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Random Sieves and Generalized Leader-election Procedures

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Abstract

A random sieve of the set of positive integers $\mathbb{N}$ is an infinite sequence of nested subsets $\mathbb{N} = \mathcal{S}_0 \supset \mathcal{S}_1 \supset \mathcal{S}_2 \supset \cdots$ such that $\mathcal{S}_k$ is obtained from $\mathcal{S}_{k-1}$ by removing elements of $\mathcal{S}_{k-1}$ with the indices outside R_k and enumerating the remaining elements in the increasing order. Here $R_1, R_2, \ldots$ is a sequence of independent copies of an infinite random set $R \subset \mathbb{N}$. We prove general limit theorems for $\mathcal{S}_n$ and related functionals, as $n \to \infty$.

Keywords: iterated random function, leader-election procedure, random sieve, stochastic-fixed point equation.

1. Introduction and motivation

Let R be an arbitrary random infinite subset of the set of positive integers $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $(R_k)_{k\in\mathbb{N}}$ independent copies of R. A random sieving of the set $\mathbb{N}$ by the set R is an infinite chain of countable sets

$$\mathbb{N} =: \mathcal{S}_0 \supset \mathcal{S}_1 \supset \cdots \supset \mathcal{S}_n \supset \cdots$$

such that the set $\mathcal{S}_k$ is obtained from $\mathcal{S}_{k-1}$, $k \in \mathbb{N}$, by removing elements of $\mathcal{S}_{k-1}$ with the indices outside R_k and enumerating the remaining elements in the increasing order. Formally, for $k \in \mathbb{N}$, if

$$\mathcal{S}_{k-1} = \left\{s_1^{(k-1)}, s_2^{(k-1)}, s_3^{(k-1)}, \ldots\right\}, \quad s_1^{(k-1)} < s_2^{(k-1)} < s_3^{(k-1)} < \cdots,$$

and

$$R_k = \left\{r_1^{(k)}, r_2^{(k)}, r_3^{(k)}, \ldots\right\}, \quad r_1^{(k)} < r_2^{(k)} < r_3^{(k)} < \cdots,$$

then

$$\mathcal{S}_k := \left\{s^{(k-1)}_{r_1^{(k)}}, s^{(k-1)}_{r_2^{(k)}}, s^{(k-1)}_{r_3^{(k)}}, \ldots\right\}. \tag{1}$$

There are several examples of random sieving that were previously analyzed in the literature.

Example 1.1. *If* $R = \{\xi+1, \xi+2, \xi+3, \ldots\}$ *with* ξ *being a random variable with values in* $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$*, then obviously*

$$\mathcal{S}_k = \{S_k + 1, S_k + 2, S_k + 3, \ldots\}, \quad k \in \mathbb{N},$$

where $(S_k)_{k\in\mathbb{N}}$ *is a standard random walk on* $\mathbb{N}$ *with generic step* ξ*. The analysis of such a sieving is the subject of the classical renewal theory on positive integers.*

Example 1.2. *For* R *being the range of an increasing integer-valued random walk, that is*

$$r_{j+1}^{(k)} - r_j^{(k)}, \quad k \in \mathbb{N}, \quad j \in \mathbb{N},$$

are independent copies of a positive integer-valued random variable, say η*, the corresponding random sieving and its connection to classical Galton–Watson processes were analyzed in Alsmeyer, Kabluchko, and Marynych (2016). An important particular case is obtained by choosing* η *to be geometrically distributed on* $\mathbb{N}$ *with a parameter* $p \in (0,1)$*. This leads to a well-known classical model called 'leader-election procedure', which has been a subject of active research during last decades; see, for example, Bruss and Grübel (2003); Fill, Mahmoud, and Szpankowski (1996); Grübel and Hagemann (2016); Janson and Szpankowski (1997); Prodinger (1993). The integers in the leader-election procedure are typically viewed as players in a game who independently toss a coin so as to determine whether they stay in the game for the next round or not. Restricted from* $\mathbb{N}$ *to* $\{1, 2, \ldots, n\}$ *the sieving can be thought of as a procedure of selecting a leader (leaders) from the group of* n *players, whence the name.*

Example 1.3. *If* R *is the set of record's positions in an infinite sample from a continuous distribution the corresponding random sieving was studied in Alsmeyer, Kabluchko, and Marynych (2017).*

The above introduced random sieving is intimately connected with several classical models in probability. First of all, let us mention a connection, already observed in Alsmeyer *et al.* (2016), with the notion of stability of point processes; see Davydov, Molchanov, and Zuyev (2011) and Zanella and Zuyev (2015). Associated with every sieving procedure is the corresponding operator on the space of point processes on $[0,\infty)$. Given a point process $\mathcal{X} := \sum_{k=1}^{\infty} \delta_{X_k}$ in $[0,\infty)$ with $0 \le X_1 \le X_2 \le \ldots$, and an a.s. infinite random set $R = \{r_1, r_2, r_3, \ldots\} \subset \mathbb{N}$, we define the thinning of $\mathcal{X}$ by R as

$$\mathcal{X} \bullet R \;:=\; \sum_{k=1}^{\infty} \delta_{X_{r_k}}.$$

This random operation transforms $\mathcal{X}$ into a 'sparser' point process $\mathcal{X} \bullet R$ by removing points of $\mathcal{X}$ with indices outside the range of R. In order to compensate such thinning, a second deterministic operation is used for rescaling. Namely, let f be a deterministic function which is 'contractive' in an appropriate sense, and set $f(\mathcal{X}) := \sum_{k=1}^{\infty} \delta_{f(X_k)}$. For example, one can take $f(x) = ax$ for some $a \in (0,1)$ or $f(x) = \log(1+x)$. A point process $\mathcal{X}$ is called *f-stable with respect to thinning by a random set R*; see Alsmeyer *et al.* (2016), if

$$\mathcal{X} \overset{\mathrm{d}}{=} f(\mathcal{X} \bullet R), \tag{2}$$

where $\overset{\mathrm{d}}{=}$ denotes equality in distribution. From this viewpoint a natural problem is to describe the set of point processes which are f-stable with respect to thinning by a random set R. In the setting of Example 1.2, that is, for sieving by a random walk, this problem has been solved in Alsmeyer *et al.* (2016). To the best of our knowledge no other cases have been addressed so far.

