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Supremum Distribution of Weighted Sum of Random Processes from Orlicz Spaces of Exponential Type with Continuous Deviation

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Abstract

The paper studies distribution of sum of random processes from Orlicz spaces of exponential type weighted by continuous functions, in particular, processes from spaces $\mathrm{Sub}_\varphi(\Omega)$, $\mathrm{SSub}_\varphi(\Omega)$ and class $V(\varphi, \psi)$ are considered. Such spaces and classes of random variables and corresponding stochastic processes provide generalizations of Gaussian and sub-Gaussian random variables and processes and are important for various applications, for example, in queuing theory and financial mathematics.

We derive the estimates for the distribution of supremum of weighted sum of such processes deviated by a continuous monotone function using the entropy method. As examples, weighted sum of Wiener and weighted sum of fractional Brownian motion processes with different Hurst indices from classes $V(\varphi, \psi)$ are considered. Corresponding estimates of the probability of exceeding by trajectories of such weighted sums a positive level determined by a linear function are obtained. In the insurance risk theory, such a problem arises during estimating a ruin probability of the corresponding risk process with a constant premium income, and in the communications theory, it appears for the buffer overflow probability for a single server with a constant service rate.

Keywords: sub-Gaussian processes, supremum distribution, sum of weighted processes, generalized fractional Brownian motion, Wiener process, entropy method.

1. Introduction

This paper is devoted to the investigation of important classes of exponential type Orlicz spaces of random variables, namely, φ-sub-Gaussian random variables and more general classes $V(\varphi, \psi)$. Recall that a random variable ξ is sub-Gaussian if its moment generating function is majorized by that of a Gaussian centered random variable η, that is,

$$\mathbf{E}\exp\{\lambda\xi\} \le \mathbf{E}\exp\{\lambda\eta\} = \exp\{\sigma^2\lambda^2/2\},$$

where σ^2 is the variance of η. Sub-Gaussian random variables were introduced by Kahane (1960) and were further widely studied together with other general classes of random variables and processes from Orlich spaces. For a comprehensive review, refer to the classic monograph by Buldygin and Kozachenko (2000). Kozachenko and Ostrovskii (1985) presented Banach spaces of the sub-Gaussian type, namely spaces of φ-sub-Gaussian random variables and processes that naturally generalize spaces of sub-Gaussian random variables. $\mathrm{Sub}_{\varphi}(\Omega)$ spaces (spaces of φ-sub-Gaussian random variables) are spaces of centered random variables with certain exponential moments. For a more in-depth understanding of these spaces, refer to the book of Vasylyk, Kozachenko, and Yamnenko (2008). Additionally, Kozachenko and Vasylyk (2001) introduced more general classes $V(\varphi, \psi)$ of random processes.

Applying entropy methods for stochastic processes from these classes allows one to investigate the behavior of their extrema, to derive estimates for various functionals of such processes and random fields, to treat their sample paths properties, see, for example, Dozzi, Kozachenko, Mishura, and Ralchenko (2018); Yamnenko (2017); Hopkalo and Sakhno (2021); Sakhno (2022).

Here we generalize results obtained by Kozachenko and Yamnenko (2014) for weighted sum of independent processes from such classes. Our main interest is focused on studying the distribution of the following functionals

$$\sup_{t\in T}\left(\sum_{i=1}^{n} w_i(t)X_i(t) - f(t)\right), \quad \inf_{t\in T}\left(\sum_{i=1}^{n} w_i(t)X_i(t) - f(t)\right),$$

where $f(t)$ is a continuous function which, for example, can describe intensity of queue serving or premium income for an insurance company.

The results obtained are applied to weighted sum of fractional Brownian motion (FBM) processes. It is well-known that long-range dependence and self-similarity properties make the FBM process a popular model in various problems of queuing theory and financial mathematics.

The paper is organized as follows. Section 2 is devoted to the general theory of random variables and processes from Orlicz spaces of exponential type and based on the works Buldygin and Kozachenko (2000); Kozachenko and Vasylyk (2001); Vasylyk *et al.* (2008); Vasylyk and Yamnenko (2007); Yamnenko, Kozachenko, and Bushmitch (2014). Section 3 contains a generalization of results from the papers Kozachenko and Yamnenko (2014) and the last section contains applications to generalized Wiener and FBM processes from classes $V(\varphi, \psi)$.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a standard probability space and (T, ρ) a pseudometric space equipped with pseudometric ρ. Note that a pseudometric possesses all the properties of a metric except the following one: if $\rho(t,s) = 0$, then $t = s$. This means that the set $\{(t,s)\colon \rho(t,s) = 0\}$ can be wider than the diagonal $\{(t,s)\colon t = s\}$ (see Buldygin and Kozachenko (2000) for more details).

Recall that the metric entropy with regard to pseudometric ρ, or just metric entropy is a function

$$H(u) = H_T(u) = \begin{cases} \log N(u), & \text{if } N(u) < +\infty \\ +\infty, & \text{if } N(u) = +\infty \end{cases},$$

where $N(u) = N_T(u)$ denotes the least number of closed ρ-balls with radius u covering space (T, ρ).

A continuous even convex function φ is said to be an Orlicz N-function if it is strictly increasing for $x > 0$, $\varphi(0) = 0$ and

$$\frac{\varphi(x)}{x} \to 0 \text{ as } x \to 0 \quad \text{and} \quad \frac{\varphi(x)}{x} \to \infty \text{ as } x \to \infty.$$

Condition Q. We say that an N-function φ satisfies condition Q if $\liminf\limits_{x\to 0} \frac{\varphi(x)}{x^2} = \alpha > 0$.

Note that α can be infinite as well.

Definition 2.1. Let φ be an Orlicz N-function satisfying condition Q. The random variable ξ belongs to the space $\mathrm{Sub}_\varphi(\Omega)$ (a space of φ-sub-Gaussian random variables), if it is centered, i.e. $\mathbf{E}\xi = 0$, the moment generating function $\mathbf{E}\exp\{\lambda\xi\}$ exists for all $\lambda \in \mathbf{R}$ and there exists a constant $a > 0$ such that the following inequality

$$\mathbf{E}\exp(\lambda\xi) \le \exp(\varphi(a\lambda)) \tag{1}$$

holds for all $\lambda \in \mathbf{R}$.

