-v)B(t-s+u+v);$

The 2nd dozen: $(13)\times\ldots$.

13. $(13)(56)(27)(48) = B(u)B(v)B(t-s+v-u)\underline{B(t-s)};$
14. $(13)(56)(28)(47) = B(u)B(v)\underline{B(t-s-u)B(t-s+v)};$
15. $(13)(57)(26)(48) = B(u)B(u)B(t-s)\underline{B(t-s)};$
16. $(13)(57)(28)(46) = B(u)B(u)\underline{B(t-s-u)B(t-s+u)};$
17. $(13)(58)(26)(47) = B(u)B(u+v)\underline{B(t-s)}B(t-s+v);$
18. $(13)(58)(27)(46) = B(u)\underline{B(u+v)}B(t{-}s{+}v{-}u)B(t{-}s{+}u);$ (A4)
19. $(13)(67)(25)(48) = B(u)B(u-v)B(t-s+v)\underline{B(t-s)};$
20. $(13)(67)(28)(45) = B(u)B(u-v)\underline{B(t-s-u)B(t-s+u+v)};$
21. $(13)(68)(25)(47) = \underline{B(u)}B(u)B(t-s+v)B(t-s+v);$
22. $(13)(68)(27)(45) = \underline{B(u)}B(u)B(t-s+v-u)B(t-s+u+v);$
23. $(13)(78)(25)(46) = B(u)\underline{B(v)}B(t-s+v)B(t-s+u);$
24. $(13)(78)(26)(45) = B(u)B(v)\underline{B(t-s)}B(t-s+u+v);$

The 3rd dozen: $(14)\times\ldots$.

25. $(14)(56)(27)(38) = B(u+v)\underline{B(v)}B(t-s+v-u)B(t-s-v);$
26. $(14)(56)(28)(37) = B(u+v)B(v)B(t-s-u)\underline{B(t-s)};$
27. $(14)(57)(26)(38) = B(u+v)B(u)\underline{B(t-s)}B(t-s-v);$
28. $(14)(57)(28)(36) = \underline{B(u+v)}B(u)B(t-s-u)B(t-s+u-v);$
29. $(14)(58)(26)(37) = B(u+v)B(u+v)B(t-s)\underline{B(t-s)};$
30. $(14)(58)(27)(36) = \underline{B(u+v)}B(u{+}v)B(t{-}s{+}v{-}u)B(t{-}s{+}u{-}v);$ (A5)
31. $(14)(67)(25)(38) = B(u+v)B(u-v)\underline{B(t-s+v)B(t-s-v)};$
32. $(14)(67)(28)(35) = B(u+v)B(u-v)\underline{B(t-s-u)B(t-s+u)};$
33. $(14)(68)(25)(37) = B(u+v)B(u)B(t-s+v)\underline{B(t-s)};$
34. $(14)(68)(27)(35) = \underline{B(u+v)}B(u)B(t-s+v-u)B(t-s+u);$
35. $(14)(78)(25)(36) = B(u+v)\underline{B(v)}B(t-s+v)B(t-s+u-v);$
36. $(14)(78)(26)(35) = B(u+v)B(v)\underline{B(t-s)}B(t-s+u);$

The 4th dozen: $(23)\times\ldots$.

37. $(23)(56)(17)(48) = B(u-v)B(v)B(t-s-u)\underline{B(t-s)};$

38. $(23)(56)(18)(47) = B(u-v)B(v)\underline{B(t-s-u-v)B(t-s+v)}$;

39. $(23)(57)(16)(48) = B(u-v)B(u)B(t-s-v)\underline{B(t-s)}$;

40. $(23)(57)(18)(46) = B(u-v)B(u)\underline{B(t-s-u-v)B(t-s+u)}$;

41. $(23)(58)(16)(47) = B(u-v)B(u+v)\underline{B(t-s-v)B(t-s+v)}$;

42. $(23)(58)(17)(46) = B(u-v)B(u+v)\underline{B(t-s-u)B(t-s+u)}$; (A6)

43. $(23)(67)(15)(48) = B(u-v)B(u-v)B(t-s)\underline{B(t-s)}$;

44. $(23)(67)(18)(45) = B(u-v)B(u-v)\underline{B(t-s-u-v)B(t-s+u+v)}$;

45. $(23)(68)(15)(47) = B(u-v)B(u)\underline{B(t-s)}B(t-s+v)$;

46. $(23)(68)(17)(45) = B(u-v)B(u)\underline{B(t-s-u)B(t-s+u+v)}$;

47. $(23)(78)(15)(46) = B(u-v)B(v)\underline{B(t-s)}B(t-s+u)$;

48. $(23)(78)(16)(45) = B(u-v)B(v)\underline{B(t-s-v)B(t-s+u+v)}$;

The 5th dozen: $(24)\times\ldots$.