Another well-known probabilistic concept related to random sieves is theory of iterated random function systems; see Diaconis and Freedman (1999) and also Marynych and Molchanov (2021) for sieving procedures of such systems. Classic theory of iterated random function systems is mainly concerned with a contractive random mapping Φ defined on some complete separable metric space. For contractive mappings under mild additional assumptions the sequence of *forward iterations* $\Phi_k \circ \Phi_{k-1} \circ \cdots \circ \Phi_1(x_0)$ converges in distribution, as $k \to \infty$, to a random limit for every starting point x_0, whereas *backward iterations* $\Phi_1 \circ \Phi_2 \circ \cdots \circ \Phi_k(x_0)$ converge a.s. The most prominent example of this kind is called *perpetuity* in which Φ is an affine mapping on $\mathbb{R}$, that is, $\Phi(x) = Ax + B$, $x \in \mathbb{R}$ with (A, B) being an arbitrary random vector. A criterion for convergence of such iterations can be found in Goldie and Maller (2000). To see the connection with our model one may look at the countable sets R_k, $k \in \mathbb{N}$, and $\mathcal{S}_k$, $k \in \mathbb{N}_0$, as random mappings from $\mathbb{N}$ to $\mathbb{N}$ or, equivalently, as elements of $\mathbb{N}^\infty$. Taking this point of view, we write

$$R_k(x) = r_x^{(k)} \quad \text{and} \quad \mathcal{S}_k(x) = s_x^{(k)} \quad \text{for} \quad k, x \in \mathbb{N},$$

and recast equation (1) as follows

$$\mathcal{S}_k = \mathcal{S}_{k-1} \circ R_k, \quad k \in \mathbb{N},$$

where $\circ$ denotes superposition of two mappings defined in the usual way $(f \circ g)(x) = f(g(x))$. Upon iterating we write

$$\mathcal{S}_k(x) = R_1 \circ R_2 \circ \cdots \circ R_k(x), \quad k \in \mathbb{N}, \quad x \in \mathbb{N}. \tag{3}$$

Despite seeming backward form of the process $\mathcal{S}_k$ in (3), $\mathcal{S}_k$, regarded as an element of $\mathbb{N}^\infty$, is actually a k-fold *forward iteration* of i.i.d. random mappings $\phi_k : \mathbb{N}^\infty \mapsto \mathbb{N}^\infty$ defined by

$$\mathbb{N}^\infty \ni (x_1, x_2, \ldots) = \mathbf{x} \stackrel{\phi_k}{\longmapsto} \mathbf{x} \circ R_k = (x_{r_1^{(k)}}, x_{r_2^{(k)}}, \ldots) \in \mathbb{N}^\infty$$

and applied to the starting point Id, the identity mapping. Thus,

$$\mathcal{S}_k = \phi_k \circ \phi_{k-1} \circ \cdots \circ \phi_1 \circ \mathrm{Id}, \quad k \in \mathbb{N}. \tag{4}$$

In contrast to contractive random mappings, every random sieving of $\mathbb{N}$ is, in a sense, expanding. More precisely,

$$\mathcal{S}_n(x) = R_1 \circ R_2 \circ \cdots \circ R_n(x) \stackrel{a.s.}{\longrightarrow} \infty, \quad n \to \infty, \tag{5}$$

for every $x \geq x_0 := x_0(R)$, where (nonrandom) x_0 is defined by[1]

$$x_0 := \inf\{x \in \mathbb{N} : \mathbb{P}\{R(x) = x\} < 1\}.$$

To see this just note that the left-hand side of (5) is a.s. nondecreasing in n because $\mathbb{P}\{R(x) \geq x\} = 1$ and diverges to infinity in probability because $\mathbb{P}\{R(x) > x\} > 0$ for $x \geq x_0$.

In view of (5) it is natural to ask what is a correct normalization of $\mathcal{S}_n(x) = R_1 \circ R_2 \circ \cdots \circ R_n(x)$ ensuring convergence to a nondegenerate limit and how to characterize the limit as a function of x, provided it exists. To get a better feeling of what a possible answer can be, let us look at the aforementioned Examples 1.1, 1.2 and 1.3. In what follows we denote by $\Longrightarrow$ weak convergence of probability measures on $\mathbb{R}^\infty$ endowed with the product topology.

In the setting of Example 1.1 $R(x) = x + \xi$ and $\mathcal{S}_n(x) = x + S_n$, $x \in \mathbb{N}$, where S_n is a random walk. The question of convergence in distribution of $\mathcal{S}_n(x)$, as $n \to \infty$, is the most classical topic in probability theory and is fully understood. If existent, the limit of properly centered and/or normalized $\mathcal{S}_n(x)$ is a stable distribution and does not depend on x. However, as we shall see, this type of behavior is not representative for random sieves of $\mathbb{N}$.

In the setting of Example 1.2 R is the range of a random walk on $\mathbb{N}$. Limit theorems for this type of random sieving were established in Alsmeyer *et al.* (2016). If $\mathbb{E}\eta \log \eta < \infty$, then with $\mu := \mathbb{E}\eta$ we have

$$\left(\mu^{-n} \mathcal{S}_n(x)\right)_{x \in \mathbb{N}} \Longrightarrow \left(\mathcal{Z}_{f_1}(x)\right)_{x \in \mathbb{N}}, \quad n \to \infty, \tag{6}$$

[1] For $x < x_0$ we have $R_1 \circ R_2 \circ \cdots \circ R_n(x) = x$ for all $n \in \mathbb{N}$ because $\mathbb{P}\{R(x) = x\} = 1$ for $x < x_0$.

