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Bivariate Poisson Generalized Lindley Distributions and the Associated BINAR(1) Processes

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Abstract

This paper proposes new bivariate distributions based on the Poisson generalized Lindley distribution as marginal. These models include the basic bivariate Poisson generalized Lindley (BPGL) and the Sarmanov-based bivariate Poisson generalized Lindley (SPGL) distributions. Subsequently, we introduce the BPGL and SPGL distributions as joint innovation distributions in a novel bivariate first-order integer-valued autoregressive process (BINAR(1)) based on binomial thinning. The model parameters in the BPGL and SPGL distributions are estimated using the method of maximum likelihood (ML) while we apply the conditional maximum likelihood (CML) for the BINAR(1) process. We conduct some simulation experiments to assess the small and large sample performances. Further, we implement the new BINAR(1)s to the Pittsburgh crime series data and they show better fitting criteria than other competing BINAR(1) models in the literature.

Keywords: Poisson generalized Lindley distribution, binomial thinning, over-dispersion, moments, maximum likelihood estimation, simulation, BINAR(1) process.

1. Introduction

Modelling the probabilistic behaviour of two random variables simultaneously demanded the construction of bivariate random variables. Bivariate distributions may be thus considered extensions of univariate distributions. Many approaches have been discussed in the literature for constructing bivariate random variables. Balakrishnan and Lai (2009) mentions many of those. Constructing discrete and continuous bivariate rv using the mixture methodology is discussed frequently in the statistical literature (see Karlis and Xekalaki (2005); Lai (2006); Sarabia Alegría and Gómez Déniz (2008) etc). The main advantage of this method is that it will have simple expressions for its marginal probability density function (pdf) and hence its moments, correlation, etc. Another method is to use a new class of distribution for the construction. The Sarmanov family (Sarmanov (1966)) of distributions can be used to construct discrete and continuous bivariate distributions with a flexible covariance structure. Ting Lee

(1996) studied various properties regarding the family and constructed bivariate distributions using various marginal distributions and construction methods. Also, the Farlie-Gumbel-Morgenstern (FGM) copula is a special case of the Sarmanov family. The first-order integer-valued autoregressive (INAR(1)) model is suitable for time series count datasets that exhibit over-dispersion. In many statistical and applied fields, such as medical insurance, sports, and finance, we can see those kinds of datasets. Since the initial works of McKenzie (1985) and Al-Osh and Alzaid (1987) on INAR(1) process with Poisson innovations, a significant number of works has been introduced in literature having univariate innovation distributions (see, Lívio, Khan, Bourguignon, and Bakouch (2018); Altun (2020a); Eliwa, Altun, El-Dawoody, and El-Morshedy (2020); Altun and Khan (2022); Irshad, Chesneau, D'cruz, and Maya (2021) etc).

Irshad, D'cruz, Maya, and Khan (2023) proposed a discrete univariate distribution, the two-parameter Poisson generalized Lindley (TPPGL) distribution, by mixing the Poisson distribution with the new generalized Lindley (NGL) distribution by Abouammoh, Alshangiti, and Ragab (2015). They introduced a count regression model and an INAR(1) model based on the TPPGL distribution (INAR(1)TPPGL) for modelling over-dispersed datasets. Also, it is proved that INAR(1)TPPGL provides a better model than many other recently developed INAR(1) models based on some model selection criterion which immensely motivates us to propose a BINAR(1) process with bivariate TPPGL innovations. Hence in this paper, discrete bivariate distributions based on the TPPGL distribution are constructed by using the mixture methodology (the basic bivariate Poisson generalized Lindley (BPGL)) as well as using the Sarmanov family of distributions (the Sarmanov bivariate Poisson generalized Lindley (SPGL)). Most importantly, these established distributions are mounted as innovations for BINAR(1)(that is., BINAR(1)BPGL and BINAR(1)SPGL), and both processes are compared with some other recently proposed BINAR(1) models as well as discussed in Pedeli and Karlis (2011). We first define the development of the TPPGL distribution and associated INAR(1) process. Then bivariate versions are constructed and adapted to BINAR(1) with bivariate TPPGL innovations (BPGL and SPGL) by inducing a cross-correlation between the counting series by assuming that the paired TPPGL innovations are jointly distributed.

The remaining parts of the paper is organized as follows: Section 2 reviews the TPPGL distribution and the associated INAR(1) process. The BPGL distribution, its corresponding BINAR(1) process, related properties, estimation of parameters, and its simulation study is given in Section 3. Similarly, Section 4 derives the SPGL distribution with its associated BINAR(1) process, corresponding properties, estimation of parameters, and simulation study. The practical importance of the proposed BINAR(1)s are studied in Section 5.