Theorem 2.1. *The space $\mathrm{Sub}_\varphi(\Omega)$ is a Banach space with respect to the norm*

$$\tau_\varphi(\xi) = \inf\{a \ge 0 \colon \mathbf{E}\exp(\lambda\xi) \le \exp(\varphi(a\lambda)), \lambda \in \mathbf{R}\}$$

and the inequality

$$\mathbf{E}\exp(\lambda\xi) \le \exp(\varphi(\lambda\tau_\varphi(\xi))) \tag{2}$$

holds for all $\lambda \in \mathbf{R}$. Moreover, for all $r > 0$ there exists constant $c_r > 0$ such that

$$(\mathbf{E}\xi^r)^{1/r} \le c_r\tau_\varphi(\xi). \tag{3}$$

When $\varphi(x) = x^2/2$ the space $\mathrm{Sub}_\varphi(\Omega)$ is actually the space of sub-Gaussian random variables and is denoted by $\mathrm{Sub}(\Omega)$.

Theorem 2.2. *Let $\xi \in \mathrm{Sub}_\varphi(\Omega)$. Then for all $\varepsilon > 0$ the following inequality holds true*

$$\mathbf{P}\{|\xi| > \varepsilon\} \le 2\exp\left\{-\varphi\left(\frac{\varepsilon}{\tau_\varphi(\xi)}\right)\right\}.$$

A random process $X = \{X(t), t \in T\}$ is called a *φ-sub-Gaussian process* if $X(t) \in \mathrm{Sub}_\varphi(\Omega)$ for all $t \in T$.

Definition 2.2. A family of random variables Δ from the space $\mathrm{Sub}_\varphi(\Omega)$ is called *strictly* $\mathrm{Sub}_\varphi(\Omega)$, if there exists a constant $C_\Delta > 0$ such that for arbitrary finite set $I : \xi_i \in \Delta, i \in I$, and for any $\lambda_i \in \mathbf{R}$ the following inequality takes place

$$\tau_\varphi\Big(\sum_{i\in I}\lambda_i\xi_i\Big) \le C_\Delta\left(\mathbf{E}\Big(\sum_{i\in I}\lambda_i\xi_i\Big)^2\right)^{1/2}. \tag{4}$$

If Δ is a family of strictly $\mathrm{Sub}_\varphi(\Omega)$ random variables, then linear closure $\overline{\Delta}$ of the family Δ in the space $L_2(\Omega)$ is also strictly $\mathrm{Sub}_\varphi(\Omega)$ family of random variables. Linearly closed families of strictly $\mathrm{Sub}_\varphi(\Omega)$ random variables form a space of *strictly φ-sub-Gaussian* random variables. This space is denoted by $\mathrm{SSub}_\varphi(\Omega)$.

When $\varphi(x) = x^2/2$ the space $\mathrm{SSub}_\varphi(\Omega)$ is called the space of *strictly sub-Gaussian* random variables and is denoted as $\mathrm{SSub}(\Omega)$. The space of jointly Gaussian random variables belongs to the space $\mathrm{SSub}(\Omega)$ and $\tau^2(\xi) = \mathbf{E}\xi^2$, i.e. $C_\Delta = 1$.

A random process $X = \{X(t), t \in T\}$ is a *strictly φ-sub-Gaussian process* if the corresponding family of random variables belongs to the space $\mathrm{SSub}_\varphi(\Omega)$.

Example 2.1. *We call the process $B^H = (B^H(t), t \in T)$ strictly φ-sub-Gaussian generalized fractional Brownian motion (GFBM) with Hurst index $H \in (0,1)$, if B^H is a strictly φ-sub-Gaussian process with stationary increments and the following covariance function*

$$R_H(t,s) = \left(t^{2H} + s^{2H} - |s-t|^{2H}\right)/2. \tag{5}$$

Consider a sequence of independent strictly φ-sub-Gaussian random variables $\{\eta_n, n = 1, 2, \ldots\}$ for which $\mathbf{E}\eta_n = 0$, $\mathbf{E}\eta_n^2 = 1$, and φ is such an N-function, that function $\varphi(\sqrt{\cdot})$ is convex and

$\tau_\varphi(\eta_n) \leq \tau < +\infty$. *Then the process* $B^H(t) = \sum\limits_{n=1}^{\infty} \lambda_n \eta_n \psi_n(t)$ *is a centered strictly* φ*-sub-Gaussian random process with covariance function* R_H *from (5), if* λ_n *are eigenvalues and* ψ_n *are corresponding eigenfunctions of the following integral equation*

$$\psi(s) = \lambda^{-2} \int_0^T R_H(t,s)\psi(t)\,\mathrm{d}t.$$

N-function φ is subordinated by an Orlicz N-function ψ ($\varphi \prec \psi$) if there exist such numbers $x_0 > 0$ and $k > 0$ that $\varphi(x) < \psi(kx)$ for $x > x_0$.

Definition 2.3. Let $\varphi \prec \psi$ be two Orlicz N-functions. Random process $X = \{X(t), t \in T\}$ belongs to class $V(\varphi, \psi)$ if for all $t \in T$ the random variable $X(t)$ is from $\mathrm{Sub}_\psi(\Omega)$ and for all $s, t \in T$ increments $(X(t) - X(s))$ belong to the family $\mathrm{Sub}_\varphi(\Omega)$.

Example 2.2. *Sub-Gaussian random processes belong to the class* $V(\varphi, \varphi)$ *with* $\varphi(x) = x^2/2$.

Example 2.3. *Let*

$$X(t) = \xi_0 + \sum_{k=1}^{\infty} \xi_k f_k(t),$$

where φ *is such an Orlicz N-function that* $\varphi(\sqrt{\cdot})$ *is a convex function. Let* ξ_0 *be a* ψ*-sub-Gaussian random variable and* $\{\xi_k, k = 1, 2, \ldots\}$ *be a sequence of* φ*-sub-Gaussian random variables such that* $\sum\limits_{k=1}^{\infty} \tau_\varphi(\xi_k)|f_k(t)| < \infty$. *Then the process* $X(t)$ *belongs to the class* $V(\varphi, \psi)$.

Condition F. A continuous function $f = \{f(t), t \in T\}$ satisfies condition F if

$$|f(u) - f(v)| \leq \delta(\rho(u,v)),$$

where $\delta = \{\delta(s), s > 0\}$ is some monotonically increasing nonnegative function.

Let $B \subset T$ be a compact set.

Condition Σ**.** We say that independent separable random processes $X_i = \{X_i(t), t \in B\}$, $i = \overline{1,n}$, from classes $V(\varphi_i, \psi_i)$ satisfy condition Σ if norms $\gamma_i(u) = \tau_\psi(X_i(u)) < \infty$ and there exist such continuous monotone increasing functions $\{\sigma_i(h), h \geq 0\}$ that $\sigma_i(h) \to 0$ when $h \to 0$, $\sup\limits_{\rho(t,s) \leq h} \tau_{\varphi_i}(X_i(t) - X_i(s)) \leq \sigma_i(h)$, and $\sigma(h) = \sup\limits_{1 \leq i \leq n} \sigma_i(h) < \infty$.