where $(\mathcal{Z}_{f_1}(x))$ is a certain random walk with a.s. positive i.i.d. steps; see Theorem 2.1 in Alsmeyer *et al.* (2016). Moreover, $(\mathcal{Z}_{f_1}(x))$ satisfies a stochastic fixed-point equation

$$(\mathcal{Z}_{f_1}(x))_{x\in\mathbb{N}} \overset{\mathrm{d}}{=} \left(\mu^{-1}\mathcal{Z}_{f_1}(R(x))\right)_{x\in\mathbb{N}}, \tag{7}$$

where R on the right-hand side is independent of $(\mathcal{Z}_{f_1}(x))_{x\in\mathbb{N}}$. On the other hand, assume that the distribution of η has infinite mean and satisfies Davies' assumption; see Davies (1978):

$$x^{-\alpha-\gamma(x)} \;\le\; \mathbb{P}\{\eta\ge x\} \;\le\; x^{-\alpha+\gamma(x)}, \quad x\ge x_0, \tag{8}$$

for some $0<\alpha<1$, $x_0\ge 0$, and a nonincreasing, non-negative function $\gamma(x)$ such that $x^{\gamma(x)}$ is nondecreasing and $\int_{x_0}^{\infty}\gamma(\exp(e^x))\,\mathrm{d}x<\infty$. Then Theorem 2.8 in Alsmeyer *et al.* (2016) says (after exponentiation) that

$$\left((\mathcal{S}_n(x))^{\alpha^n}\right)_{x\in\mathbb{N}} \Longrightarrow (\mathcal{Z}_{f_2}(x))_{x\in\mathbb{N}}, \quad n\to\infty, \tag{9}$$

where $(\mathcal{Z}_{f_2}(x))_{x\in\mathbb{N}}$ is the running maximum process of i.i.d. random variables with values in $(1,\infty)$. Here the limit satisfies a stochastic fixed-point equation

$$(\mathcal{Z}_{f_2}(x))_{x\in\mathbb{N}} \overset{\mathrm{d}}{=} \left(\mathcal{Z}^{\alpha}_{f_2}(R(x))\right)_{x\in\mathbb{N}}, \tag{10}$$

where again R on the right-hand side is independent of $(\mathcal{Z}_{f_2}(x))_{x\in\mathbb{N}}$.
In the setting of Example 1.3 $R(n)$ is the position of nth record, $n\in\mathbb{N}$, in an infinite sample from a continuous distribution. In this case Theorem 2.8 in Alsmeyer *et al.* (2017) says that

$$(L_n(\mathcal{S}_n(x)))_{x\in\mathbb{N}} \Longrightarrow (\mathcal{Z}_{f_3}(x))_{x\in\mathbb{N}}, \quad n\to\infty, \tag{11}$$

where $L_n(x)$ is the n-fold iteration of the function $x\mapsto\log(1+x)$ with itself and the limit satisfies a stochastic fixed-point equation

$$(\mathcal{Z}_{f_3}(x))_{x\in\mathbb{N}} \overset{\mathrm{d}}{=} (\log(1+\mathcal{Z}_{f_3}(R(x))))_{x\in\mathbb{N}},$$

where, as before, R on the right-hand side is independent of $(\mathcal{Z}_{f_3}(x))_{x\in\mathbb{N}}$.

A common feature of limit relations (6), (9) and (11) is that they can be written in a unified way as follows:

$$\Big(\underbrace{f_i\circ\cdots\circ f_i}_{n\text{ times}}\circ R_1\circ R_2\circ\cdots\circ R_n(x)\Big)_{x\in\mathbb{N}} \Longrightarrow (\mathcal{Z}_{f_i}(x))_{x\in\mathbb{N}}, \quad n\to\infty, \quad i=1,2,3, \tag{12}$$

where $f_i:[0,\infty)\mapsto[0,\infty)$ is a strictly increasing unbounded concave function given by

$$f_i(t):=\begin{cases}\mu^{-1}t, & \text{if } i=1 \text{ and (6) holds},\\ t^{\alpha}, & \text{if } i=2 \text{ and (9) holds},\\ \log(1+t), & \text{if } i=3 \text{ and (11) holds}.\end{cases} \tag{13}$$

Furthermore, the fixed-point equations satisfied by the limits take the form

$$(\mathcal{Z}_{f_i}(x))_{x\in\mathbb{N}} \overset{\mathrm{d}}{=} (f_i(\mathcal{Z}_{f_i}(R(x))))_{x\in\mathbb{N}}, \quad i=1,2,3.$$

The main goal of this paper is to establish general conditions ensuring, for a given random set R, existence of a normalizing function f such that

$$\Big(\underbrace{f\circ\cdots\circ f}_{n\text{ times}}\circ R_1\circ R_2\circ\cdots\circ R_n(x)\Big)_{x\in\mathbb{N}} \Longrightarrow (\mathcal{Z}_f(x))_{x\in\mathbb{N}}, \quad n\to\infty, \tag{14}$$

for some limit $(\mathcal{Z}_f(x))_{x\in\mathbb{N}}$. The fact that whenever convergence (14) holds the limit should satisfy a stochastic fixed-point equation

$$(\mathcal{Z}_f(x))_{x\in\mathbb{N}} \stackrel{\mathrm{d}}{=} (f(\mathcal{Z}_f(R(x))))_{x\in\mathbb{N}}, \tag{15}$$

is obvious. Note that (15) is a particular case of (2) when restricted to point processes on $\mathbb{N}$.

As a byproduct we also derive limit theorems for two related characteristics which describe the speed of sieving:

- $N_M^{(n)}$ the number of integers among $1,2,\dots,M$ which remain after n steps, formally,

$$N_M^{(n)} := \mathrm{card}\,(\mathcal{S}_n \cap \{1,2,\dots,M\}), \quad M\in\mathbb{N}, \quad n\in\mathbb{N}_0; \tag{16}$$

- $T(M)$ the number of steps until all integers $1,2,\dots,M$ have been removed, namely,

$$T(M) \ := \ \inf\{n\in\mathbb{N} : N_M^{(n)} = 0\}, \quad M\in\mathbb{N}. \tag{17}$$

Note that the above quantities are connected via duality relations:

$$\{N_M^{(n)} \geq k\} = \{\mathcal{S}_n(k) \leq M\} \quad \text{and} \quad \{T(M)\leq k\} = \{\mathcal{S}_k(1) > M\}, \quad k,M\in\mathbb{N}, \quad n\in\mathbb{N}_0. \tag{18}$$

In particular, $T(M)$ is a.s. finite if and only if $x_0(R)=1$, see Eq. (5) above.