2. Development of the TPPGL distribution and the associated INAR(1) process

The NGL distribution with parameters α and θ introduced by Abouammoh *et al.* (2015) has a pdf given by

$$f(x) = \frac{\theta^{\alpha} x^{\alpha-2}}{(\theta+1)\Gamma(\alpha)}(x+\alpha-1)\mathrm{e}^{-\theta x}, \quad \theta \geq 0, \alpha \geq 1, x \geq 0.$$

Recently, Irshad *et al.* (2023) introduced a mixture distribution by mixing the Poisson distribution with the NGL distribution, that is, if X denote the random variable having the TPPGL distribution such that

$$X|\lambda \sim P(\lambda)$$

and

$$\lambda|\alpha, \theta \sim NGL(\alpha, \theta),$$

where $\lambda > 0, \theta \geq 0, \alpha \geq 1$. Then the unconditional probability mass function (pmf) of X

having the TPPGL distribution is

$$P(X=x)=\frac{\theta^{\alpha}\Gamma(x+\alpha-1)}{\Gamma(\alpha)x!(\theta+1)^{x+\alpha+1}}\left(x+(\alpha-1)(\theta+2)\right),\quad x=0,1,2,\ldots\,.$$

Thus the probability generating function (pgf), moment generating function (mgf), and hence the mean and variance of X are obtained such that

$$G_X(s)=\frac{\theta^{\alpha}}{1+\theta}\frac{2-s+\theta}{(1-s+\theta)^{\alpha}},$$

$$M_X(t)=\frac{\theta^{\alpha}}{1+\theta}\frac{2-e^{t}+\theta}{(1-e^{t}+\theta)^{\alpha}},$$

$$\mathrm{E}(X)=\frac{1+(\alpha-1)(\theta+1)}{\theta(1+\theta)}$$

and

$$\mathrm{Var}(X)=\frac{\alpha(1+\theta)}{\theta^{2}}-\frac{2+\theta}{(1+\theta)^{2}}.$$

Also, the dispersion index (DI) of X is obtained as

$$\mathrm{DI}(X)=1+\frac{1}{\theta}+\frac{1}{(1+\theta)(\alpha+(\alpha-1)\theta)},$$

which indicates a clear case of over-dispersion (since DI>1). Therefore, the TPPGL distribution is used as an innovation distribution for the INAR(1) process as the INAR(1)TPPGL process and it is proved that INAR(1)TPPGL gives better results for AIC, empirical mean and variance, than many other recently proposed INAR(1) processes based on real count datasets. The process $\{X_t\}_{t\in\mathbb{Z}}$ which follows

$$X_t=p\circ X_{t-1}+\varepsilon_t,\qquad 0\le p<1,$$

where ε_t is an independent and identically distributed sequence of random variables having the TPPGL distribution is said to be the INAR(1)TPPGL process. Also, ε_t is independent of X_{t-k} for all $k\ge 1$ and counting series in the binomial thinning operator $\circ$. Now the one-step transition probability of the INAR(1)TPPGL process is

$$\begin{aligned}\mathrm{P}\left(X_t=k\mid X_{t-1}=l\right) &= \sum_{i=1}^{\min(k,l)}\binom{l}{i}p^{i}(1-p)^{l-i}\\ &\quad\frac{\theta^{\alpha}\Gamma((k-i)+\alpha-1)((k-i)+(\alpha-1)(\theta+2))}{\Gamma(\alpha)(k-i)!(\theta+1)^{(k-i)+\alpha+1}},\quad k,l\ge 0.\end{aligned}\tag{2.1}$$

The mean, variance, DI, conditional mean and conditional variance are given by

$$\mathrm{E}(X_t)=\frac{1+(\alpha-1)(\theta+1)}{\theta(1+\theta)(1-p)},$$

$$\mathrm{Var}(X_t)=\frac{p(1+(\alpha-1)(\theta+1))}{(1-p^{2})\theta(1+\theta)}+\frac{\alpha(1+\theta)}{(1-p^{2})\theta^{2}}-\frac{2+\theta}{(1-p^{2})(1+\theta)^{2}},$$

$$\mathrm{DI}(X_t)=1+\frac{1}{\theta(1+p)}+\frac{1}{(1+p)(1+\theta)(\alpha+(\alpha-1)\theta)},$$

$$\mathrm{E}\left(X_t\mid X_{t-1}\right)=pX_{t-1}+\frac{1+(\alpha-1)(\theta+1)}{\theta(1+\theta)}$$

and

$$\operatorname{Var}\left(X_t \mid X_{t-1}\right)=p(1-p)X_{t-1}+\frac{\alpha(1+\theta)}{\theta^2}-\frac{2+\theta}{(1+\theta)^2}.$$

Here also, DI greater than 1 implying over-dispersion of the process.

