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Polynomial Recurrence of Time-inhomogeneous Markov Chains

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Abstract

This paper is devoted to establishing conditions that guarantee the existence of a p-th moment of the time it takes for a time-inhomogeneous Markov chain to hit some set C. We modified the well-known Drift Condition from the theory of homogeneous Markov chains. We demonstrated that the inhomogeneous Markov chain may be polynomially recurrent while exhibiting different dynamics from its homogeneous counterpart.

Keywords: polynomial recurrence, renewal theory, inhomogeneous Markov chain, drift condition.

1. Introduction

In this paper, we investigate the recurrence properties of time-inhomogeneous Markov chains. The main object of interest is a polynomial moment of the return time σ_C to some specific set C. We aim at finding conditions that ensure $\mathbb{E}_x[\sigma_C^p] < \infty$ for some p.

One principal application of recurrence properties of time-inhomogeneous Markov chains is developing a coupling for two time-inhomogeneous Markov chains. The coupling method was introduced by W. Doeblin almost a century ago, see Doeblin (1938). The classical coupling method for homogeneous chains is described in the books of Lindvall (1991) and Thorisson (2000). The paper Golomoziy (2015) contains results that generalize Lindvall's Theorem about the finiteness of the first moment of the coupling time with some practical applications.

The coupling method for inhomogeneous chains requires some modifications in its construction. See Golomoziy and Kartashov (2013a), Golomoziy and Kartashov (2013b) and Golomoziy and Kartashov (2016) for examples of building so-called maximal coupling on a discrete state space with applications to stability of transition probabilities.

Coupling constructions for time-inhomogeneous Markov chains with values in general phase space are used in Douc, Moulines, and Rosenthal (2004b) with applications to convergence rates and Golomoziy (2014) with applications to stability. An application of these results to actuarial mathematics can be found in Golomoziy, Kartashov, and Kartashov (2016).

Typical results involving the coupling method rely on properties of consecutive visits to some so-called "small" set C (see Meyn and Tweedie (1993) for a detailed discussion of small sets and homogeneous theory overall). The shorter the renewal intervals are, the better coupling properties are. The theory of homogeneous Markov chains has many results that describe renewal intervals, and of course, the Key Renewal Theorem plays the central role in this theory (see Meyn and Tweedie (1993), Douc, Moulines, Priouret, and Soulier (2018)).

In the present paper, we focused on a polynomial recurrence, a partial case of a more general - subgeometric recurrence. It was studied in the papers Douc, Fort, Moulines, and Soulier (2004a), Andrieu, Fort, and Vihola (2015) and Fort and Roberts (2005) and further developed in the book Douc *et al.* (2018).

This paper is organized as follows. Section 2 contains the main definitions and notation. In Section 3, we present the main result, which states the existence of a polynomial moment of the return time. Section 4 contains an example which demonstrates how the conditions of the main Theorem could be verified. Finally, Section 5 includes technical auxiliary results that are used in the proof of the main Theorem.

2. Definitions and notation

Assume $\{X_n, n \geq 0\}$ is a time-inhomogeneous Markov chain defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the values in a general state space $(E, \mathcal{E})$. Denote its transition probability by

$$P_n(x, A) = \mathbb{P}\{X_{n+1} \in A | X_n = x\}.$$

The sequence of transition probabilities generates a canonical probability space $(E^\infty, \mathcal{E}^\infty, \mathbb{P}_x^t)$ for each $t \in \mathbb{N}$ and $x \in \mathbb{N}$, where

$$\mathbb{P}_x^t\{A_1, \ldots, A_m\} = \mathbb{P}\left\{X_{t+1} \in A_1, \ldots, X_{t+m} \in A_m | X_t = x\right\},$$

and $A_1, \ldots, A_m \in \mathcal{E}$.

Let us fix some set $C \in \mathcal{E}$, and denote the first return time by

$$\sigma_C = \inf\{n \geq 1 : X_n \in C\}. \quad (1)$$

If used in context of $\mathbb{P}_x^t$ we will understand σ_C as

$$\sigma_C = \inf\{n \geq 1 : X_{t+n} \in C\}.$$

Such a notation allows us to significantly simplify derivations by omitting unnecessary indices at a cost of some abuse of formality.

Our goal is to establish the finiteness of the p-th moment of the return time, i.e.

$$\sup_{x \in C} \mathbb{E}_x^t \left[\sigma_C^p\right] < \infty,$$

and

$$\mathbb{E}_x^t \left[\sigma_C^p\right] < \infty,$$

for some $p \geq 1$. Similar results for geometric recurrence can be found in Golomoziy (2022). We would like to emphasize that the bounds we derive are not uniform in x. Uniform in x estimates are related to the uniform ergodicity in time-homogeneous case, and studied in details in Douc *et al.* (2018) and Meyn and Tweedie (1993) in the homogeneous case. The papers Golomoziy (2020) and Golomoziy and Mishura (2020) are devoted to the stability of time-inhomogeneous case when a uniform minorization condition holds true, which is equivalent to the uniform ergodicity in the homogeneous theory.

As we mentioned in the Introduction, polynomial recurrence plays a critical role in developing coupling for a pair of different time-inhomogeneous chains. An interested reader may find more details on how recurrence is related to the properties of coupling in the papers Golomoziy and Kartashov (2012), Golomoziy (2019) and Golomoziy (2017).