For example, if $X_i(t)$ are continuous processes in corresponding norms $\tau_{\varphi_i}(\cdot)$ then $\sigma_i(h) = \sup\limits_{\rho(t,s) \leq h} \tau_{\varphi_i}(X(t) - X(s))$ satisfies condition Σ.

Condition R. Let $r = \{r(u), u \geq 1\}$ be a continuous function such that $r(u) > 0$ when $u > 1$ and function $s(t) = r(\exp\{t\}), t \geq 0$, is convex. We say that independent separable random processes $X_i = \{X_i(t), t \in B\}$ from classes $V(\varphi_i, \psi_i)$ satisfy condition R if they satisfy condition Σ and the following entropy integral is finite for all $p \in (0,1)$

$$I_\sigma(\beta, p) = \frac{1}{\beta p} \int\limits_0^{\beta p} r\left(\mathsf{N}_B(\sigma^{(-1)}(u))\right) \mathrm{d}u < \infty, \tag{6}$$

where $\beta > 0$ is such a number that $\beta \leq \sigma\left(\inf\limits_{s \in B} \sup\limits_{t \in B} \rho(t,s)\right)$.

The next result is a slightly reformulated version of Theorem 6 in Kozachenko and Yamnenko (2014).

Theorem 2.3. *Let independent separable random processes $X_i = \{X_i(t), t \in B\}$ from classes $V(\varphi_i, \psi_i)$ satisfy Condition R and a continuous function $f = \{f(t), t \in T\}$ satisfy Condition F. Then for all $p \in (0;1)$ and $x > 0$ the following inequalities hold true*

$$\mathbf{P}\left\{\sup_{t\in B}\left(\sum_{i=1}^n X_i(t) - f(t)\right) > x\right\} = \mathbf{P}\left\{\inf_{t\in B}\left(\sum_{i=1}^n X_i(t) - f(t)\right) < -x\right\} \leq Z_r(p,\beta,x),$$
$$\mathbf{P}\left\{\sup_{t\in B}\left|\sum_{i=1}^n X_i(t) - f(t)\right| > x\right\} \leq 2Z_r(p,\beta,x),$$

where $\theta_\psi(\lambda,p) = \sup\limits_{u\in B}\left((1-p)\sum\limits_{i=1}^n \psi_i\left(\frac{\lambda\gamma_i(u)}{1-p}\right) - \lambda f(u)\right)$ *and* $Z_r(p,\beta,x) = r^{(-1)}\left(I_\sigma(\beta,p)\right) \times$ $\times \inf\limits_{\lambda>0} \exp\left\{\theta_\varphi(\lambda,p) + p\sum\limits_{i=1}^n \varphi_i\left(\frac{\lambda\beta}{1-p}\right) + \lambda\left(\sum\limits_{k=2}^\infty \delta\left(\sigma^{(-1)}(\beta p^{k-1})\right) - x\right)\right\}.$

The following example can be easily obtained as a partial case of Theorem 10 in Yamnenko *et al.* (2014)

Example 2.4. *Let $X_i = (X_i(t), t \in [a,b])$, $i = \overline{1,n}$, be independent strictly φ_i-sub-Gaussian generalized FBM processes with Hurst parameter $H \in (0,1)$, $C > 0$ be some constant. Then for all numbers a, b $(0 \leq a < b < \infty)$, $p \in (0,1)$, $\beta \in \left(0, \left(\frac{b-a}{2}\right)^H\right]$ and $\lambda > 0$ the following inequality holds true*

$$\mathbf{P}\left\{\sup_{a\leq t\leq b}\left(C_\Delta^{-1}\sum_{i=1}^n X_i(t) - Ct\right) > \varepsilon\right\} \leq (b-a)\left(\frac{e}{\beta p}\right)^{1/H} \times$$
$$\times \exp\left\{-\lambda\varepsilon + \frac{\lambda C(\beta p)^{1/H}}{1-p^{1/H}} + p\sum_{i=1}^n \varphi_i\left(\frac{\lambda\beta}{1-p}\right) + (1-p)\theta_\varphi(\lambda,C,p)\right\},$$

where $\theta_\varphi(\lambda,C,p) = \sup\limits_{a\leq u\leq b}\left(\sum\limits_{i=1}^n \varphi_i\left(\frac{\lambda u^H}{1-p}\right) - \frac{\lambda C u}{1-p}\right)$ *and C_Δ is maximum defining constant of all the spaces* $\mathrm{SSub}_{\varphi_i}(\Omega)$.

It should be noted that GFBM processes were firstly introduced in Kozachenko, Vasylyk, and Sottinen (2002) as weakly self-similar stationary increment $SSub_\varphi(\Omega)$-processes.

3. Main results

Our next results are based on generalization of Theorem 2.3 from Kozachenko and Yamnenko (2014).

Condition $W\Sigma$. We say that independent separable random processes $X_i = \{X_i(t), t \in B\}$, $i = \overline{1,n}$, from classes $V(\varphi_i,\psi_i)$ weighted by continuous functions $\{w_i(t), t \in B\}$ satisfy condition $W\Sigma$ if norms $\gamma_i(u) = \tau_\psi(X_i(u)) < \infty$ and there exist such continuous monotone increasing functions $\{\sigma_i(h), h \geq 0\}$ that $\sigma_i(h) \to 0$ when $h \to 0$,

$$\sup_{\rho(t,s)\leq h} \tau_{\varphi_i}(w_i(t)X_i(t) - w_i(s)X_i(s)) \leq \sigma_i(h),$$

and $\sigma(h) = \sup\limits_{1\leq i\leq n} \sigma_i(h) < \infty$.

Condition $RW\Sigma 1$. Let $r = \{r(u), u \geq 1\}$ be a continuous function such that $r(u) > 0$ when $u > 1$ and function $s(t) = r(\exp\{t\}), t \geq 0$, is convex. We say that independent separable random processes $X_i = \{X_i(t), t \in B\}$ from classes $V(\varphi_i,\psi_i)$ satisfy condition $RW\Sigma$ if they satisfy condition $W\Sigma$ and the entropy integral (6) is finite.