Let us introduce the following shorthand notation: given a sequence of either deterministic or random functions $f_n : X\to X$, where X is an arbitrary set, we put

$$f^{(k\uparrow n)} := f_k\circ\cdots\circ f_n \quad \text{and} \quad f^{(n\downarrow k)} := f_n\circ\cdots\circ f_k$$

for $k\leq n$. For $n<k$, we stipulate that $f^{(k\uparrow n)}$ and $f^{(n\downarrow k)}$ denote the identity map on X. Further, we shall use the notation

$$f^{\circ(n)} = \underbrace{f\circ\cdots\circ f}_{n \text{ times}}, \quad n\in\mathbb{N},$$

for the n-fold iteration of f with itself.

2. Limit theorems for random sieves

Let $\mathcal{F}$ be a family of nondecreasing unbounded concave functions $f:[0,\infty)\to[0,\infty)$ such that $f(0)=0$. Let $\mathcal{F}_R$ be a subset of functions $f\in\mathcal{F}$ that also satisfy

$$\mathbb{E}f(R(n))\leq n, \quad n\in\mathbb{N}. \tag{19}$$

The key role of the family $\mathcal{F}_R$ is revealed by Proposition 2.2 below. However, we shall first need to ensure that $\mathcal{F}_R$ is non-empty.

Lemma 2.1. *For an arbitrary infinite random set $R\subseteq\mathbb{N}$ we have $\mathcal{F}_R\neq\varnothing$.*

A proof of Lemma 2.1 will be given in Section 4.

Proposition 2.2. *Let $\mathcal{G}_n$ be a sigma-algebra generated by the mappings $R_1,R_2,\dots,R_n$, $n\in\mathbb{N}$, $\mathcal{G}_0$ the trivial sigma-algebra. For every $f\in\mathcal{F}_R$ and $x\in\mathbb{N}$, the sequence*

$$f^{\circ(n)}(R^{(n\downarrow 1)}(x)), \quad n\in\mathbb{N},$$

is a positive supermartingale with respect to the filtration $(\mathcal{G}_n)_{n\in\mathbb{N}_0}$ and, thus, converges a.s.

Proof. By Jensen's inequality, since $f^{\circ(n-1)} : [0,\infty) \to [0,\infty)$ is concave,

$$\mathbb{E}\left(f^{\circ(n)}(R^{(n\downarrow 1)}(x))|\mathcal{G}_{n-1}\right) = \mathbb{E}\left(f^{\circ(n-1)}(f(R^{(n)}(R^{((n-1)\downarrow 1)}(x))))|\mathcal{G}_{n-1}\right)$$
$$\leq f^{\circ(n-1)}\left(\mathbb{E}\left(f(R^{(n)}(R^{((n-1)\downarrow 1)}(x)))|\mathcal{G}_{n-1}\right)\right) \overset{(19)}{\leq} f^{\circ(n-1)}(R^{((n-1)\downarrow 1)}(x)).$$

Thus, $(f^{\circ(n)}(R^{(n\downarrow 1)}(x)))_{n\in\mathbb{N}}$ is a positive supermartingale. □

We are ready to formulate our first main result.

Theorem 2.3. *Let $f \in \mathcal{F}_R$. Then*

$$\left(f^{\circ(n)} \circ R^{(1\uparrow n)}(x)\right)_{x\in\mathbb{N}} \Longrightarrow (\mathcal{Z}_f(x))_{x\in\mathbb{N}}, \quad n\to\infty,$$

where $(\mathcal{Z}_f(x))_{x\in\mathbb{N}}$ satisfies (15).

Proof. The result follows from Proposition 2.2 upon noticing that

$$\left(f^{\circ(n)} \circ R^{(1\uparrow n)}(x)\right)_{x\in\mathbb{N}} \overset{\mathrm{d}}{=} \left(f^{\circ(n)} \circ R^{(n\downarrow 1)}(x)\right)_{x\in\mathbb{N}},$$

for every fixed $n\in\mathbb{N}$. □

Theorem 2.3 is not completely satisfactory, since the limit $(\mathcal{Z}_f(x))_{x\in\mathbb{N}}$ might be trivial (a.s. equal to a fixed point of f) if $f\in\mathcal{F}_R$ is chosen incorrectly. To avoid such trivialities we naturally want to take $f\in\mathcal{F}_R$ 'as large as possible'. To formalize the latter notion, we endow the set $\mathcal{F}_R$ with a pointwise partial order $\preceq$:

$$f_1 \preceq f_2 \iff f_1(t) \leq f_2(t) \quad \text{for all} \quad t\geq 0.$$

Note that, for every $n\in\mathbb{N}$ and $f\in\mathcal{F}_R$,

$$n\leq R(n) \implies f(n) \leq \mathbb{E}f(R(n)) \leq n,$$

and, therefore,

$$\sup_{f\in\mathcal{F}_R} f(t) \leq \sup_{f\in\mathcal{F}_R} f(\lceil t\rceil) \leq \lceil t\rceil \leq t+1, \quad t\geq 0, \tag{20}$$

which shows that all the functions in $\mathcal{F}_R$ are uniformly locally bounded.

It is clear that $f\preceq g$ implies $\mathbb{P}\{\mathcal{Z}_f(x)\leq \mathcal{Z}_g(x)\}=1$, for all $x\in\mathbb{N}$. Recall that a function f^* is called a maximal element of $(\mathcal{F}_R,\preceq)$ if $f^*\preceq g$, for some $g\in\mathcal{F}_R$, implies $g\preceq f^*$. Let $\mathcal{M}_R$ be the set of maximal elements in $(\mathcal{F}_R,\preceq)$. The next proposition, whose proof is postponed to Section 4, demonstrates that the set $\mathcal{M}_R$ is non-empty.

Proposition 2.4. *Every chain in the partially ordered set $(\mathcal{F}_R,\preceq)$ possesses an upper bound. Thus, $(\mathcal{F}_R,\preceq)$ contains at least one maximal element.*

The elements of $\mathcal{M}_R$ seem to be the best candidates for deriving (14) with a non-trivial limit. However, Proposition 2.4 is a result on existence and does not provide any way to find at least one element of $\mathcal{M}_R$ explicitly. Therefore, we shall formulate the second theorem, which provides us with sufficient conditions for (14) with a non-trivial limit.