3. Main result

To achieve the goal outlined above we will modify the Drift Condition or the Foster-Lyapunov criterion which are well-known in the homogeneous theory. Generally speaking, we will follow an approach described in Douc *et al.* (2018), Chapter 16, but adopted to the time-inhomogeneous case. This approach requires us to introduce a special Drift Condition that guarantees the finiteness of a p-th moment of σ_C (see the formal definition in (1)). For doing so we define $\alpha \in (0,1)$, such that $p = \frac{1}{1-\alpha}$, for example, $p = 2$ requires $\alpha = 1/2$ and $\alpha = 2/3$ corresponds to $p = 3$.

Condition D. Assume there exist a measurable function $V : E \to [1,\infty]$, a constant $c > 0$ and a sequence of positive real numbers $\{\alpha_n, n \geq 0\} \subset (0,\alpha]$, such that

$$P_n V \leq V - cV^{\alpha_n} + b\mathbb{1}_C,$$

and

$$\sup_{x\in C} V(x) < \infty.$$

We assume, that "most" of α_n are equal to α, but "sometimes", $\alpha_n < \alpha$.

Next, we define the sequence $\{n_k, k \geq 0\}$ in the following way

$$n_0 = \min\{n \geq 0 : \alpha_n = \alpha\},$$

$$n_{k+1} = \min\{n > n_k : \alpha_n = \alpha\}.$$

In other words, $\{n_k, k \geq 0\}$ is a sequence of indices, such that $\alpha_{n_k} = \alpha$. By convention we will assume that $\min\{\emptyset\} = \infty$. We should treat the sequence $\{n_k, k \geq 0\}$ in the same context as σ_C with regard to the starting index t. So that, when considering a chain that starts at $X_t = x$ we will assume

$$n_0 = \min\{n \geq 0 : \alpha_{t+n} = \alpha\},$$

$$n_{k+1} = \min\{n > n_k : \alpha_{t+n} = \alpha\}.$$

Theorem 1. *Let $\{X_n, n \geq 0\}$ be a Markov chain described above. Assume that **Condition D** holds true along with the following conditions. 1. Denote the lower limit of the sequence $\{\alpha_n, n \geq 1\}$ by β and assume*

$$\beta := \underline{\lim}\alpha_n > \max\left\{2 - \frac{1}{\alpha}, 0\right\}.$$

2. Indices j, such that $\alpha_j < \alpha$ are so sparse that

$$\sum_{k=1}^{\infty} k^{-\rho} m_k < \infty,$$

where $m_k = n_{k+1} - n_k - 1$,

$$\rho = \left(\frac{1-\varepsilon}{1-\beta} - \frac{\alpha}{1-\alpha}\right)\left(1 + \frac{1-\varepsilon}{1-\beta}\right)^{-1} \in (0,1),$$

and ε is an arbitrary small constant such that

$$\frac{1-\varepsilon}{1-\beta} > \frac{\alpha}{1-\alpha}.$$

Then the following inequality holds true for all $x \in E$ and $t \geq 0$

$$c^{\frac{\alpha}{1-\alpha}} \mathbb{E}_x^t \left[((1-\alpha)\sigma_C)^{\frac{1}{1-\alpha}}\right] \leq \left[V(x) + b(2-\alpha)^{\frac{\alpha}{1-\alpha}}\mathbb{1}_C(x) + bB(x)\sum_{k=1}^{\infty}\left(\frac{1-\alpha}{k} + 1\right)^{\frac{\alpha}{1-\alpha}} k^{\frac{\alpha}{1-\alpha} - \frac{\gamma}{1-\beta}} m_k\right] < \infty,$$

where $m_k = n_{k+1} - n_k - 1$, *and*

$$B(x) = (1-\beta)^{\frac{-1}{1-\beta}} \left(V(x) + \sup_{y\in C} V(y) + b(2-\beta)^{\frac{\beta}{1-\beta}} \right). \tag{2}$$

Remark 1. *Condition 2 is a condition on "sparsity" of indices i such that $\alpha_i < \alpha$. A typical example of such a sparsity is a situation in which $\alpha_i < \alpha$ only holds when i are equal to n^s for an appropriate $s > 1$. The latter clearly implies that all $n_k < \infty$ and $n_k \to \infty$, $k \to \infty$.*

Proof. For a Markov chain $\tilde{X}_k = X_{n_k}$ defined in Section 5, Theorem 2 provides an estimate

$$((1-\alpha)c)^{\frac{\alpha}{1-\alpha}} \mathbb{E}^t_x \left[\sum_{k=0}^{\tilde{\sigma}_C - 1} k^{\frac{\alpha}{1-\alpha}} \right] \le V(x) + br(1)\mathbb{1}_C(x) + b\mathbb{E}^t_x \left[\sum_{k=1}^{\tilde{\sigma}_C - 1} r(k+1) m_k \right]. \tag{3}$$

In order to estimate the last term in (3) we use Lemma 4. Conditions 1 and 2 of the present theorem applied to Lemma 4 allows us to obtain the inequality

$$\mathbb{E}^t_x \left[\sum_{k=1}^{\tilde{\sigma}_C - 1} r(k+1) m_k \right] \le B(x) \sum_{k=1}^{\infty} \left(1 + \frac{1-\alpha}{k} \right)^{\frac{\alpha}{1-\alpha}} k^{-\rho} m_k.$$

Finally we employ the elementary inequality (16) to conclude the proof. □

Remark 2. *We refer the reader to the paper Golomoziy (2023) for a similar result that guarantees geometric recurrence for time-inhomogeneous autoregression. We would like to emphasize the similarities in behavior of time-inhomogeneous Markov chains which are polynomial or geometrically recurrent. In both cases inhomogeneous chains may drift to the set C with the varying speed. However, as long as deviations from the required (geometric or exponential) speed are rare enough, a chain remains recurrent, even though such deviation may occur infinitely often.*

4. Random walk on a half-line

In this section we demonstrate one immediate application of Theorem 1 to a time-inhomogeneous random walk on a half line, which extends a known homogeneous result. To this end, we will adopt derivations from Douc *et al.* (2018), Example 16.1.13.