3. The BPGL distribution and the corresponding BINAR(1) process

Following Gomez-Deniz, Sarabia, and Balakrishnan (2012), the BPGL distribution based on the TPPGL distribution is developed in this section.

Proposition 1. *Suppose $\boldsymbol{X}=(X_1,X_2)$ denotes a bivariate random vector which possesses the BPGL distribution such that,*

$$X_i \mid \Lambda=\lambda \sim P\left(\lambda\phi_i\right), i=1,2, \text{ independent}$$

and

$$\Lambda \sim NGL(\alpha,\theta),$$

$\phi_i>0,\theta\geq 0,\alpha\geq 1$. *Then the unconditional pmf of X having the BPGL distribution is*

$$P(X_1=x_1,X_2=x_2)=\frac{\theta^\alpha\phi_1^{x_1}\phi_2^{x_2}\Gamma\left(\alpha+x_1+x_2-1\right)\left(\left(\alpha-1\right)\left(\theta+\phi_1+\phi_2+1\right)+x_1+x_2\right)}{\left(\theta+\phi_1+\phi_2\right)^{\alpha+x_1+x_2}\left(\theta+1\right)x_1!x_2!\Gamma(\alpha)}, \tag{3.1}$$

where $x_1,x_2=0,1,2,...,\phi_1,\phi_2,\theta>0$ *and* $\alpha\geq 1$.

Proof. By the procedure defined in Gomez-Deniz *et al.* (2012), the pmf of the BPGL distribution is

$$\begin{aligned} P\left(X_1=x_1,X_2=x_2\right) &= \int_0^\infty P\left(X_1=x_1,X_2=x_2 \mid \Lambda=\lambda\right)dF_\lambda(\lambda) \\ &= \int_0^\infty \frac{\left(\lambda\phi_1\right)^{x_1}e^{-\lambda\phi_1}}{x_1!}\frac{\left(\lambda\phi_2\right)^{x_2}e^{-\lambda\phi_2}}{x_2!}\frac{\theta^\alpha x^{\alpha-2}}{(\theta+1)\Gamma(\alpha)}(\lambda+\alpha-1)\mathrm{e}^{-\theta\lambda}d\lambda \\ &= \frac{\theta^\alpha\phi_1^{x_1}\phi_2^{x_2}\Gamma\left(\alpha+x_1+x_2-1\right)\left(\left(\alpha-1\right)\left(\theta+\phi_1+\phi_2+1\right)+x_1+x_2\right)}{\left(\theta+\phi_1+\phi_2\right)^{\alpha+x_1+x_2}\left(\theta+1\right)x_1!x_2!\Gamma(\alpha)}. \end{aligned}$$

Hence it is proven. □

The random vector $\boldsymbol{X}$ with the pmf (3.1) is hereafter denoted as $\boldsymbol{X}\sim\mathrm{BPGL}(\theta,\alpha,\phi_1,\phi_2)$.

Remark 3.1. If $\alpha=2$, then the pmf of $\mathrm{BPGL}(\theta,\alpha,\phi_1,\phi_2)$ reduces to that of the basic bivariate Poisson-Lindley (BBPL) distribution with parameters θ,ϕ_1 and ϕ_2 of Gomez-Deniz *et al.* (2012).

Proposition 2. *Suppose $\boldsymbol{X}\sim BPGL(\theta,\alpha,\phi_1,\phi_2)$, then*

a) the marginal pmf of X_i,

$$P(X_i=x_i)=\frac{\theta^\alpha\phi_i^{x_i}\Gamma\left(\alpha+x_i-1\right)\left(\left(\alpha-1\right)\left(\theta+\phi_i+1\right)+x_i\right)}{(\theta+1)(\theta+\phi_i)^{\alpha+x_i}\Gamma(\alpha)x_i!},\ x_i=0,1,2,\ldots\ .$$

Remark 3.2. If $\phi_i=1$ for $i=1,2$, $X_i\sim\mathrm{TPPGL}(\alpha,\theta)$.