Theorem 3.1. *Let independent separable random processes $X_i = \{X_i(t), t \in B\}$ from classes $V(\varphi_i, \psi_i)$ weighted by continuous functions $\{w_i(t), t \in B\}$ satisfy Condition $RW\Sigma 1$ and a continuous function $f = \{f(t), t \in B\}$ satisfy Condition F. Then for all $p \in (0;1)$ and $x > 0$ the following inequalities take places*

$$\mathbf{P}\left\{\sup_{t\in B}\left(\sum_{i=1}^{n} w_i(t)X_i(t) - f(t)\right) > x\right\} \le Z_r(p,\beta,x),$$
$$\mathbf{P}\left\{\inf_{t\in B}\left(\sum_{i=1}^{n} w_i(t)X_i(t) - f(t)\right) < -x\right\} \le Z_r(p,\beta,x),$$
$$\mathbf{P}\left\{\sup_{t\in B}\left|\sum_{i=1}^{n} w_i(t)X_i(t) - f(t)\right| > x\right\} \le 2Z_r(p,\beta,x),$$

where $\theta_\psi(\lambda,p) = \sup\limits_{u\in B}\left((1-p)\sum\limits_{i=1}^{n}\psi_i\left(\frac{\lambda w_i(u)\gamma_i(u)}{1-p}\right) - \lambda f(u)\right)$ *and* $Z_r(p,\beta,x) = r^{(-1)}\left(I_\sigma(\beta,p)\right) \times$
$\times \inf\limits_{\lambda>0}\exp\left\{\theta_\psi(\lambda,p) + p\sum\limits_{i=1}^{n}\varphi_i\left(\frac{\lambda\beta}{1-p}\right) + \lambda\left[\sum\limits_{k=2}^{\infty}\delta\left(\sigma^{(-1)}(\beta p^{k-1})\right) - x\right]\right\}.$

Proof. Let V_{ε_k} denote a set of the centers of closed balls with radii ε_k, which forms minimal covering of the space (B,ρ). Number of elements in the set V_{ε_k} is equal to $N_B(\varepsilon_k)$. It follows from theorem 2.2 and condition $W\Sigma$ that for any $\varepsilon > 0$

$$\mathbf{P}\{|w_i(t)X_i(t) - w_i(s)X_i(s)| > \varepsilon\}$$
$$\le 2\exp\left\{-\varphi_i\left(\frac{\varepsilon}{\tau_{\varphi_i}(w_i(t)X_i(t) - w_i(s)X_i(s))}\right)\right\} \le 2\exp\left\{-\varphi_i\left(\frac{\varepsilon}{\sigma_i(\rho(t,s))}\right)\right\}.$$

Therefore $w_i(t)X_i(t)$ is continuous in probability and the weighted sum of processes with a drift $X(t) = \sum\limits_{i=1}^{n} w_i(t)X_i(t) - f(t)$ is continuous in probability as well. Hence the set $V = \bigcup\limits_{k=1}^{\infty} V_{\varepsilon_k}$ is a set of separability of the process X and with probability one

$$\sup_{t\in B} X(t) = \sup_{t\in V} X(t). \tag{7}$$

Consider a mapping $\alpha_m = \{\alpha_m(t), m = 0,1\ldots\}$ of the set $V = \bigcup V_{\varepsilon_k}$ into the set V_{ε_m}, where $\alpha_m(t)$ is such a point from the set V_{ε_m} that $\rho(t,\alpha_m(t)) < \varepsilon_m$. If $t \in V_{\varepsilon_m}$ then $\alpha_m(t) = t$. If there exist several such points from the set V_{ε_m} that $\rho(t,\alpha_m(t)) < \varepsilon_m$ then we choose one of them and denote it $\alpha_m(t)$. The family of mappings $\{\alpha_m, m \ge 0\}$ is called the α-procedure for choosing points in V_k (see Buldygin and Kozachenko (2000), p. 94).

It follows from theorem 2.1 and condition ΣW that there exists such a constant $c_2 > 0$ that

$$\mathbf{P}\left\{|w_i(t)X_i(t) - w_i(\alpha_m(t))X_i(\alpha_m(t))| > p^{\frac{m}{2}}\right\}$$
$$\le \frac{\mathbf{E}[w_i(t)X_i(t) - w_i(\alpha_m(t))X_i(\alpha_m(t))]^2}{p^m} \le \frac{c_2^2\tau_\varphi^2(w_i(t)X_i(t) - w_i(\alpha_m(t))X_i(\alpha_m(t)))}{p^m}$$
$$\le \frac{c_2^2\sigma_i^2(\varepsilon_m)}{p^m} \le \frac{c_2^2\sigma^2(\varepsilon_m)}{p^m} = c_2^2\beta^2 p^m.$$

This inequality implies that $\sum\limits_{m=1}^{\infty}\mathbf{P}\left\{|w_i(t)X_i(t) - w_i(\alpha_m(t))X_i(\alpha_m(t))| > p^{\frac{m}{2}}\right\} < \infty$. Therefore it follows from the Borel-Kantelli lemma that $w_i(t)X_i(t) - w_i(\alpha_m(t))X_i(\alpha_m(t)) \to 0$ as $m \to \infty$ with probability one. Since f is a continuous function then $X(t) - X(\alpha_m(t)) \to 0$ as $m \to \infty$ with probability one as well. Since the set V is countable, then $X(t) - X(\alpha_m(t)) \to 0$ as $m \to \infty$ for all t simultaneously.

Let t be an arbitrary point from the set V. Denote by $t_m = \alpha_m(t)$, $t_{m-1} = \alpha_{m-1}(t_m), \ldots,$ $t_1 = \alpha_1(t_2)$ for any $m \geq 1$. Since for all $m \geq 2$

$$X(t) = X(t_1) + \sum_{k=2}^{m}(X(t_k) - X(t_{k-1})) + X(t) - X(\alpha_m(t))$$

we have

$$\sup_{t\in V} X(t) \leq \max_{u\in V_{\varepsilon_1}} X(u) + \sum_{k=2}^{m} \max_{u\in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u)) + X(t) - X(\alpha_m(t)). \tag{8}$$

It follows from (7) and (8) that with probability one

$$\sup_{t\in T} X(t) \leq \liminf_{m\to\infty} \left(\max_{u\in V_{\varepsilon_1}} X(u) + \sum_{k=2}^{m} \max_{u\in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u))) \right). \tag{9}$$

Let $\{q_k, k = 1, 2, \ldots\}$ be such a sequence that $q_k > 1$ and $\sum\limits_{k=1}^{\infty} q_k^{-1} \leq 1$. It follows from the Hölder's inequality, the Fatou's lemma and (9) that for all $\lambda > 0$

$$\begin{aligned} &\mathbf{E}\exp\left\{\lambda \sup_{t\in T} X(t)\right\} \\ &\leq \left(\mathbf{E}\exp\left\{q_1\lambda \max_{u\in V_{\varepsilon_1}} X(u)\right\}\right)^{\frac{1}{q_1}} \prod_{k=2}^{\infty}\left(\mathbf{E}\exp\left\{q_k\lambda \max_{u\in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u)))\right\}\right)^{\frac{1}{q_k}}. \end{aligned} \tag{10}$$

Some details are omitted here since the same derivation can be found also in Kozachenko and Yamnenko (2014).