Let us consider a family of independent random variables $\{W_n, n \ge 0\}$ with values in $\mathbb{R}$. Denote by Γ_n the distribution of W_n. Assume that

$$\sup_n \mathbb{E}\left[W_n\right] < 0, \tag{4}$$

and

$$\sup_n \mathbb{E}\left[\left(W_n^+\right)^m\right] < \infty,$$

for some $m > 1$.

We are interested in properties of the process

$$X_{n+1} = (X_n + W_{n+1})^+, n \ge 0, X_0 = x \ge 0.$$

Clearly, $\{X_n, n \ge 0\}$ is an inhomogeneous Markov chain. We select $\alpha = \frac{m-1}{m}$. We will verify **Condition D** with some set C of the form $C = [0, z]$. Thus, we demonstrate how the properties of increments W_n are connected with recurrent properties of the chain $\{X_n, n \ge 0\}$. In this case Conditions 1 and 2 of Theorem 1 are automatically satisfied, since all alphas are equal to $\frac{m-1}{m}$.

In order to verify **Condition D** we use (4) to find a $\chi > 0$ satisfying

$$\sup_n \mathbb{E}\left[W_n \mathbb{1}_{W_n > -\chi}\right] < 0.$$

We use the test function

$$V(x) = (1+x)^m, x \geq 0.$$

For $x \geq \chi$ we can write

$$P_n V(x) \leq V(x-\chi)\Gamma_n(-\chi) + \int_{-x_0}^{\infty} V(x+y)\Gamma_n(dy). \tag{5}$$

For the function V we can write the following inequalities (see Douc *et al.* (2018), Example 16.1.13)

$$V(x) - V(x-\chi) \geq \frac{\chi m}{(\chi+1)^{m-1}}(x+1)^{m-1},$$

$$V(x+y) \leq V(x) + m(x+1)^{m-1} + m(x+1)^{m-1}y + \frac{1}{2}m(m-1)(x+1)^{m-2}(y+1)^m, y \geq 0$$

and

$$V(x+y) \leq V(x) + m(x+1)^{m-1} + m(x+1)^{m-1}y + \frac{1}{2}m(m-1)(x+1)^{m-2}\chi^2, y \in [-\chi, 0).$$

Plugging these into (5) we obtain

$$P_n V(x) \leq V(x)\Gamma_n(-\chi) - \frac{\chi m}{(\chi+1)^{m-1}}(x+1)^{m-1}\Gamma_n(-\chi) + V(x)(1-\Gamma_n(-\chi))$$

$$+ m(x+1)^{m-1}\int_{-\chi}^{\infty}(1+y)\Gamma_n(dy) + \frac{1}{2}m(m-1)(x+1)^{m-2}\left(\Gamma_n((-\chi,0)) + \int_0^{\infty}(y+1)^m\Gamma_n(dy)\right)$$

$$= V(x) - c_1(x+1)^{m-1} + c_2(x+1)^{m-2} = V(x) - (x+1)^{m-1}\left(c_1 - \frac{c_2}{x+1}\right),$$

where

$$c_1 = c_1(\chi, m, n) = \frac{\chi m}{(\chi+1)^{m-1}}\Gamma_n(-\chi) + m\int_{-\chi}^{\infty}(1+y)\Gamma_n(dy),$$

$$c_2 = c_2(\chi, m, n) = \frac{1}{2}m(m-1)\left(\Gamma_n((-\chi,0)) + \int_0^{\infty}(y+1)^m\Gamma_n(dy)\right).$$

Clearly, we can select $z = \max\{\chi, \frac{c_2}{c_1} - 1\}$ so that $c := c_1 - \frac{c_2}{x+1} > 0$ when $x > z$, and thus

$$P_n V(x) \leq V(x) - cV^\alpha(x) + b\mathbb{1}_{[0,z]}(x),$$

for some constant $b > 0$.

5. Auxiliary results

In this section we provide some auxilliary results which play a crucial role in establishing a polynomial recurrence.

Let us now define a random process $\tilde{X}_k = X_{n_k}$, $k \geq 0$. Note that $\tilde{X}_k$ is an inhomogeneous Markov chain with the transition probability

$$\tilde{P}_k(x, A) = \mathbb{P}\left\{X_{n_{k+1}} \in A | X_{n_k} = x\right\} = \left(\prod_{j=n_k}^{n_{k+1}-1} P_j\right)(x, A).$$

Let us define the sequence of functions $\{V_n, n \geq 0\}$, $V_n \colon E \to [1, \infty)$ such that

$$V_n = \left((1-\alpha)cn + V^{1-\alpha}\right)^{\frac{1}{1-\alpha}} - ((1-\alpha)cn)^{\frac{1}{1-\alpha}}, \tag{6}$$

where V is defined in **Condition D**.

We start with the following observation.