b) the conditional pmf of $X_2=x_2|X_1=x_1$ for given $x_1=0,1,2,...,$

$$\begin{aligned} P(X_2=x_2|X_1=x_1) &= \frac{\Gamma(x_1+x_2+\alpha-1)\phi_2^{x_2}(\theta+\phi_1)^{\alpha+x_1}}{\Gamma(x_1+\alpha-1)\Gamma(x_2+1)(\theta+\phi_1+\phi_2)^{(\alpha+x_1+x_2)}} \\ &\quad\times\frac{(x_1+x_2+(\alpha-1)(1+\theta+\phi_1+\phi_2))}{(x_1+(\alpha-1)(1+\theta+\phi_1))}, x_2>0. \end{aligned}$$

Similarly, the pmf of $X_1|X_2$ can also be obtained.

c) the joint pgf of $\boldsymbol{X}$ *is*

$$\begin{aligned} G_{\boldsymbol{X}}(s_1, s_2) &= \mathrm{E}(\mathrm{s}_1^{\mathrm{X}_1} \mathrm{s}_2^{\mathrm{X}_2}) \\ &= \frac{\theta^{\alpha} \left(\theta - (s_1 - 1)\, \phi_1 - (s_2 - 1)\, \phi_2 + 1\right)}{(\theta + 1) \left(\theta - (s_1 - 1)\, \phi_1 - (s_2 - 1)\, \phi_2\right)^{\alpha}}. \end{aligned}$$

Following proposition 2, we can find the moment-related properties.
The mean of X_i,

$$E(X_i) = \frac{(\alpha\theta + \alpha - \theta)\phi_i}{\theta(\theta + 1)}, \ i = 1, 2.$$

Also,

$$\mathrm{E}(X_i^2) = \frac{\phi_i \left[(\alpha - 1)\theta^2 + \alpha\theta + \alpha\phi_i((\alpha - 1)\theta + \alpha + 1)\right]}{\theta^2(\theta + 1)}, \ i = 1, 2.$$

Hence the variance of X_i,

$$\mathrm{Var}(X_i) = \phi_i \left(\frac{\alpha}{\theta^2(\theta + 1)} - \frac{1}{\theta(\theta + 1)^2} \right) + \phi_i^2 \left(\frac{\alpha}{\theta^2} - \frac{1}{(\theta + 1)^2} \right), \ i = 1, 2,$$

and the covariance of X_1 and X_2,

$$\mathrm{Cov}(X_1, X_2) = \frac{\phi_1\phi_2 \left(\alpha(\theta + 1)^2 - \theta^2\right)}{\theta^2(\theta + 1)^2},$$

which will be always positive.

3.1. Estimation of parameters of the BPGL distribution

Here the method of ML is used to estimate the unknown parameters. Suppose (X_{1i}, X_{2i}), $i = 1, 2, ..., n$ be the observations of a random sample from BPGL$(\theta, \alpha, \phi_1, \phi_2)$. Then the log of the likelihood function is

$$\begin{aligned} L(\boldsymbol{\beta}) &\propto \sum_{i=1}^{n} x_{1i} \log \phi_1 + \sum_{i=1}^{n} x_{2i} \log \phi_2 - (\sum_{i=1}^{n} x_{1i} + \sum_{i=1}^{n} x_{2i}) \log(\theta + \phi_1 + \phi_2) \\ &+ \sum_{i=1}^{n} (\log(\Gamma(\alpha + x_{1i} + x_{2i} - 1)) + \log((\alpha - 1)(\theta + \phi_1 + \phi_2 + 1) + x_{1i} + x_{2i}) \\ &- \log(x_{1i}!) - \log(x_{2i}!)), \end{aligned} \tag{3.2}$$

where $\boldsymbol{\beta} = (\theta, \alpha, \phi_1, \phi_2)$. The ML estimators of $\boldsymbol{\beta}$ are obtained by maximizing (3.2) using numerical methods with the help of statistical software. The asymptotic normality and consistency properties of the ML estimators follow from the well-known Lindberg-Levy central limit theorem. Therefore, the ML estimators of $\boldsymbol{\beta}$, $\hat{\boldsymbol{\beta}}$ is such that, $(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \sim \mathrm{N}(0, I^{-1})$, where I^{-1} is the inverse of the Fisher information matrix. Here we have used the quasi-Newton approach, the BFGS algorithm available in the `optim` function of the R programming language, to find the ML estimators for the $\boldsymbol{\beta}$.