Note that every function f in $\mathcal{F}_R$ is strictly increasing on $[0,\infty)$ and continuous on $(0,\infty)$. Put $x_f := f(0+)$. There exists a unique strictly increasing convex function $f^{\leftarrow}$ defined on $[x_f,+\infty)$ such

$$f^{\leftarrow}(f(x)) = x, \quad x>0 \quad \text{and} \quad f(f^{\leftarrow}(x)) = x, \quad x\geq x_f.$$

Extend $f^{\leftarrow}$ to a convex function on $[0,\infty)$ by putting $f^{\leftarrow}(x)=0$, for $x\in[0,x_f)$. For $n\in\mathbb{N}$ and $f\in\mathcal{F}_R$, introduce a stochastic process $X_n:=(X_n(x))_{x\geq 1}$ defined by

$$X_n(x):=f^{\circ(n)}\circ R_n(\lfloor (f^{\leftarrow})^{\circ(n-1)}(x)\rfloor),\quad x\geq 0,$$

where we stipulate $R(0)=0$. Note that

$$\left(f^{\circ(n)}\circ R^{(n\downarrow 1)}(x)\right)_{x\in\mathbb{N}}=(X^{(n\downarrow 1)}(x))_{x\in\mathbb{N}},\quad n\in\mathbb{N}.$$

The processes X_n, $n\in\mathbb{N}$, are independent but **not** identically distributed. By Proposition 2.2

$$(X^{(n\downarrow 1)}(x))_{x\in\mathbb{N}}\overset{a.s.}{\longrightarrow}(\mathcal{Z}_f(x))_{x\in\mathbb{N}},\quad n\to\infty.$$

The next theorem says, in essence, that if $(X_n)_{n\in\mathbb{N}}$ converges uniformly to the identity function, as $n\to\infty$, then the solution $\mathcal{Z}_f(x)$ satisfies a strong law of large numbers, as $x\to\infty$. The proof will be given in Section 4.

Theorem 2.5. *Suppose that for some* $f\in\mathcal{F}_R$

$$\frac{f(R(x))}{x}\overset{a.s.}{\longrightarrow}1,\quad x\to\infty.$$

Further, assume that the series

$$\sum_{n=1}^{\infty}\sup_{x\geq 0}\frac{|X_n(x)-x|}{1+x},\tag{21}$$

which is comprised of independent random variables, converges almost surely. Then

$$\frac{\mathcal{Z}_f(x)}{x}\overset{a.s.}{\longrightarrow}1,\quad x\to\infty.\tag{22}$$

In particular, the nondecreasing process $\mathcal{Z}_f$ *is a.s. unbounded.*

Remark 2.6. *Necessary and sufficient conditions for the a.s. convergence of the series defined in* (21) *are given by the celebrated Kolmogorov three series theorem. Put*

$$W_n:=\sup_{x\geq 0}\frac{|X_n(x)-x|}{1+x},\quad n\in\mathbb{N}.$$

The series (21) *converges a.s. if and only if for every* $A>0$ *the following three series converge*

$$\sum_{n\geq 1}\mathbb{P}\{W_n\geq A\},\quad \sum_{n\geq 1}\mathbb{E}\left(W_n\mathbb{1}_{\{W_n\leq A\}}\right)\quad\textit{and}\quad \sum_{n\geq 1}\mathrm{Var}\left(W_n\mathbb{1}_{\{W_n\leq A\}}\right).$$

Remark 2.7. *The denominator* $1+x$ *in* (21) *can be replaced by an arbitrary function* $p:[0,\infty)\to[0,\infty)$ *such that* $p(x)\sim x$*, as* $x\to\infty$*, and* p *is bounded away from zero.*

Using Theorem 2.5 and duality relations (18) we immediately obtain the following limit theorems for the functionals $N_M^{(n)}$ and $T(M)$ defined by (16) and (17), respectively. For the limit process $(\mathcal{Z}_f(x))_{x\in\mathbb{N}}$ in Theorem 2.5 define the counting process

$$\mathcal{Z}_f^{\#}(y):=\mathrm{card}\{x\in\mathbb{N}:\mathcal{Z}_f(x)\leq y\},\quad y>0,$$

and put also

$$y_0:=\inf\{y>0:\lim_{n\to\infty}(f^{\leftarrow})^{\circ(n)}(y)=+\infty\}.$$

Proposition 2.8. *Under the assumptions of Theorem 2.5 it holds:*

$$\left(N^{(n)}_{\lfloor (f^{\leftarrow})^{\circ(n)}(y)\rfloor}\right)_{y>y_0}\overset{\text{f.d.d.}}{\longrightarrow}\left(\mathcal{Z}_f^{\#}(y)\right)_{y>y_0},\quad n\to\infty.$$

Furthermore, if $x_0=1$*, then for every fixed* $y>y_0$ *and* $x\in\mathbb{Z}$*,*

$$\lim_{n\to\infty}\mathbb{P}\{T(\lfloor (f^{\leftarrow})^{\circ(n)}(y)\rfloor)-n\leq x\}=\mathbb{P}\{\mathcal{Z}_f(1)>f^{\circ(x)}(y)\},$$

where $f^{\circ(x)}=(f^{\leftarrow})^{\circ(|x|)}$ *if* $x<0$.

Remark 2.9. *The distribution of $X^{(n\downarrow 1)}$, $n \in \mathbb{N}$, can be regarded as a convolution of probability measures on the semigroup of self-maps on $[0,\infty)$ endowed with the composition operation. General results on convergence of convolutions of probability measures on semigroups can be found in (Högnäs and Mukherjea 2013, Section 2.4). However, they do not seem to be applicable in our setting.*

As was pointed out to us by the referee, another closely related concept is known in theory of dynamical systems. One can think of the sequence $X^{(n\downarrow 1)}$, $n \in \mathbb{N}$, as a random discrete dynamical system which is non-autonomous in a sense that X_n's are not identically distributed. This type of dynamical systems has received some attention in the literature but from a different perspective than considered here; see, for example, Cui and Langa (2017) and (Kloeden, Pötzsche, and Rasmussen 2013, Section 8).