Lemma 1. *Let $\{V_n, n \geq 0\}$ be the sequence of functions defined by (6). Then the following inequality holds true*

$$V_{n+1} - c\left((1-\alpha)(n+1)c + V^{1-\alpha}\right)^{\frac{\alpha}{1-\alpha}} \leq V_n - c((1-\alpha)cn)^{\frac{\alpha}{1-\alpha}}.$$

Proof. First we note that

$$V_n = \left((1-\alpha)cn + V^{1-\alpha}\right)^{\frac{1}{1-\alpha}} - ((1-\alpha)cn)^{\frac{1}{1-\alpha}} = \frac{1}{1-\alpha}\int_0^{V^{1-\alpha}} ((1-\alpha)cn + z)^{\frac{\alpha}{1-\alpha}}\, dz.$$

Having this equality in mind we can write

$$\begin{aligned}
V_{n+1} - V_n &= \frac{1}{1-\alpha}\int_0^{V^{1-\alpha}} \left(((1-\alpha)c(n+1) + z)^{\frac{\alpha}{1-\alpha}} - ((1-\alpha)cn + z)^{\frac{\alpha}{1-\alpha}}\right) dz \\
&= \frac{1}{1-\alpha}\int_0^{V^{1-\alpha}}\int_0^1 \frac{d}{ds}((1-\alpha)(n+s)c + z)^{\frac{\alpha}{1-\alpha}}\, ds dz \\
&= \frac{\alpha c}{1-\alpha}\int_0^{V^{1-\alpha}}\int_0^1 ((1-\alpha)(n+s)c + z)^{\frac{\alpha}{1-\alpha}-1}\, ds dz \\
&= c\int_0^1 \left[\left((1-\alpha)(n+s)c + V^{1-\alpha}\right)^{\frac{\alpha}{1-\alpha}} - ((1-\alpha)(n+s)c)^{\frac{\alpha}{1-\alpha}}\right] ds \\
&\leq c\left((1-\alpha)(n+1)c + V^{1-\alpha}\right)^{\frac{\alpha}{1-\alpha}} - c((1-\alpha)cn)^{\frac{\alpha}{1-\alpha}}.
\end{aligned}$$

□

The next lemma helps us to calculate the conditional expectation $\mathbb{E}\left[V_{n+1}(X_n)|\mathcal{F}_n\right]$.

Lemma 2. *Let the sequence V_n be defined in (6), $\alpha_n = \alpha$ and* ***Condition D*** *hold true. Then the following inequality holds true*

$$P_n V_{n+1}(X_n) \leq V_n(X_n) - c((1-\alpha)cn)^{\frac{\alpha}{1-\alpha}} + br(n+1)\mathbb{1}_C(X_n),$$

where

$$r(n) = \left[(1-\alpha)cn + 1\right]^{\frac{\alpha}{1-\alpha}}. \tag{7}$$

Proof. First, we note that for every $n \geq 0$ the function

$$g_n(x) = ((1-\alpha)cn + x^{1-\alpha})^{\frac{1}{1-\alpha}} - ((1-\alpha)cn)^{\frac{1}{1-\alpha}}, x \geq 0$$

is concave, since for $x \geq 0$

$$g_n''(x) = \alpha x^{-\alpha-1}\left((1-\alpha)cn + x^{1-\alpha}\right)^{\frac{\alpha}{1-\alpha}}\left(\frac{1}{(1-\alpha)cnx^{\alpha-1} + 1} - 1\right) < 0.$$

Thus, we can use Jensen's inequality to write

$$P_n V_{n+1}(X_n) = \mathbb{E}\left[V_{n+1}(X_{n+1})|\mathcal{F}_n\right] = \mathbb{E}\left[g_{n+1}(V(X_{n+1}))|\mathcal{F}_n\right] \leq g_{n+1}\left(\mathbb{E}\left[V(X_{n+1})|\mathcal{F}_n\right]\right)$$

$$= g_{n+1}\left(P_n V_{n+1}(X_n)\right) = \left[(1-\alpha)(n+1)c + (P_n V(X_n))^{1-\alpha}\right]^{\frac{1}{1-\alpha}} - ((1-\alpha)(n+1)c)^{\frac{1}{1-\alpha}}.$$

Next we use the fact that g_{n+1} is increasing along with **Condition D** to derive

$$P_n V_{n+1}(X_n) \leq g_{n+1}\left(P_n V(X_n)\right) \leq g_{n+1}\left(V(X_n) - cV^{\alpha}(X_n) + b\mathbb{1}_C(X_n)\right)$$

$$= \left[(1-\alpha)(n+1)c + \left(V(X_n) - cV^{\alpha}(X_n) + b\mathbb{1}_C(X_n)\right)^{1-\alpha}\right]^{\frac{1}{1-\alpha}} - ((1-\alpha)(n+1)c)^{\frac{1}{1-\alpha}},$$

Next, we make the following observation. For any differentiable, concave function $g(x), x \geq 0$ the following inequality holds true

$$g(x+y) \leq g(x) + yg'(x), \tag{8}$$

as long as $x \geq 0$ and $x + y \geq 0$. We put $x = V(X_n)$ and $y = -cV^{\alpha}(X_n) + b\mathbb{1}_C(X_n)$ and apply inequality (8) to the function g_{n+1}. So we have

$$P_n V_{n+1}(X_n) \leq \left[(1-\alpha)(n+1)c + V^{1-\alpha}(X_n)\right]^{\frac{1}{1-\alpha}} - ((1-\alpha)(n+1)c)^{\frac{1}{1-\alpha}}$$

$$+ (-cV^{\alpha}(X_n) + b\mathbb{1}_C(X_n)) \left[(1-\alpha)(n+1)c + V^{1-\alpha}(X_n)\right]^{\frac{\alpha}{1-\alpha}} V^{-\alpha}(X_n)$$

$$= V_{n+1}(X_n) - c\left[(1-\alpha)(n+1)c + V^{1-\alpha}(X_n)\right]^{\frac{\alpha}{1-\alpha}} + b\mathbb{1}_C(X_n)\left[\frac{(1-\alpha)(n+1)c}{V^{1-\alpha}(X_n)} + 1\right]^{\frac{\alpha}{1-\alpha}}.$$