Consider each of the factors in the right-hand side of inequality (10) separately. It follows from (2) that $\mathbf{E}\exp\{q_1\lambda w_i(u)X_i(u)\} \leq \exp\{\psi_i(q_1\lambda w_i(u)\gamma_i(u))\}$ and

$$\mathbf{E}\exp\{q_k\lambda[w_i(u)X_i(u) - w_i(\alpha_{k-1}(u))X_i(\alpha_{k-1}(u))]\} \leq \exp\{\varphi_i(q_k\lambda\sigma_i(\varepsilon_{k-1}))\}.$$

Therefore,

$$\begin{aligned} &\left(\mathbf{E}\exp\left\{q_1\lambda \max_{u\in V_{\varepsilon_1}} X(u)\right\}\right)^{1/q_1} \\ &\leq \left(\sum_{u\in V_{\varepsilon_1}} \mathbf{E}\exp\left\{q_1\lambda \sum_{i=1}^{n} w_i(u)X_i(u)\right\}\exp\left\{-q_1\lambda f(u)\right\}\right)^{1/q_1} \\ &\leq \left(\sum_{u\in V_{\varepsilon_1}} \prod_{i=1}^{n} \mathbf{E}\exp\left\{q_1\lambda w_i(u)X_i(u)\right\}\exp\left\{-q_1\lambda f(u)\right\}\right)^{1/q_1} \\ &\leq \left(N_B(\varepsilon_1)\right)^{1/q_1} \exp\left\{\frac{1}{q_1}\sup_{u\in B}\left(\sum_{i=1}^{n}\psi_i(q_1\lambda w_i(u)\gamma_i(u)) - q_1\lambda f(u)\right)\right\}. \end{aligned} \tag{11}$$

Since $|f(u) - f(v)| \leq \delta(\rho(u,v))$ due to Condition F, we obtain that

$$\begin{aligned} &\left(\mathbf{E}\exp\left\{q_k\lambda \max_{u\in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u)))\right\}\right)^{1/q_k} \\ &\leq \left(N_B(\varepsilon_k) \max_{u\in V_{\varepsilon_k}} \mathbf{E}\exp\left\{q_k\lambda \sum_{i=1}^{n}[w_i(u)X_i(u) - w_i(\alpha_{k-1}(u))X_i(\alpha_{k-1}(u))]\right\}\right. \\ &\left.\times\exp\left\{-q_k\lambda(f(u) - f(\alpha_{k-1}(u)))\right\}\right)^{1/q_k} \end{aligned}$$

$$\leq \left(N_B(\varepsilon_k)\right)^{1/q_k} \left(\max_{u \in V_{\varepsilon_k}} \exp\left\{ \sum_{i=1}^{n} \varphi_i(q_k \lambda \sigma(\varepsilon_{k-1})) + q_k \lambda \delta(\rho(u, \alpha_{k-1}(u))) \right\} \right)^{1/q_k}$$
$$\leq \left(N_B(\varepsilon_k)\right)^{1/q_k} \exp\left\{ q_k^{-1} \sum_{i=1}^{n} \varphi_i(q_k \lambda \beta p^{k-1}) + \lambda \delta\left(\sigma^{(-1)}(\beta p^{k-1})\right) \right\}. \quad (12)$$

From (10) after substitution of $q_k = p^{1-k}/(1-p)$, $k \geq 1$, we have

$$\mathbf{E} \exp\left\{ \lambda \sup_{t \in B} X(t) \right\}$$
$$\leq \exp\left\{ \sum_{k=2}^{\infty} (1-p)p^{k-1} \sum_{i=1}^{n} \varphi_i\left(\frac{\lambda \beta}{1-p}\right) + \lambda \sum_{k=2}^{\infty} \delta\left(\sigma^{(-1)}(\beta p^{k-1})\right) \right\}$$
$$\times \exp\left\{ \theta_\psi(\lambda, p) + \sum_{k=1}^{\infty} (1-p)p^{k-1} H_B\left(\sigma^{(-1)}(\beta p^k)\right) \right\}. \quad (13)$$

And finally in the same way as in theorem 3.5 in Vasylyk, Kozachenko, and Yamnenko (2005) we obtain

$$\exp\left\{ \sum_{k=1}^{\infty} (1-p)p^{k-1} H_B\left(\sigma^{(-1)}(\beta p^k)\right) \right\} \leq r^{(-1)}\left(I_\sigma(\beta, p)\right). \quad (14)$$

So, from the Chebyshev's inequality and (10)–(14) we obtain the assertion of the theorem. □

Condition $RW\Sigma 2$. Let $r = \{r(u), u \geq 1\}$ be a continuous function such that $r(u) > 0$ when $u > 1$ and function $s(t) = r(\exp\{t\}), t \geq 0$, is convex. We say that independent separable random processes $X_i = \{X_i(t), t \in B\}$ from classes $V(\varphi_i, \psi_i)$ satisfy condition $RW\Sigma$ if they satisfy condition $W\Sigma$ and the following entropy integral is finite for all $p \in (0,1)$

$$J_\sigma(\beta, p) = \frac{1}{p(1-p)} \int_0^{\beta p^2} \frac{r(N(\sigma^{(-1)}(u)))}{\phi^{(-1)}(\ln N(\sigma^{(-1)}(u)))} \, du$$

where $\phi(u) = \sup_{1 \leq i \leq n} \varphi_i(u)$.

Theorem 3.2. *Let independent separable random processes $X_i = \{X_i(t), t \in B\}$ from classes $V(\varphi_i, \psi_i)$ weighted by continuous functions $\{w_i(t), t \in B\}$ satisfy Condition $RW\Sigma 2$ and a continuous function $f = \{f(t), t \in B\}$ satisfy Condition F. Then for all $p \in (0;1)$ and $x > 0$ the following inequalities take places*

$$\mathbf{P}\left\{ \sup_{t \in B} \left(\sum_{i=1}^{n} w_i(t) X_i(t) - f(t) \right) > x \right\} \leq Z_r(p, \beta, x),$$
$$\mathbf{P}\left\{ \inf_{t \in B} \left(\sum_{i=1}^{n} w_i(t) X_i(t) - f(t) \right) < -x \right\} \leq Z_r(p, \beta, x),$$
$$\mathbf{P}\left\{ \sup_{t \in B} \left| \sum_{i=1}^{n} w_i(t) X_i(t) - f(t) \right| > x \right\} \leq 2Z_r(p, \beta, x),$$

where

$$Z_r(p, \beta, x) = \inf_{\lambda > 0} \left(r^{(-1)}\left(\lambda J_\sigma(\beta, p)\right) \right)^{n+1}$$
$$\times \exp\left\{ W(\lambda, p, \beta) + np\varphi\left(\frac{\lambda \beta}{1-p}\right) + \lambda \left(\sum_{k=2}^{\infty} \delta\left(\sigma^{(-1)}\left(\beta p^{k-1}\right)\right) - x \right) \right\},$$
$$W(\lambda, p, \beta) = \inf_{v \geq (1-p)^{-1}} \left(\frac{1}{v} H(\sigma^{(-1)}(\beta p)) + \sup_{u \in B} \left(\frac{\sum_{i=1}^{n} \psi_i(\lambda w_i(u) \gamma_i(u) v)}{v} - \lambda f(u) \right) \right).$$