3.2. Simulation study for the BPGL distribution

The estimates obtained with the ML method are analysed using a simulation study to check the performance of BPGL distribution in small and large samples. So we took N=1000 replications each of sample sizes n=50, 100, 200, 400, 800 for two sets of parameters ($\theta = 0.2, \alpha = 1.1, \phi_1 = 0.15, \phi_2 = 0.2$) and ($\theta = 0.5, \alpha = 1.4, \phi_1 = 0.8, \phi_2 = 0.9$) and hence ML estimates (MLEs), bias and mean square errors (MSEs) were calculated and reported through box plots as in Figures 1, 2, 3, 4, 5 and 6. That is, Figures from 1 to 3 portrayed through box plots for the set of parameter values, $\theta = 0.2,\ \alpha = 1.1,\ \phi_1 = 0.15,\ \phi_2 = 0.2$ and Figures from 4 to 6 for the set of parameter values, $\theta = 0.5,\ \alpha = 1.4,\ \phi_1 = 0.8,\ \phi_2 = 0.9$. To make the results more comparable and interpretable, $\log 10-$scale of the y-axis for plots in Figures 1 and 4 is used. The plotted figures show that for the MLEs, their bias and MSEs corresponding to each parameter decrease as the sample size increases.

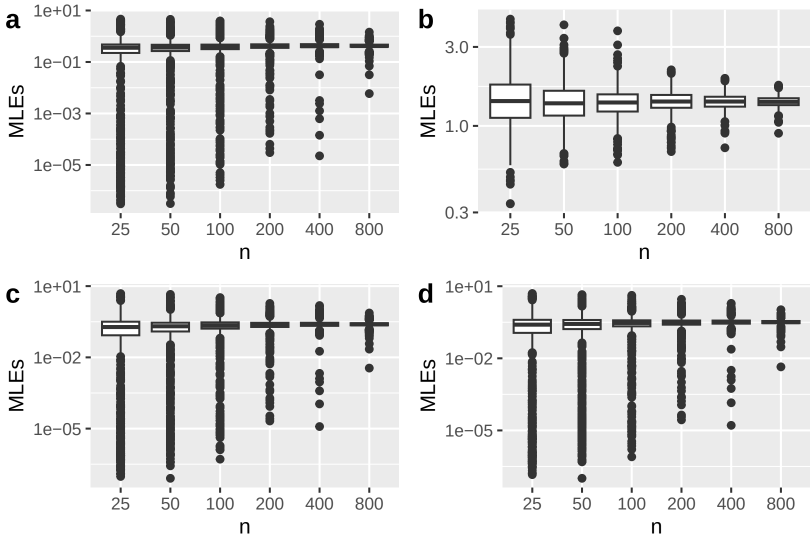

Figure 1: Sample size (x axis) against MLEs (y axis) boxplots for parameters (a)θ, (b)α, (c) ϕ_1 and (d) ϕ_2

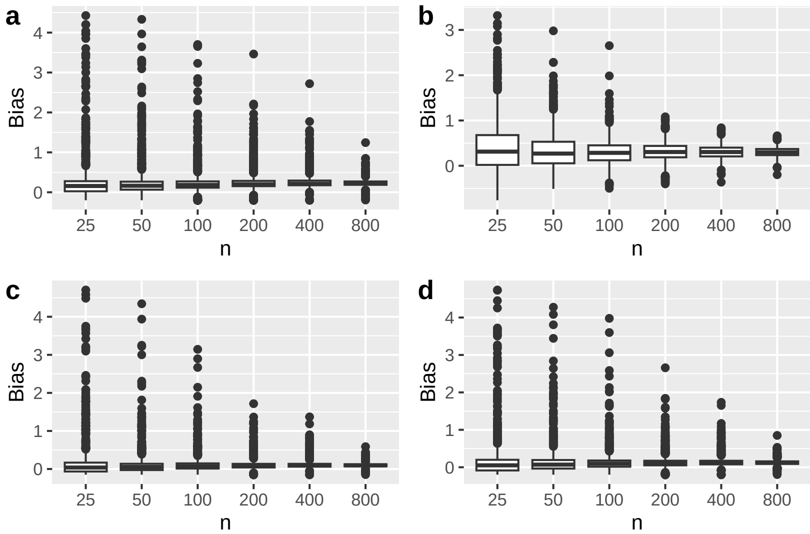

Figure 2: Sample size (x axis) against bias (y axis) boxplots for parameters (a)θ, (b)α, (c) ϕ_1 and (d) ϕ_2