Next we use Lemma 1 and the fact $V \geq 1$ to get the inequality

$$P_n V_{n+1}(X_n) \leq V_n - c((1-\alpha)n)^{\frac{\alpha}{1-\alpha}} + b\mathbb{1}_C(X_n)\left[(1-\alpha)(n+1)c + 1\right]^{\frac{\alpha}{1-\alpha}}.$$

□

Lemma 2 demonstrates the presence of a "drift of the size" $O\left(n^{\frac{\alpha}{1-\alpha}}\right)$ towards the set C, at the transition from time n to $n+1$ whenever $\alpha_n = \alpha$. As we see later, this fact is crucial for establishing the finiteness of $\mathbb{E}_x\left[\sigma_C^{\frac{1}{1-\alpha}}\right]$. However, in the case $\alpha_n < \alpha$, one may only expect a drift of smaller size, namely $O(n^{\frac{\alpha_n}{1-\alpha_n}})$, which is insufficient for establishing the finiteness of a desired moment. To solve this problem, we will "skip" the timesteps n such that $\alpha_n < \alpha$, and require that such timesteps are rare enough (see Condition 2 of Theorem 1). The following Lemma demonstrates how to execute such a "skip".

Lemma 3. *Let the sequence V_n be defined in (6), $m > 0$ be some integer, $\alpha_n = \alpha$, $\alpha_{n+i} < \alpha$ for $i = \overline{1,m}$ and $\alpha_{n+m+1} = \alpha$. Assume also that **Condition D** holds true. Then the following inequality holds true for any $k \geq 1$*

$$\left(\prod_{j=0}^{m-1} P_{n+j}\right) V_k(X_n) \leq V_{k-1}(X_n) - c((1-\alpha)(k-1)c)^{\frac{\alpha}{1-\alpha}} + br(k)\left(\sum_{i=0}^{m}\left(\prod_{j=0}^{i-1} P_{n+j}\right)(X_n, C)\right),$$

where $\left(\prod_{j=0}^{-1} P_{n+j}\right)(x, C) = \mathbb{1}_C(x)$ by convention and $r(n)$ is defined in (7).

Proof. To simplify the notation, let us denote the n-th step transition probability by

$$P_{n,i} = \prod_{j=0}^{i-1} P_{n+j}.$$

We start with P_{n+m-1} and, following the line of the proof of Lemma 2, we apply the Jensen inequality and **Condition D** to get the following inequality

$$P_{n+m-1} V_k(X_n)$$

$$\leq \left[(1-\alpha)ck + (V(X_n) - cV^{\alpha_{n+m-1}}(X_n) + b\mathbb{1}_C(X_n))^{1-\alpha}\right]^{\frac{1}{1-\alpha}} - ((1-\alpha)ck)^{\frac{1}{1-\alpha}}.$$

Next, we apply elementary inequality (8) to obtain

$$P_{n+m-1} V_k(X_n) \leq \left[(1-\alpha)ck + V^{1-\alpha}(X_n)\right]^{\frac{1}{1-\alpha}} - ((1-\alpha)ck)^{\frac{1}{1-\alpha}}$$

$$+\left(-cV^{\alpha_{n+m-1}}(X_n)+b\mathbb{1}_C(X_n)\right)\left[(1-\alpha)ck+V^{1-\alpha}(X_n)\right]^{\frac{\alpha}{1-\alpha}}V^{-\alpha}(X_n)$$
$$=V_k(X_n)-c\left[(1-\alpha)ck+V^{1-\alpha}(X_n)\right]^{\frac{\alpha}{1-\alpha}}V^{\alpha_{n+m-1}-\alpha}(X_n)$$
$$+b\mathbb{1}_C(X_n)\left[\frac{(1-\alpha)ck}{V^{1-\alpha}(X_n)}+1\right]^{\frac{\alpha}{1-\alpha}}\leq V_k(X_n)+br(k)\mathbb{1}_C(X_n).$$

Now we put

$$A_i=P_{n,i-1}V_k(X_n).$$

We have then the following inequality

$$A_m\leq P_{n,m-1}\left(V_k(X_n)+br(k)\mathbb{1}_C(X_n)\right)=A_{m-1}+br(k)P_{n,m-2}(X_n,C).$$

Following the same derivations as before we obtain

$$A_m\leq A_1+br(k)\left(\sum_{j=1}^{m-1}P_{n,j}(X_n,C)\right).$$

But since $\alpha_n=\alpha$ we can apply Lemma 2 to A_1 and obtain

$$A_1=P_nV_k(X_n)\leq V_{k-1}-c\left((1-\alpha)(k-1)c\right)^{\frac{\alpha}{1-\alpha}}+br(k)\mathbb{1}_C(X_n).$$

□

In the next theorem we establish the finiteness of the $\frac{\alpha}{1-\alpha}$-th moment of $\tilde{\sigma}_C$.

Theorem 2. *Assume* ***Condition D*** *holds true, and $n_k<\infty$ for all $k\geq 0$. Then for all $x\in E$ and $t\geq 0$*

$$((1-\alpha)c)^{\frac{\alpha}{1-\alpha}}\mathbb{E}_x^t\left[\sum_{k=0}^{\tilde{\sigma}_C-1}k^{\frac{\alpha}{1-\alpha}}\right]\leq V(x)+br(1)\mathbb{1}_C(x)+b\mathbb{E}_x^t\left[\sum_{k=1}^{\tilde{\sigma}_C-1}r(k+1)m_k\right],\qquad(9)$$

where $m_k=n_{k+1}-n_k-1$.