Proof. The proof of theorem 3.1 is repeated until the selection of the sequence q_k in (13). After substituting (11) and (12) in (13) we obtain

$$\begin{aligned}
&\mathbf{E}\exp\left\{\lambda\sup_{t\in B}X(t)\right\}\\
&\le \exp\left\{\sum_{k=2}^{\infty}\frac{1}{q_k}\left[H(\varepsilon_k)+\sum_{i=1}^{n}\varphi_i\left(q_k\lambda\beta p^{k-1}\right)\right]\right\}\exp\left\{\lambda\sum_{k=2}^{\infty}\delta\left(\sigma^{(-1)}(\beta p^{k-1})\right)\right\}\\
&\times\exp\left\{\frac{1}{q_1}\left[H(\varepsilon_1)+\sup_{u\in B}\left(\sum_{i=1}^{n}\psi_i\left(q_1\lambda w_i(u)\gamma_i(u)\right)-q_1\lambda f(u)\right)\right]\right\}. \qquad (15)
\end{aligned}$$

Let $q_1 = v$, where v is such a number that $v\ge\frac{1}{1-p}$ and

$$q_k=\frac{1}{\lambda\beta p^{k-1}}\phi^{(-1)}\left(\phi\left(\frac{\lambda\beta}{1-p}\right)+H(\varepsilon_k)\right),\ k=2,3\ldots \qquad (16)$$

where $\phi(u)=\sup\limits_{1\le i\le n}\varphi_i(u)$. Since

$$\frac{1}{q_k}\le\frac{\lambda\beta p^{k-1}}{\phi^{(-1)}\left(\phi\left(\frac{\lambda\beta}{1-p}\right)\right)}=p^{k-1}(1-p)$$

as $k=2,3\ldots$, then $\sum\limits_{k=1}^{\infty}\frac{1}{q_k}\le\sum\limits_{k=1}^{\infty}p^{k-1}(1-p)=1$.

For the sequence q_k defined in (16) consider

$$\begin{aligned}
\tilde{Z} &= \sum_{k=2}^{\infty}\frac{1}{q_k}\left[H(\varepsilon_k)+\sum_{i=1}^{n}\varphi_i\left(q_k\lambda\beta p^{k-1}\right)\right]\\
&= \sum_{k=2}^{\infty}\frac{H(\varepsilon_k)}{q_k}+\sum_{k=2}^{\infty}\frac{1}{q_k}\sum_{i=1}^{n}\varphi_i\left(\lambda\beta p^{k-1}\frac{\phi^{(-1)}\left(\phi\left(\frac{\lambda\beta}{1-p}\right)+H(\varepsilon_k)\right)}{\lambda\beta p^{k-1}}\right)\\
&\le \sum_{k=2}^{\infty}\frac{H(\varepsilon_k)}{q_k}+\sum_{k=2}^{\infty}\frac{nH(\varepsilon_k)}{q_k}+n\phi\left(\frac{\lambda\beta}{1-p}\right)\sum_{k=2}^{\infty}\frac{1}{q_k}\le(n+1)\sum_{k=2}^{\infty}\frac{H(\varepsilon_k)}{q_k}+n\phi\left(\frac{\lambda\beta}{1-p}\right)p.
\end{aligned}$$

Consider

$$\begin{aligned}
&\exp\left\{(n+1)\sum_{k=2}^{\infty}\frac{1}{q_k}H\left(\sigma^{(-1)}(\beta p^k)\right)\right\}\\
&=\left(r^{(-1)}\left(r\left(\exp\left\{\sum_{k=2}^{\infty}q_k^{-1}\ln N\left(\sigma^{(-1)}(\beta p^k)\right)\right\}\right)\right)\right)^{n+1}\\
&\le\left(r^{(-1)}\left(\sum_{k=2}^{\infty}q_k^{-1}s\left(\ln N\left(\sigma^{(-1)}(\beta p^k)\right)\right)\right)\right)^{n+1}\\
&\le\left(r^{(-1)}\left(\lambda\sum_{k=2}^{\infty}\beta p^{k-1}\frac{r(N(\sigma^{(-1)}(\beta p^k)))}{\phi^{(-1)}(\ln N(\sigma^{(-1)}(\beta p^k)))}\right)\right)^{n+1}. \qquad (17)
\end{aligned}$$

Using convexity of the function $s(t)=r(\exp\{t\})$ from Condition $RW\Sigma2$ it can be shown (see, for example, Kozachenko and Vasylyk (2001)) that

$$\sum_{k=2}^{\infty}\beta p^{k-1}\frac{r(N(\sigma^{(-1)}(\beta p^k)))}{\phi^{(-1)}(\ln N(\sigma^{(-1)}(\beta p^k)))}\le\frac{1}{p(1-p)}\int_0^{\beta p^2}\frac{r(N(\sigma^{(-1)}(u)))}{\phi^{(-1)}(\ln N(\sigma^{(-1)}(u)))}\,\mathrm{d}u. \qquad (18)$$

Therefore the assertion of the theorem follows from (15) – (18) and the Chebyshev's inequality. □

4. Applications to Wiener and FBM processes

As an example let's consider a weighted sum of generalized Wiener processes and then fractional Brownian motions from classes $V(\varphi_i, \psi_i)$. Recall that $B^H = (B^H(t), t \in T)$ is a generalized fractional Brownian motion process from the class $V(\varphi, \psi)$ if B^H is strictly ψ-sub-Gaussian process with stationary strictly φ-sub-Gaussian increments and the covariance function as in (5), where $\varphi \prec \psi$ are two subordinated Orlicz N-functions.