3. Examples

3.1. A martingale

Assume that there exists $\mu > 1$ such that $\mathbb{E}R(x) = \mu x$, for all $x \in \mathbb{N}$. In this special case $\mathcal{M}_R$ contains a function $x \mapsto \mu^{-1}x$. Furthermore, for every fixed $x \in \mathbb{N}$, the sequence $(\mu^{-n}R^{(n\downarrow 1)}(x))$ is a positive martingale rather than just a positive supermartingale.

Suppose that

$$\frac{R(x)}{x} \overset{a.s.}{\longrightarrow} \mu, \quad x \to \infty, \tag{23}$$

and, further,

$$\sum_{n=1}^{\infty} \sup_{x\geq 0} \frac{|R_n(\lfloor \mu^{n-1}x \rfloor) - \mu^n x|}{\mu^n(1+x)} < \infty.$$

The latter is equivalent to

$$\sum_{n=1}^{\infty} \sup_{x\geq 0} \frac{|R_n(\lfloor \mu^{n-1}x \rfloor) - \mu\lfloor \mu^{n-1}x \rfloor|}{\mu^n(1+x)} < \infty. \tag{24}$$

A simple sufficient condition for (23) and (24) is a Marcinkiewicz–Zygmund type strong law

$$\frac{R(x) - \mu x}{x^{1-\delta}} \overset{a.s.}{\longrightarrow} 0, \quad x \to \infty,$$

for some $\delta \in (0,1)$. In particular, if R is the standard random walk with $\mathbb{E}(R(1))^{1+\varepsilon} < \infty$, for some $\varepsilon > 0$, then (23) and (24) hold true.

Summarizing, under the above assumptions the limit relation (14) holds true with $f(x) = \mu^{-1}x$.

3.2. Records

This model has been treated in details in Alsmeyer *et al.* (2017). Here we just demonstrate that our Theorem 2.5 is powerful enough to recover the results of Alsmeyer *et al.* (2017).

Let $U_1, U_2, \ldots$ be a sequence of independent copies of a random variable with the uniform distribution on $[0,1]$. Let $R(k)$ be the index of k-th record in the sample. Thus,

$$R(1) = 1, \quad R(k) := \inf\{j > R(k-1) : U_j > U_{R(k-1)}\}, \quad k \geq 2. \tag{25}$$

According to the next lemma, whose proof will be given in Section 4, the function $x \mapsto \log(1+x)$ belongs to $\mathcal{F}_R$.

Lemma 3.1. *For all $x \in \mathbb{N}$, we have $\mathbb{E}\log(1+R(x)) \leq x$.*

The relation

$$\sum_{n=1}^{\infty} \sup_{x\geq 0} \frac{|(1+\log(\cdot))^{\circ(n)} \circ R_n(\lfloor(\exp(\cdot)-1)^{\circ(n-1)}(x)\rfloor)-x|}{1+x} < \infty \quad \text{a.s.} \tag{26}$$

can be checked using the mean value theorem for differentiable functions in conjunction with the fact that

$$\frac{|\log(1+R(x))-x|}{x^{1/2+\varepsilon}} \overset{a.s.}{\longrightarrow} 0, \quad x\to\infty; \tag{27}$$

see Theorem 2(v) in Gut (1990) for a stronger version of (27). We refer the reader to calculations on pp. 4364–4366 in Alsmeyer *et al.* (2017) for a derivation of an L_1-version of (26). Thus, (14) holds true with $f(x)=\log(1+x)$.

4. Proofs

Proof of Lemma 2.1. We need to find a concave strictly increasing and unbounded function f such that (19) holds. Such a function will be constructed by finding a sequence $0=t_0<t_1<t_2<\cdots$ such that $(t_k-t_{k-1})_{k\in\mathbb{N}}$ is nondecreasing and defining f by a linear interpolation of the points $(t_k,\alpha k)$, $k=0,1,2,\ldots$, for some $\alpha\in(0,1)$. The resulting function is obviously strictly increasing and unbounded. Furthermore, it is concave, since $(t_k-t_{k-1})_{k\in\mathbb{N}}$ is assumed nondecreasing.

In order to find (t_k) we argue as follows. Fix $\alpha\in(0,1)$, put

$$h(t) := \sup_{n\geq 1} \frac{\alpha\mathbb{P}\{R(n)\geq t\}}{n-\alpha}, \quad t\geq 0,$$

and $t'_k := \inf\{t\geq 0 : h(t)\leq 2^{-k}\}$. Define the sequence (t_k) recursively as follows

$$t_0 := 0, \quad t_k := \max\{t'_k, t_{k-1}+\max\{t_j-t_{j-1} : j=1,\ldots,k-1\}\}, \quad k\in\mathbb{N}.$$

Then, for every $n\in\mathbb{N}$,

$$\begin{aligned}
\mathbb{E}f(R(n)) &= \mathbb{E}\left(\sum_{k\geq 1} f(R(n))\mathbb{1}_{\{t_{k-1}\leq R(n)<t_k\}}\right) \leq \sum_{k\geq 1}\alpha k\mathbb{P}\{t_{k-1}\leq R(n)<t_k\} \\
&= \sum_{k\geq 1}\alpha k\left(\mathbb{P}\{R(n)\geq t_{k-1}\}-\mathbb{P}\{R(n)\geq t_k\}\right) = \alpha+\alpha\sum_{k\geq 1}\mathbb{P}\{R(n)\geq t_k\} \\
&\leq \alpha+(n-\alpha)\sum_{k\geq 1}h(t_k) \leq \alpha+(n-\alpha)\sum_{k\geq 1}h(t'_k) \leq \alpha+(n-\alpha)\sum_{k\geq 1}2^{-k}\leq n.
\end{aligned}$$