Proof. Without loss of generality, we will assume that $t=0$ and omit the upper index t for brevity.

$$\mathcal{V}_k=\left((1-\alpha)ck+V^{1-\alpha}(X_{n_k})\right)^{\frac{1}{1-\alpha}}-\left((1-\alpha)ck\right)^{\frac{1}{1-\alpha}}.$$

Clearly $\{\mathcal{V}_k,k\geq 0\}$ is a Markov chain.

In order to establish the desired result we will use the Comparison Theorem (see Douc *et al.* (2018), Chapter 4). To verify its condition we consider two cases.

1. Assume $n_{k+1}=n_k+1$, so that $\alpha_{n_k}=\alpha_{n_k+1}=\alpha$. In this case

$$\mathcal{V}_k=\left((1-\alpha)ck+V^{1-\alpha}(X_{n_k})\right)^{\frac{1}{1-\alpha}}-\left((1-\alpha)ck\right)^{\frac{1}{1-\alpha}},$$

and

$$\mathcal{V}_{k+1}=\left((1-\alpha)(k+1)c+V^{1-\alpha}(X_{n_k+1})\right)^{\frac{1}{1-\alpha}}-\left((1-\alpha)(k+1)c\right)^{\frac{1}{1-\alpha}}.$$

We can apply Lemma 2 and arrive at the inequality

$$\mathbb{E}\left[\mathcal{V}_{k+1}|\tilde{\mathfrak{F}}_k\right]\leq\mathcal{V}_k-\left((1-\alpha)ck\right)^{\frac{\alpha}{1-\alpha}}+br(k+1)\mathbb{1}_C(X_{n_k}).\qquad(10)$$

2. Assume $n_{k+1}=n_k+m+1$, so that $\alpha_{n_k}=\alpha$, $\alpha_{n_k+i}<\alpha$, $i=\overline{1,m}$, $\alpha_{n_{k+1}}=\alpha_{n_k+m+1}=\alpha$. In this case

$$\mathcal{V}_k=\left((1-\alpha)ck+V^{1-\alpha}(X_{n_k})\right)^{\frac{1}{1-\alpha}}-\left((1-\alpha)ck\right)^{\frac{1}{1-\alpha}},$$

and
$$\mathcal{V}_{k+1} = \left((1-\alpha)(k+1)c + V^{1-\alpha}(X_{n_k+m+1})\right)^{\frac{1}{1-\alpha}} - ((1-\alpha)(k+1)c)^{\frac{1}{1-\alpha}}.$$
We can apply Lemma 3 and obtain
$$\mathbb{E}\left[\mathcal{V}_{k+1}|\tilde{\mathcal{F}}_k\right] \le \mathcal{V}_k - ((1-\alpha)ck)^{\frac{\alpha}{1-\alpha}} + br(k+1)\left(\sum_{i=0}^{m}\left(\prod_{j=0}^{i-1} P_{n_k+j}\right)(X_{n_k}, C)\right). \tag{11}$$

Combining inequalities (10) and (11) we come to the following inequality
$$\mathbb{E}\left[\mathcal{V}_{k+1}|\tilde{\mathcal{F}}_k\right] + \mathcal{Z}_k \le \mathcal{V}_k + \mathcal{Y}_k, \tag{12}$$
where
$$\mathcal{Z}_k = ((1-\alpha)ck)^{\frac{\alpha}{1-\alpha}},$$
$$\mathcal{Y}_k = br(k+1)\left(\sum_{i=0}^{n_{k+1}-n_k-1}\left(\prod_{j=0}^{i-1} P_{n_k+j}\right)(X_{n_k}, C)\right).$$
Since all $\mathcal{V}_k$, $\mathcal{Z}_k$ and $\mathcal{Y}_k$ are non-negative, (12) allows us to apply the Comparison Theorem, which renders
$$((1-\alpha)c)^{\frac{\alpha}{1-\alpha}}\mathbb{E}_x\left[\sum_{k=0}^{\tilde{\sigma}_C-1} k^{\frac{\alpha}{1-\alpha}}\right] \le \mathcal{V}_0 + \mathbb{E}_x\left[\sum_{k=0}^{\tilde{\sigma}_C-1}\mathcal{Y}_k\right]$$
$$\le V(x) + br(1)\mathbb{1}_C(x) + b\mathbb{E}_x\left[\sum_{k=1}^{\tilde{\sigma}_C-1} r(k+1)(n_{k+1}-n_k-1)\right].$$

□

Next we want to establish the finiteness of the last term in (9).

Lemma 4. *Assume the conditions of Theorem 2 hold. Assume additionaly that*
1. The lower limit β of the sequence $\{\alpha_n, n \ge 0\}$ is positive and satisfies $(1-\beta)^{-1} > \frac{\alpha}{1-\alpha}$, or in other words
$$\beta := \underline{\lim}\alpha_n > \max\left\{2 - \frac{1}{\alpha}, 0\right\},$$
2. Indices j, such that $\alpha_j < \alpha$ are so sparse that
$$\sum_{k=1}^{\infty} k^{-\rho} m_k < \infty,$$
where $m_k = n_{k+1} - n_k - 1$,
$$\rho = \left(\frac{1-\varepsilon}{1-\beta} - \frac{\alpha}{1-\alpha}\right)\left(1 + \frac{1-\varepsilon}{1-\beta}\right)^{-1} \in (0,1),$$
and ε is an arbitrary small constant such that
$$\frac{1-\varepsilon}{1-\beta} > \frac{\alpha}{1-\alpha}.$$
Then the following inequality holds true for all $x \in E$ and $t \ge 0$
$$\mathbb{E}_x^t\left[\sum_{k=1}^{\tilde{\sigma}_C-1} r(k+1)m_k\right] \le B(x)\sum_{k=1}^{\infty}\left(1 + \frac{1-\alpha}{k}\right)^{\frac{\alpha}{1-\alpha}} k^{-\rho} m_k,$$
where $B(x)$ is defined in (2) and V, b are from ***Condition D****.*

Remark 3. *Note, that ρ was selected in such a way that*

$$\frac{(1-\varepsilon)(1-\rho)}{1-\beta} - \frac{\alpha}{1-\alpha} = \rho. \tag{13}$$

We will use this equation later in the proof.