Theorem 4.1. *Let $B_i = (B_i(t), t \in [a, b])$, $i = \overline{1, n}$, $0 < a < b < \infty$ be independent generalized Wiener processes from the class $V(\varphi_i, \psi_i)$ and let $c > 0$ be a constant. Let $w_i = \{w_i(t), t \in [a, b]\}$ be continuous weighting functions such that $|w_i(t)t^{1/2} - w_i(s)s^{1/2}| \leq \nu_i(t - s)$ where $\nu_i(u)$ is some continuous monotone increasing function, $\nu_i(0) = 0$, $\nu_i(h) \leq \eta_i h^{1/2}$, $\eta_i > 0$ be some constants. Then for all $x > 0$ the following inequality holds true*

$$\mathbf{P}\left\{\sup_{a\leq t\leq b}\left(\sum_{i=1}^n w_i(t)B_i(t) - ct\right) > x\right\} = \mathbf{P}\left\{\inf_{a\leq t\leq b}\left(\sum_{i=1}^n w_i(t)B_i(t) - ct\right) < -x\right\}$$

$$\leq \inf_{p\in(0,1);\ \lambda>0;\ 0<\beta\leq D\left(\frac{b-a}{2}\right)^{1/2}} (b-a)\left(\frac{eD}{\beta p}\right)^2 \times$$

$$\times \exp\left\{\frac{\lambda c(\beta p)^2}{D^2(1-p^2)} + p\sum_{i=1}^n \varphi_i\left(\frac{\lambda\beta}{1-p}\right) + (1-p)\theta_\psi(\lambda, p) - \lambda x\right\},$$

where

$$\theta_\psi(\lambda, p) = \sup_{a\leq u\leq b}\left(\sum_{i=1}^n \psi_i\left(\frac{\lambda w_i(u)u^H}{1-p}\right) - \frac{\lambda cu}{1-p}\right),$$

C_Δ is the maximal constant from definition 2.2 of the space $SSub_{\varphi_i}(\Omega)$, $W_i = \sup_{t\in[a,b]} w_i(t)$ and $D = C_\Delta \max\limits_{1\leq i\leq n}\left(\eta_i + \frac{b^{1/2}}{a^{1/2}}W_i^2\right)$.

Proof. Recall that Wiener process is also known as Brownian motion (or FBM with Hurst index $H = 0.5$) and $\mathbf{E}B_i(t)B_i(s) = \min\{t, s\}$. Using Euclidean metrics $\rho(t, s) = |t - s|$ consider for $s < t$

$$\tau_{\varphi_i}(w_i(t)B_i(t) - w_i(s)B_i(s)) \leq C_\Delta\left(\mathbf{E}(w_i(t)B_i(t) - w_i(s)B_i(s))^2\right)^{\frac{1}{2}}$$

$$= C_\Delta\left(\mathbf{E}(w_i^2(t)Z_i^2(t) - 2w_i(s)w_i(t)Z_i(t)Z_i(s) + w_i^2(s)Z_i^2(s))\right)^{\frac{1}{2}}$$

$$= C_\Delta\left(w_i^2(t)t - 2sw_i(s)w_i(t) + w_i^2(s)s\right)^{\frac{1}{2}}$$

$$= C_\Delta\left((w_i(t)t^{1/2} - s^{1/2}w_i(s))^2 + 2s^{1/2}w_i(s)w_i(t)(t^{1/2} - s^{1/2})\right)^{\frac{1}{2}}$$

$$\leq C_\Delta\left(\nu_i^2(t-s) + 2b^{1/2}W_i^2(t-s)/(t^{1/2} + s^{1/2})\right)^{\frac{1}{2}}$$

$$\leq C_\Delta\left(\nu_i^2(t-s) + \frac{b^{1/2}}{a^{1/2}}W_i^2(t-s)\right)^{\frac{1}{2}} \leq C_\Delta\left(\eta_i + \frac{b^{1/2}}{a^{1/2}}W_i^2\right)(t-s)^{\frac{1}{2}}.$$

Let's apply theorem 3.1. Put $\gamma_i(u) = C_\Delta u^{1/2}$ and $\sigma_i(h) = C_\Delta\left(\eta_i + \frac{b^{1/2}}{a^{1/2}}W_i^2\right)(t-s)^{1/2}$, thus $\sigma(h) = Dh^{1/2}$, where $D = C_\Delta \max\limits_{1\leq i\leq n}\left(\eta_i + \frac{b^{1/2}}{a^{1/2}}W_i^2\right)$, then $0 < \beta \leq \sigma\left(\frac{b-a}{2}\right) = D\left(\frac{b-a}{2}\right)^{1/2}$. Also we have that Condition F becomes valid for $\delta(h) = ch$ since $|f(u) - f(v)| = |cu - cv| = c|u - v|$. Then

$$\theta_\psi(\lambda, p) = \sup_{a\leq u\leq b}\left(\sum_{i=1}^n \psi_i\left(\frac{\lambda C_\Delta w_i(u)u^H}{1-p}\right) - \frac{\lambda cu}{1-p}\right), \tag{19}$$

$$\sum_{k=2}^\infty \delta\left(\sigma^{(-1)}(\beta p^{k-1})\right) = \sum_{k=2}^\infty c\left(\frac{\beta p^{k-1}}{D}\right)^2 = \frac{c(\beta p)^2}{D^2(1-p^2)}. \tag{20}$$

Let's choose $r(u) = u^{\alpha}, u \geq 1, 0 < \alpha < 1/2$, as the respective function in Condition $RW\Sigma$. Since the following bounds take place for metric entropy $\ln\left(\max\left\{\frac{b-a}{2u},1\right\}\right) \leq H(u) \leq \ln\left(\frac{b-a}{2u}+1\right)$, then for $u \leq \beta$ the following estimate is fulfilled

$$r\left(N\left(\sigma^{(-1)}(u)\right)\right) \leq r\left(\frac{b-a}{2\sigma^{(-1)}(u)}+1\right) = \left(\frac{b-a}{2u^2/D^2}+1\right)^{\alpha} \leq \frac{(b-a)^{\alpha}}{(u/D)^{2\alpha}}.$$

Since $\beta p < \beta \leq D\left(\frac{b-a}{2}\right)^{1/2}$ then

$$r^{(-1)}\left(I_{\sigma}(\beta,p)\right) \leq \left(\frac{1}{\beta p}\int\limits_0^{\beta p}\frac{(b-a)^{\alpha}}{(u/D)^{2\alpha}}\,\mathrm{d}u\right)^{\frac{1}{\alpha}} = \frac{(b-a)D^2}{\beta^2p^2}\Big(1-2\alpha\Big)^{-\frac{1}{\alpha}}. \tag{21}$$

Infimum of the right side in (21) is attained as $\alpha \to 0$ and $\lim\limits_{\alpha\to 0}\Big(1-2\alpha\Big)^{-\frac{1}{\alpha}} = e^2$. So, from (19)-(21) we obtain the assertion of the theorem. □