49. $(24)(56)(17)(38) = B(u)\underline{B(v)}B(t-s-u)B(t-s-v)$;

50. $(24)(56)(18)(37) = B(u)B(v)B(t-s-u-v)\underline{B(t-s)}$;

51. $(24)(57)(16)(38) = \underline{B(u)}B(u)B(t-s-v)B(t-s-v)$;

52. $(24)(57)(18)(36) = \underline{B(u)}B(u)B(t-s-u-v)B(t-s+u-v)$; (A7)

53. $(24)(58)(16)(37) = B(u)B(u+v)B(t-s-v)\underline{B(t-s)}$;

54. $(24)(58)(17)(36) = B(u)\underline{B(u+v)}B(t-s-u)B(t-s+u-v)$;

55. $(24)(67)(15)(38) = B(u)B(u-v)\underline{B(t-s)}B(t-s-v)$;

56. $(24)(67)(18)(35) = B(u)B(u-v)\underline{B(t-s-u-v)B(t-s+u)}$;

57. $(24)(68)(15)(37) = B(u)B(u)B(t-s)\underline{B(t-s)}$;

58. $(24)(68)(17)(35) = B(u)B(u)\underline{B(t-s-u)B(t-s+u)}$;

59. $(24)(78)(15)(36) = B(u)B(v)\underline{B(t-s)}B(t-s+u-v)$;

60. $(24)(78)(16)(35) = B(u)B(v)\underline{B(t-s-v)B(t-s+u)}$;

The 6th dozen: $(34)\times\ldots$.

61. $(34)(56)(17)(28) = \underline{B(v)}B(v)B(t-s-u)B(t-s-u)$;

62. $(34)(56)(18)(27) = \underline{B(v)}B(v)B(t-s-u-v)B(t-s+v-u)$;

63. $(34)(57)(16)(28) = \underline{B(v)}B(u)B(t-s-v)B(t-s-u)$;

64. $(34)(57)(18)(26) = B(v)B(u)\underline{B(t-s-u-v)B(t-s+u)}$;

65. $(34)(58)(16)(27) = \underline{B(v)}B(u+v)B(t-s-v)B(t-s+v-u)$;

66. $(34)(58)(17)(26) = B(v)B(u+v)\underline{B(t-s-u)B(t-s+u)}$; (A8)

67. $(34)(67)(15)(28) = B(v)B(u-v)\underline{B(t-s)}B(t-s-u)$;

68. $(34)(67)(18)(25) = B(v)B(u-v)\underline{B(t-s-u-v)B(t-s+v)}$;

69. $(34)(68)(15)(27) = B(v)B(u)\underline{B(t-s)}B(t-s+v-u)$;
70. $(34)(68)(17)(25) = B(v)B(u)\underline{B(t-s-u)B(t-s+v)}$;
71. $(34)(78)(15)(26) = B(v)B(v)B(t-s)\underline{B(t-s)}$;
72. $(34)(78)(16)(25) = B(v)B(v)\underline{B(t-s-v)B(t-s+v)}$.

The 3rd set consists of the terms which do not contain $t-s$ in the c.f. argument. There are 9 such terms and they can be obtained as terms of the product

$$((12)(34) + (13)(24) + (14)(23))((56)(78) + (57)(68) + (58)(67)),$$

that is

$$\begin{aligned}
&1.\ (12)(34)(56)(78) = \underline{B(v)}B(v)B(v)B(v);\\
&2.\ (12)(34)(57)(68) = \underline{B(v)}B(v)B(u)B(u);\\
&3.\ (12)(34)(58)(67) = \underline{B(v)}B(v)B(u+v)B(u-v);\\
&4.\ (13)(24)(56)(78) = B(u)B(u)\underline{B(v)}B(v);\\
&5.\ (13)(24)(57)(68) = \underline{B(u)}B(u)B(u)B(u);\\
&6.\ (13)(24)(58)(67) = B(u)B(u)\underline{B(u+v)}B(u-v);\\
&7.\ (14)(23)(56)(78) = B(u+v)B(u-v)\underline{B(v)}B(v);\\
&8.\ (14)(23)(57)(68) = \underline{B(u+v)}B(u-v)B(u)B(u);\\
&9.\ (14)(23)(58)(67) = \underline{B(u+v)}B(u-v)B(u+v)B(u-v).
\end{aligned} \tag{A9}$$

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