Proof. In this proof we will follow the notations from Theorem 2 and omit the upper index t. We start with the representation

$$\mathbb{E}_x\left[\sum_{k=1}^{\tilde{\sigma}_C-1} r(k+1)m_k\right] = \sum_{k=1}^{\infty} r(k+1)m_k \mathbb{P}_x\left\{\tilde{\sigma}_C > k\right\}. \tag{14}$$

Note, that

$$P_nV + V^{\beta} \le P_nV + V^{\alpha_n} \le V + b\mathbb{1}_C.$$

Thus, we may conclude that **Condition D** holds true when all α_n are replaced with β. In such case all $m_k = m_k(\beta) = 0$ and Theorem 2 renders

$$\mathbb{E}_x\left[\sigma_C^{\frac{1}{1-\beta}}\right] \le (1-\beta)^{-\frac{1}{1-\beta}}\left(V(x) + b(2-\beta)^{\frac{\beta}{1-\beta}}\mathbb{1}_C(x)\right). \tag{15}$$

Here we used an elementary inequality

$$x^{\frac{1}{1-\beta}} = \frac{1}{1-\beta}\int_0^x y^{\frac{\beta}{1-\beta}}dy \le \frac{1}{1-\beta}\sum_{k=0}^{x} k^{\frac{\beta}{1-\beta}} = \frac{1}{1-\beta}\sum_{k=0}^{x-1}(k+1)^{\frac{\beta}{1-\beta}}. \tag{16}$$

In order to prove the finiteness of (14) we would like to bound $\mathbb{P}_x\{\tilde{\sigma}_C > k\}$ by $\mathbb{P}_x\{\sigma_C > f(k)\}$ for some function f, and then apply the Markov inequality and (15) to write a bound of the form

$$\mathbb{P}_x\{\sigma_C > f(k)\} \le \mathbb{E}_x\left[\sigma_C^{\frac{1}{1-\beta}}\right](f(k))^{-\frac{1}{1-\beta}}.$$

The idea behind finding an appropriate function f comes from Condition 2. Since $\sum_{k=0}^{\infty} k^{-\rho}m_k$ is finite and $\rho < 1$ we may conclude that a set $\{k : m_k > 0\}$ is very sparse. One may expect (informally, of course) that $m_k > 0$ only for indices of the form $k = j^s$, where $s > 1$ is some fixed constant. This, in turn, implies that intervals of indices i such that $m_i = 0$ should be long enough so that we will be looking for a function f of the form $f(N) = N^{\gamma}$ for some constant γ to be defined. To execute this plan, we will introduce a special notation for the maximal number of indices in a row, smaller than N, such that $m_i = 0$.

Let us denote by $L_k = \max\{n_j - n_i + 1, k \ge j > i, \alpha_{n_j} = \ldots = \alpha_{n_i} = \alpha\}$ - the length of a maximal time subinterval of $[0, k]$ in which all $\alpha_j = \alpha$. In order to illustrate the notation, let us consider the following example.

n:	1	2	3	4	5	6	7	8	9	10	11
is $\alpha_n = \alpha$	+	+	+	-	-	+	+	+	+	-	+
n_k	1	2	3	x	x	**4**	**5**	**6**	**7**	x	8
m_k	0	0	2	x	x	0	0	0	1	x	0

In this example we have $k = 8$, $n_k = 11$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_6 = \alpha_7 = \alpha_8 = \alpha_9 = \alpha_{11} = \alpha$ and $\alpha_4, \alpha_5, \alpha_{10}$ are all smaller than α. Here $L_k = n_7 - n_4 + 1 = 9 - 6 + 1 = 4$, and all $m_k = 0$ except $m_3 = 2$ and $m_9 = 1$. Assume we are interested in the event $\{\tilde{\sigma}_C > k\}$ ($k = 8$ in this example). This means that the chain $(X_{n_k})_{k\ge0}$ never hits C for $k = \overline{1,8}$. But $(X_n)_{n\ge0}$ may still visit C at the moments $n \in \{4, 5, 10\}$. However, we know that during at least $L_k = 4$

timesteps in the $[0, n_k] = [0, 11]$ time interval, the chain $(X_n)_{n\geq 0}$ never hits C. Should it never hit C until $n = n_4 = 6$ we will have the inequality

$$\mathbb{P}_x\{\tilde{\sigma}_C > k\} \leq \mathbb{P}_x\{\sigma_C > k\} \leq \mathbb{P}_x\{\sigma_C > L_k\}.$$