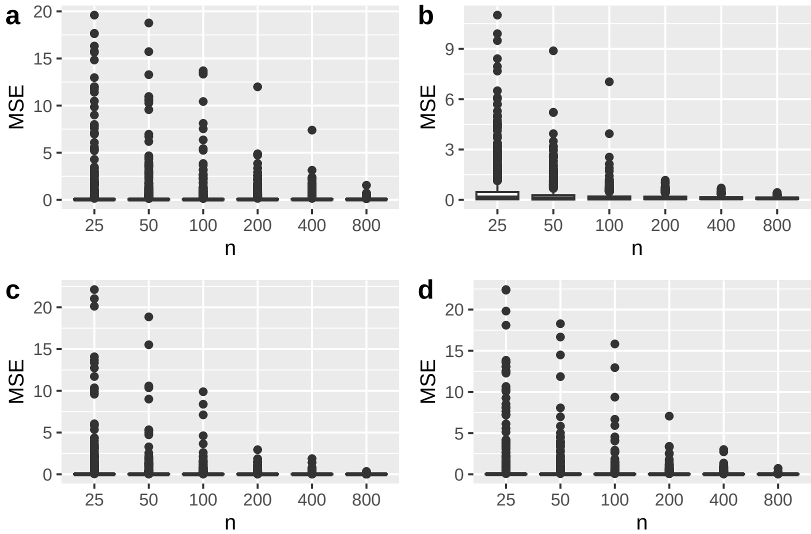

Figure 3: Sample size (x axis) against MSEs (y axis) for parameters (a)θ, (b)α, (c) ϕ_1 and (d) ϕ_2

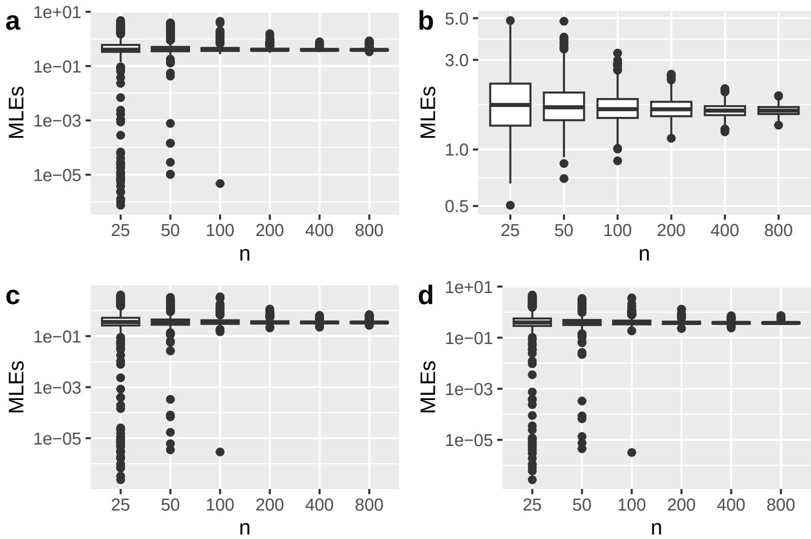

Figure 4: Sample size (x axis) against MLEs (y axis) boxplots for parameters (a)θ, (b)α, (c) ϕ_1 and (d) ϕ_2

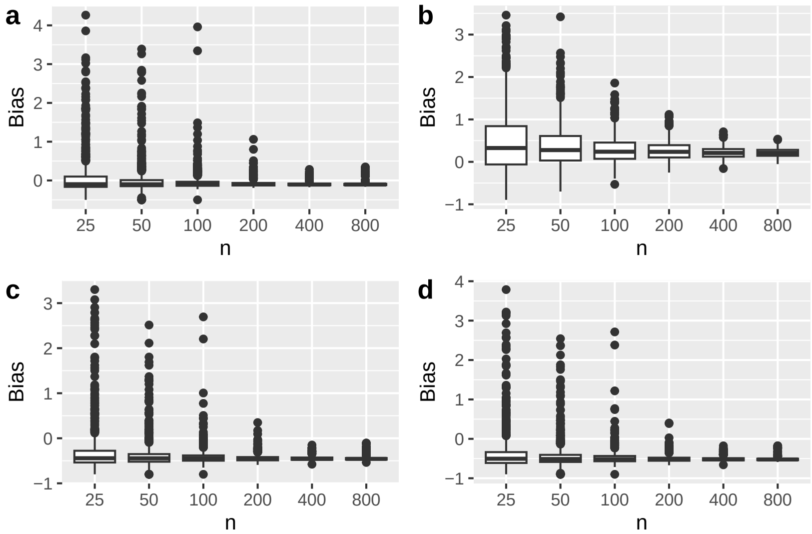

Figure 5: Sample size (x axis) against bias (y axis) boxplots for parameters (a)θ, (b)α, (c) ϕ_1 and (d) ϕ_2