The proof is complete. □

Proof of Proposition 2.4. Let $\mathcal{C}$ be a chain (totally ordered subset) in $(\mathcal{F}_R,\preceq)$. Define a function $f^*_{\mathcal{C}} : [0,\infty)\to[0,\infty)$ as a pointwise supremum:

$$f^*_{\mathcal{C}}(t) := \sup_{f\in\mathcal{C}} f(t), \quad t\geq 0.$$

Note that (20) implies that $f^*_{\mathcal{C}}$ is locally bounded. Obviously, $f^*_{\mathcal{C}}$ is nondecreasing, unbounded and is an upper bound for the chain $\mathcal{C}$. We need to prove that $f^*_{\mathcal{C}}\in\mathcal{F}_R$ which amounts to checking that $f^*_{\mathcal{C}}$ is concave and

$$\mathbb{E}f^*_{\mathcal{C}}(R(n))\leq n, \quad n\in\mathbb{N}. \tag{28}$$

We shall first prove concavity. Fix $t_1,t_2\geq 0$ and $\varepsilon>0$. Then

$$f^*_{\mathcal{C}}(t_1)\leq f_1(t_1)+\varepsilon \quad \text{and} \quad f^*_{\mathcal{C}}(t_2)\leq f_2(t_2)+\varepsilon,$$

for some $f_1, f_2 \in \mathcal{C}$. Since $\mathcal{C}$ is a chain we have either $\max(f_1, f_2) = f_1$ or $\max(f_1, f_2) = f_2$. Without loss of generality assume the latter. Then, for every $\lambda \in [0, 1]$,

$$\begin{aligned}\lambda f^*_{\mathcal{C}}(t_1) + (1-\lambda) f^*_{\mathcal{C}}(t_2) &\le \lambda f_1(t_1) + (1-\lambda) f_2(t_2) + \varepsilon \\ &\le \lambda f_2(t_1) + (1-\lambda) f_2(t_2) + \varepsilon \le f_2(\lambda t_1 + (1-\lambda)t_2) + \varepsilon \le f^*_{\mathcal{C}}(\lambda t_1 + (1-\lambda)t_2) + \varepsilon.\end{aligned}$$

Sending $\varepsilon \to 0+$ yields concavity of $f^*_{\mathcal{C}}$ on $[0, \infty)$.

In order to prove (28) note that, for every $m \in \mathbb{N}$ and every $\varepsilon > 0$, there exists $f_m \in \mathcal{C}$ such that

$$f^*_{\mathcal{C}}(m) \le f_m(m) + \varepsilon, \quad m \in \mathbb{N}.$$

Thus, with $\widetilde{f} := \sup_{k\ge 1} f_k$, we have

$$f^*_{\mathcal{C}}(m) \le \widetilde{f}(m) + \varepsilon, \quad m \in \mathbb{N}.$$

It remains to note that

$$\widetilde{f}(m) = \lim_{n\to\infty} \sup_{k=1,\dots,n} f_k(m) = \lim_{n\to\infty} f_{p_n}(m), \quad m \in \mathbb{N},$$

where p_n is the index of the $\preceq$-maximal function in $\{f_1, \dots, f_n\} \subset \mathcal{C}$ which exists since $\mathcal{C}$ is a chain. Thus, for every fixed $n \in \mathbb{N}$, by the monotone convergence theorem,

$$\mathbb{E} f^*_{\mathcal{C}}(R(n)) \le \mathbb{E}\widetilde{f}(R(n)) + \varepsilon = \mathbb{E}\left(\lim_{m\to\infty} f_{p_m}(R(n))\right) + \varepsilon = \lim_{m\to\infty} \mathbb{E} f_{p_m}(R(n)) + \varepsilon \le n + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain (28). The second claim of the lemma follows from Zorn's lemma. □

We now turn to the proof of Theorem 2.5. We start with an auxiliary lemma.

Lemma 4.1. *Let $h : [0, \infty) \mapsto [0, \infty)$ be a nondecreasing unbounded concave function. Let $(x_n)_{n\in\mathbb{N}} \subset (0, \infty)$ and $(y_n)_{n\in\mathbb{N}} \subset (0, \infty)$ be two asymptotically equivalent sequences, that is, $\lim_{n\to\infty} x_n/y_n = 1$. Then $\lim_{n\to\infty} h(x_n)/h(y_n) = 1$.*

Proof. Firstly, note that concavity of h and $h(0) \ge 0$ jointly imply

$$h(\lambda x) = h(\lambda x + (1-\lambda)0) \ge \lambda h(x) + (1-\lambda)h(0) \ge \lambda h(x), \quad \lambda \in [0,1], \quad x \ge 0. \tag{29}$$

Fix $\varepsilon \in (0, 1)$. There exists $n_0(\varepsilon)$ such that $1 - \varepsilon \le x_n/y_n \le 1 + \varepsilon$, for $n \ge n_0(\varepsilon)$. Using (29) and monotonicity of h we obtain

$$\frac{h(x_n)}{h(y_n)} \le \frac{h((1+\varepsilon)y_n)}{h((1+\varepsilon)^{-1}(1+\varepsilon)y_n)} \le (1+\varepsilon)\frac{h((1+\varepsilon)y_n)}{h((1+\varepsilon)y_n)} = 1 + \varepsilon, \quad n \ge n_0(\varepsilon).$$

Thus,

$$\limsup_{n\to\infty} \frac{h(x_n)}{h(y_n)} \le 1 + \varepsilon.$$

Similarly, one can check that lim inf is greater or equal than $1 - \varepsilon$, which completes the proof. □

Proof of Theorem 2.5. The proof consists of several steps.