Otherwise, say if $X_4 \in C$ we have

$$\mathbb{P}_x\{\tilde{\sigma}_C > k\} \leq \sup_{y\in C} \mathbb{P}_y\{\sigma_C > u + L_k\} \leq \sup_{y\in C} \mathbb{P}_y\{\sigma_C > L_k\},$$

where u is the length of the time interval between the hitting time and the "beginning" of L_k (in our example, this is $n_4 = 6$, and $u = 1$). Clearly, such derivations are valid for any k and do not depend on a particular selection of $\{n_k, k \geq 0\}$. So we have the inequality

$$\mathbb{P}_x\{\tilde{\sigma}_C > k\} \leq \mathbb{P}_x\{\sigma_C > L_k\} + \sup_{y\in C} \mathbb{P}_y\{\sigma_C > L_k\}. \tag{17}$$

Next, we show that L_k is asymptotically bigger than $k^{(1-\varepsilon)(1-\rho)}$ for all $\varepsilon \in (0, 1)$. To do so, we prove that for any $\varepsilon \in (0, 1)$

$$\frac{L_k}{k^{(1-\varepsilon)(1-\rho)}} \to \infty, k \to \infty. \tag{18}$$

In fact, we are only interested in ε from Condition 2, but (18) is valid for all $\varepsilon \in (0, 1)$. Yet, from now on, we will assume that ε is a specific value selected in Condition 2.

Assume that (18) is not valid, so that

$$\underline{lim} \frac{L_k}{k^{(1-\varepsilon)(1-\rho)}} < \infty,$$

which implies the existence of some constant $c > 0$ and subsequence $\{k_j, j \geq 0\}$, $k_j \to \infty$, $j \to \infty$ such that

$$L_{k_j} \leq c k_j^{(1-\varepsilon)(1-\rho)}.$$

In this case, for a sufficiently large $k \in \{k_j, j \geq 0\}$, time interval $[0, k]$ contains at least $\frac{k}{ck^{(1-\varepsilon)(1-\rho)}} = c^{-1}k^{\rho+\varepsilon(1-\rho)}$ subintervals of lengths $ck^{(1-\varepsilon)(1-\rho)}$, so that every such interval contains at least one $m_i > 0$. From the latter, we may derive

$$\sum_{i=1}^{k} i^{-\rho} m_i \geq \sum_{i=1}^{c^{-1}k^{\rho+\varepsilon(1-\rho)}} \left(ick^{(1-\varepsilon)(1-\rho)}\right)^{-\rho} = \frac{1}{c^\rho k^{(1-\varepsilon)(1-\rho)\rho}} \sum_{i=1}^{c^{-1}k^{\rho+\varepsilon(1-\rho)}} i^{-\rho}$$

$$\geq \frac{1}{c^\rho k^{(1-\varepsilon)(1-\rho)\rho}} \int_0^{c^{-1}k^{\rho+\varepsilon(1-\rho)}} \frac{dx}{(x+1)^\rho} = \frac{\left(\frac{1}{c}k^{\rho+\varepsilon(1-\rho)} + 1\right)^{1-\rho} - 1}{(1-\rho)c^\rho k^{(1-\varepsilon)(1-\rho)\rho}}$$

$$= \frac{1}{(1-\rho)c^\rho}\left(k^{(1-\rho)(\rho+\varepsilon(1-\rho))-(1-\varepsilon)(1-\rho)\rho}\left(\frac{1}{c} + \frac{1}{k^{\rho+\varepsilon(1-\rho)}}\right)^{1-\rho} - \frac{1}{k^{(1-\varepsilon)(1-\rho)\rho}}\right)$$

$$= \frac{1}{(1-\rho)c^\rho}\left(k^{(1-\rho)\varepsilon}\left(\frac{1}{c} + \frac{1}{k^{\rho+\varepsilon(1-\rho)}}\right)^{1-\rho} - \frac{1}{k^{(1-\varepsilon)(1-\rho)\rho}}\right) \to \infty,\ k \to \infty.$$

Recall, that we selected k from the sequence $\{k_j, j \geq 0\}$ and $k_j \to \infty$, $j \to \infty$. Thus we may conclude that

$$\sum_{i=1}^{\infty} i^{-\rho} m_i = \lim_{j\to\infty} \sum_{i=1}^{k_j} i^{-\rho} m_i = \infty,$$

which contradicts Condition 2. Thus, (18) holds true and L_k is bigger than $k^{(1-\varepsilon)(1-\rho)}$ for all sufficiently large $k > 0$. Using this fact and inequality (17) we may write

$$\mathbb{P}_x\{\tilde{\sigma}_C > N\} \leq \mathbb{P}_x\{\sigma_C > N^{(1-\varepsilon)(1-\rho)}\} + \sup_{x\in C} \mathbb{P}_x\{\sigma_C > N^{(1-\varepsilon)(1-\rho)}\}. \tag{19}$$

Using (15), (19) and Markov inequality we may derive the following bound

$$\mathbb{P}_x\left\{\tilde{\sigma}_C > N\right\} \le B(x) N^{-\frac{(1-\varepsilon)(1-\rho)}{1-\beta}},$$

where $B(x)$ is defined in (2). Substituting this into (14) and using (13) we come to the following bound

$$\mathbb{E}_x\left[\sum_{k=1}^{\tilde{\sigma}_C-1} r(k+1)m_k\right] \le B(x)\sum_{k=1}^{\infty} r(k+1)m_k k^{-\frac{(1-\varepsilon)(1-\rho)}{1-\beta}}$$

$$= B(x)\sum_{k=1}^{\infty}\left((1-\alpha)+\frac{1}{k}\right)^{\frac{\alpha}{1-\alpha}} k^{\frac{\alpha}{1-\alpha}-\frac{(1-\varepsilon)(1-\rho)}{1-\beta}} m_k \le \alpha B(x)\sum_{k=1}^{\infty} k^{-\rho} m_k < \infty.$$

due to Condition 2. □

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