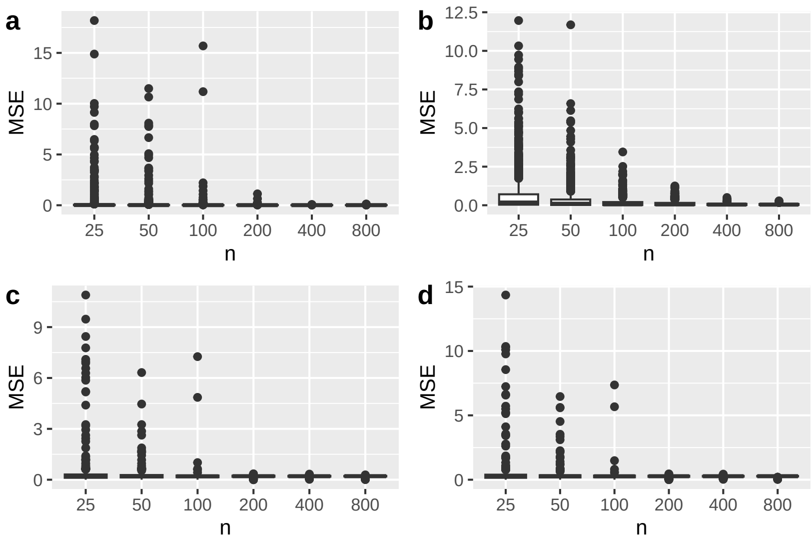

Figure 6: Sample size (x axis) against MSE (y axis) boxplots for parameters (a)θ, (b)α, (c) ϕ_1 and (d) ϕ_2

3.3. The BINAR(1) process with paired BPGL innovations

Let $\boldsymbol{Y_t} = (Y_{t,1}, Y_{t,2}); t = 1, 2, ...$ defines a BINAR(1) process where

$$Y_{t,1} = p_1 \circ Y_{t-1,1} + \epsilon_{t,1}$$
$$Y_{t,2} = p_2 \circ Y_{t-1,2} + \epsilon_{t,2}; \quad p_i \in (0,1), i = 1, 2.$$

Now suppose the innovation vector $\boldsymbol{\epsilon_t} = (\epsilon_{t,1}, \epsilon_{t,2})$ have the BPGL$(\theta, \alpha, \phi_1, \phi_2)$ and $\epsilon_{t,k}$ is independent from $Y_{s;j}, j = 1, 2$ for each t and s, $s < t$. Also, innovations are independent of the counting series in the binomial thinning operator $\circ$. Then the resulting $\boldsymbol{Y_t} = (Y_{t,1}, Y_{t,2}); t = 1, 2, ...$ will have the BINAR(1) process with the BPGL innovation denoted by BINAR(1)BPGL and the below-mentioned proposition defines some of its properties.

Proposition 3. *Suppose the bivariate random vector $\boldsymbol{Y_t} = (Y_{t,1}, Y_{t,2}); t = 1, 2, ...$ follows the BINAR(1)BPGL, then*

a) for $i = 1, 2$, the mean, variance and DI of $Y_{t,i}$,

$$\mathrm{E}(Y_{t,i}) = \frac{(\alpha\theta + \alpha - \theta)\phi_i}{(1 - p_i)\theta(\theta + 1)},$$

$$\mathrm{Var}(Y_{t,i}) = \frac{\phi_i}{1 - p_i^2}\left(p_i\left(\frac{\alpha}{\theta} - \frac{1}{(\theta+1)}\right) + \frac{\alpha}{\theta^2(\theta+1)} - \frac{1}{\theta(\theta+1)^2} + \phi_i\left(\frac{\alpha}{\theta^2} - \frac{1}{(\theta+1)^2}\right)\right)$$

and

$$\mathrm{DI}(Y_{t,i}) = 1 + \frac{\phi_i\left(\alpha(\theta+1)^2 - \theta^2\right)}{\theta(\theta+1)(p_i+1)((\alpha-1)\theta+\alpha)}.$$

implying each of the $Y_{t,i}$ is over-dispersed marginally.