Step 1. Let us show that, for every fixed $n \in \mathbb{N}$,

$$\frac{X_n(x)}{x} \overset{a.s.}{\longrightarrow} 1, \quad x \to \infty. \tag{30}$$

The above formula is equivalent to

$$\frac{f^{\circ(n)}(R_n(\lfloor (f^{\leftarrow})^{\circ(n-1)}(x)\rfloor))}{x} \overset{a.s.}{\longrightarrow} 1, \quad x \to \infty,$$

which, in turn, is the same as

$$\frac{f^{\circ(n)}(R_n(\lfloor x \rfloor))}{f^{\circ(n-1)}(x)} = \frac{f^{\circ(n-1)}(f(R^{(n)}(\lfloor x \rfloor)))}{f^{\circ(n-1)}(x)} \xrightarrow{a.s.} 1, \quad x \to \infty.$$

The latter follows from Lemma 4.1 and the fact that

$$\frac{f(R_n(\lfloor x \rfloor))}{x} \xrightarrow{a.s.} 1, \quad x \to \infty.$$

Step 2. For every fixed $n \in \mathbb{N}$,

$$\frac{X^{(n\downarrow 1)}(x)}{x} = \prod_{k=1}^{n} \frac{X^{(k\downarrow 1)}(x)}{X^{((k-1)\downarrow 1)}(x)} = \prod_{k=1}^{n} \frac{X_k(X^{((k-1)\downarrow 1)}(x))}{X^{((k-1)\downarrow 1)}(x)} \xrightarrow{a.s.} 1, \quad x \to \infty, \tag{31}$$

where the last passage follows from (30).

Step 3. Let us show that

$$\sup_{n\in\mathbb{N}} \sup_{x\geq 0} \frac{X^{(n\downarrow 1)}(x)}{1+x} < \infty \quad \text{a.s.} \tag{32}$$

We have

$$\begin{aligned}\sup_{x\geq 0} \frac{X^{(n\downarrow 1)}(x)}{1+x} &\leq \prod_{k=1}^{n} \sup_{x\geq 0} \frac{1+X_k(X^{((k-1)\downarrow 1)}(x))}{1+X^{((k-1)\downarrow 1)}(x)} \leq \prod_{k=1}^{n} \sup_{x\geq 0} \frac{1+X_k(x)}{1+x} \\ &\leq \prod_{k=1}^{n} \left(1+\sup_{x\geq 0} \frac{|X_k(x)-x|}{1+x}\right) \leq \prod_{k=1}^{\infty} \left(1+\sup_{x\geq 0} \frac{|X_k(x)-x|}{1+x}\right) < \infty \quad \text{a.s.,}\end{aligned}$$

where the last inequality follows from (21). Thus, (32) holds true.

Step 4. For $x \in \mathbb{N}$, we have the representation

$$\frac{\mathcal{Z}_f(x)}{x} = \frac{f(R_1(x))}{x} + \frac{x+1}{x} \sum_{n\geq 2} \frac{X^{(n\downarrow 1)}(x) - X^{((n-1)\downarrow 1)}(x)}{1+x},$$

where the series converges by Theorem 2.3. We need to show that

$$\lim_{x\to\infty} \sum_{n\geq 2} \frac{X^{(n\downarrow 1)}(x) - X^{((n-1)\downarrow 1)}(x)}{1+x} = \sum_{n\geq 2} \lim_{x\to\infty} \frac{X^{(n\downarrow 1)}(x) - X^{((n-1)\downarrow 1)}(x)}{1+x},$$

since the right-hand side is equal to 0 by (31). In view of the dominated convergence theorem, it suffices to check that

$$\sum_{n\geq 2} \sup_{x\geq 0} \left| \frac{X^{(n\downarrow 1)}(x) - X^{((n-1)\downarrow 1)}(x)}{1+x} \right| < \infty \quad \text{a.s.}$$

The latter follows from

$$\begin{aligned}&\sum_{n\geq 2} \sup_{x\geq 0} \left| \frac{X^{(n\downarrow 1)}(x) - X^{((n-1)\downarrow 1)}(x)}{1+x} \right| \\ &\qquad \leq \sum_{n\geq 2} \sup_{x\geq 0} \left| \frac{X^{(n\downarrow 1)}(x) - X^{((n-1)\downarrow 1)}(x)}{1+X^{((n-1)\downarrow 1)}(x)} \right| \sup_{x\geq 0} \frac{1+X^{((n-1)\downarrow 1)}(x)}{1+x} \\ &\qquad \leq \left(\sup_{n\geq 2} \sup_{x\geq 0} \frac{1+X^{((n-1)\downarrow 1)}(x)}{1+x} \right) \sum_{n\geq 2} \sup_{x\geq 0} \frac{|X_n(x)-x|}{1+x} < \infty,\end{aligned}$$

where on the last step we utilized (32) and condition (21).

The proof of Theorem 2.5 is complete. □

Proof of Lemma 3.1. We shall rely on Williamson's representation for R, see Eq. (3.4) in Gut (1990),

$$R(1) = 1, \quad R(k+1) = \lceil R(k)/U_k \rceil, \quad k \in \mathbb{N},$$

where $(U_k)_{k\in\mathbb{N}}$ are i.i.d. random variables with the uniform distribution on $[0,1]$. According to this representation, for $k \in \mathbb{N}$,

$$\mathbb{E}\log(1 + R(k+1)) = \mathbb{E}\left(\sum_{j\geq R(k)+1} \log(1+j)\frac{R(k)}{j(j-1)}\right).$$

Assume that we have proved

$$\sum_{j\geq A+1} \frac{\log(1+j)}{j(j-1)} \leq \frac{1+\log(1+A)}{A}, \quad A \in \mathbb{N}. \tag{33}$$

Then $\mathbb{E}\log(1+R(k+1)) \leq 1 + \mathbb{E}\log(1+R(k))$, $k \in \mathbb{N}$, and the claim of lemma follows by induction. It remains to check (33). To this end, note that

$$\begin{aligned}\sum_{j\geq A+1} \frac{\log(1+j)}{j(j-1)} &= \sum_{k\geq A} \frac{\log(2+j)}{j} - \sum_{j\geq A+1} \frac{\log(1+j)}{j} = \frac{\log(A+2)}{A} + \sum_{j\geq A+1} \frac{1}{j}\log\left(\frac{j+2}{j+1}\right)\\ &\leq \frac{\log(A+2)}{A} + \sum_{j\geq A+1} \frac{1}{j(j+1)} \leq \frac{\log(A+2)}{A} + \frac{1}{A+1}\\ &\leq \frac{\log(1+A)}{A} + \frac{1}{A(A+1)} + \frac{1}{A+1} = \frac{1+\log(1+A)}{A}.\end{aligned}$$

The proof of (33) and of the entire lemma is complete. □

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