Theorem 4.2. *Let $B_i = (B_i(t), t \in [a,b])$, $i = \overline{1,n}$, $0 < a < b < \infty$ be independent generalized fractional Brownian motion processes from classes $V(\varphi_i,\psi_i)$ with Hurst indices $0 < H_1 \leq \ldots \leq H_n < 1$ and let $c > 0$ be a constant. Let $w_i = \{w_i(t), t \in [a,b]\}$ be continuous weighting functions such that $\left|w_i(t)t^{2H_i} - w_i(s)s^{2H_i}\right| |w_i(t) - w_i(s)| \leq \nu_i(t-s)$ where $\nu_i(u)$ is some continuous monotone increasing function, $\nu_i(0) = 0$, $\nu_i(h) \leq \eta_i h^{2H_i}$, and $\eta_i > 0$ be some constants. Then for all $x > 0$ the following inequality holds true*

$$\begin{aligned}
&\mathbf{P}\left\{\sup_{a\leq t\leq b}\left(\sum_{i=1}^n w_i(t)B_i(t) - ct\right) > x\right\} = \mathbf{P}\left\{\inf_{a\leq t\leq b}\left(\sum_{i=1}^n w_i(t)B_i(t) - ct\right) < -x\right\}\\
&\leq \inf_{p\in(0,1);\ \lambda>0;\ 0<\beta\leq\min\left\{D\left(\frac{b-a}{2}\right)^{1/H_1},1\right\}}(b-a)\left(\frac{eD}{\beta p}\right)^{1/H_1}\times\\
&\times\exp\left\{\frac{\lambda c(\beta p)^{1/H_1}}{D^{1/H_1}(1-p^{1/H_1})} + p\sum_{i=1}^n\varphi_i\left(\frac{\lambda\beta}{1-p}\right) + (1-p)\theta_{\psi}(\lambda,p) - \lambda x\right\},
\end{aligned}$$

where

$$\theta_{\psi}(\lambda,p) = \sup_{a\leq u\leq b}\left(\sum_{i=1}^n\psi_i\left(\frac{\lambda C_{\Delta}w_i(u)u^{H_i}}{1-p}\right) - \frac{\lambda cu}{1-p}\right),$$

C_{Δ} is the maximal constant from definition 2.2 of the space $SSub_{\varphi_i}(\Omega)$, $W_i = \sup_{t\in[a,b]} w_i(t)$ and $D = C_{\Delta}\max\limits_{1\leq i\leq n}\left(\eta_i + W_i^2\right)^{1/2}$.

Proof. Let's again apply theorem 3.1 using Euclidean metrics $\rho(t,s) = |t-s|$. Taking into account (5) and (4) put $\gamma_i(u) = C_{\Delta}u^{H_i}$ and consider the norm of weighted increments of a process $B_i(t)$

$$\begin{aligned}
\tau_{\varphi_i}(w_i(t)B_i(t) - w_i(s)B_i(s)) &\leq C_{\Delta}\left(\mathbf{E}(w_i(t)B_i(t) - w_i(s)B_i(s))^2\right)^{\frac{1}{2}}\\
&= C_{\Delta}\left(w_i^2(t)\mathbf{E}B_i^2(t) - 2w_i(s)w_i(t)\mathbf{E}B_i(t)B_i(s) + w_i^2(s)\mathbf{E}B_i^2(s)\right)^{\frac{1}{2}}\\
&\leq C_{\Delta}\left(\left|w_i(t)t^{2H_i} - w_i(s)s^{2H_i}\right| |w_i(t) - w_i(s)| + w_i(t)w_i(s)|s-t|^{2H_i}\right)^{\frac{1}{2}}\\
&\leq C_{\Delta}\left(\nu_i(|t-s|) + W_i^2|t-s|^{2H_i}\right)^{\frac{1}{2}} \leq C_{\Delta}\left(\eta_i + W_i^2\right)^{\frac{1}{2}}|t-s|^{H_i}.
\end{aligned}$$

Put $\sigma_i(h) = C_\Delta \left(\eta_i + W_i^2\right)^{\frac{1}{2}} h^{H_i}$ and $\sigma(h) = \sup_{1\le i\le n} \sigma_i(h) = Dh^{H_\star(h)}$, where $H_\star(h) = H_1$ if $h \le 1$ and $H_\star(h) = H_N$ if $h \ge 1$. Then $0 < \beta \le D\left(\frac{b-a}{2}\right)^{H_\star}$. Also we have that $|f(u)-f(v)| = |cu-cv| = c|u-v|$, i.e. $\delta(h) = ch$. As function $r(u)$ let's choose $r(u) = u^\alpha, u \ge 1, 0 < \alpha < H_1$. If $\beta p \le 1$ then

$$\theta_\psi(\lambda, p) = \sup_{a\le u\le b} \left(\sum_{i=1}^{n} \psi_i \left(\frac{C_\Delta \lambda w_i(u) u^{H_i}}{1-p} \right) - \frac{\lambda C u}{1-p} \right), \tag{22}$$

$$\sum_{k=2}^{\infty} \delta\left(\sigma^{(-1)}(\beta p^{k-1})\right) = \sum_{k=2}^{\infty} c\left(\beta p^{k-1}/D\right)^{1/H_1} = \frac{c(\beta p)^{1/H_1}}{1-p^{1/H_1}}. \tag{23}$$

Since $\ln\left(\max\left\{\frac{b-a}{2u}, 1\right\}\right) \le \mathsf{H}(u) \le \ln\left(\frac{b-a}{2u}+1\right)$, then for $u \le \min\{\beta, 1\}$ the following estimate is fulfilled

$$r\left(\mathsf{N}\left(\sigma^{(-1)}(u)\right)\right) \le r\left(\frac{b-a}{2\sigma^{(-1)}(u)}+1\right) = \left(\frac{b-a}{2(u/D)^{1/H_1}}+1\right)^\alpha \le \frac{(b-a)^\alpha}{(u/D)^{\frac{\alpha}{H_1}}}.$$

Put $\beta \le \min\left\{\left(D\frac{b-a}{2}\right)^{H_1}, 1\right\}$ then $\beta p \le 1$ and

$$r^{(-1)}\left(I_\sigma(\beta, p)\right) \le \left(\frac{1}{\beta p}\int_0^{\beta p} \frac{(b-a)^\alpha}{(u/D)^{\alpha/H_1}}\,\mathrm{d}u\right)^{\frac{1}{\alpha}} = \frac{(b-a)D^{1/H_1}}{(\beta p)^{1/H_1}}\left(1-\frac{\alpha}{H_1}\right)^{-\frac{1}{\alpha}}. \tag{24}$$

Infimum of the right side of estimate (24) equals to

$$\lim_{\alpha\to 0} \frac{(b-a)D^{1/H_1}}{(\beta p)^{1/H_1}}\left(1-\frac{\alpha}{H_1}\right)^{-\frac{1}{\alpha}} = (b-a)\left(\frac{eD}{\beta p}\right)^{\frac{1}{H_1}}. \tag{25}$$

Therefore from (22)-(25) we obtain the assertion of the theorem. □

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