b) the conditional mean and the conditional variance of components of the process for $i = 1, 2$,

$$\mathrm{E}\left(Y_{t,i} \mid Y_{t-1,i}\right) = p_i X_{t-1} + \frac{\phi_i(1 + (\alpha-1)(\theta+1))}{\theta(1+\theta)}$$

and

$$\mathrm{Var}\left(Y_{t,i} \mid Y_{t-1,i}\right) = p_i(1-p_i)X_{t-1} + \phi_i\left(\frac{\alpha}{\theta^2(\theta+1)} - \frac{1}{\theta(\theta+1)^2}\right) + \phi_i^2\left(\frac{\alpha}{\theta^2} - \frac{1}{(\theta+1)^2}\right).$$

c) the covariance of $Y_{t,1}$ and $Y_{t,2}$,

$$\mathrm{Cov}(Y_{t,1}, Y_{t,2}) = \frac{\phi_1\phi_2\left(\alpha(\theta+1)^2 - \theta^2\right)}{(1 - p_1p_2)\theta^2(\theta+1)^2}.$$

d) the conditional joint pmf of the process

$$P(Y_t = y_t \mid Y_{t-1} = y_{t-1}) = \sum_{k=0}^{u}\sum_{s=0}^{v} z_1(k)z_2(s)P\left(\epsilon_{t,1} = y_{t,1} - k, \epsilon_{t,2} = y_{t,2} - s\right), \quad (3.3)$$

where $u = min(y_{t,1}; y_{t-1,1})$, $v = min(y_{t,2}; y_{t-1,2})$,

$$z_1(k) = \binom{y_{t-1,1}}{k} p_1^k (1-p_1)^{y_{t-1,1}-k},$$
$$z_2(s) = \binom{y_{t-1,2}}{s} p_2^s (1-p_2)^{y_{t-1,2}-s}$$

and $P\left(\epsilon_{t,1} = y_{t,1} - k, \epsilon_{t,2} = y_{t,2} - s\right)$ is given by substituting x_1 with $y_{t-1,1} - k$ and x_2 with $y_{t-1,2} - s$ in (3.1).

Also, the stationary condition for the model is that $0 < p_i < 1$ (Khan, Oncel Cekim, and Ozel 2020) and $\mathrm{E}(Y_{t,i})$, $\mathrm{Var}(Y_{t,i})$ for $i = 1, 2$ and $\mathrm{Cov}(Y_{t,1}, Y_{t,2})$ does not depend on t and $\mathrm{Var}(Y_{t,i})$ is finite under the conditions mentioned.

Remark 3.3. If $\alpha = 2$, then the conditional joint pmf of the BINAR(1)BPGL process reduces to that of the BINAR(1) process with BBPL innovation (BINAR(1)BBPL).

3.4. Estimation of parameters of the BINAR(1)BPGL process

Suppose $\{Y_{t,i}, t = 1,\ 2, ..., n, i = 1,\ 2\}$ is a random sample of size n taken from the BINAR(1)BPGL process. Here the CML approach is used to estimate the parametric vector $\Theta = (\theta, \alpha, \phi_1, \phi_2, p_1, p_2)$ of the BINAR(1)BPGL process. The conditional log-likelihood function of the BINAR(1)BPGL process is

$$\ell(\Theta) = \sum_{t=2}^{n} \log\left[P\left(\boldsymbol{y_t} \mid \boldsymbol{y_{t-1}}\right)\right], \tag{3.4}$$

where $\Theta = (\theta, \alpha, \phi_1, \phi_2, p_1, p_2)$ is the unknown parametric vector to be estimated and $P\left(\boldsymbol{y_t} \mid \boldsymbol{y_{t-1}}\right)$ is given by (3.3). The CML estimators are obtained by maximizing (3.4). Furthermore, the consistency and asymptotic normality of the CML estimators under standard regularity conditions are demonstrated in Andersen (1970) and Bu and McCabe (2008). That is, the CML estimator of Θ, $\hat{\Theta}$ is such that, $(\hat{\Theta} - \Theta) \sim \mathrm{N}(0,\ I^{-1})$, where I^{-1} is the inverse of the Fisher information matrix (Freeland and McCabe (2004)). Here we used the quasi-Newton approach, the BFGS algorithm available in the `optim` function of the R programming language to find the CML estimators for the parameters in the BINAR(1)BPGL process.

3.5. Simulation study for the BINAR(1)BPGL process

The CML estimates (CMLEs) obtained for the unknown parameters of the BINAR(1)BPGL process is assessed through a simulation study. Hence N=1000 samples each of sizes n=25, 50, 100 are taken for two sets of parametric values ($\theta = 0.2,\ \alpha = 1.1,\ \phi_1 = 1,\ \phi_2 = 2,\ p_1 = 0.5,\ p_2 = 0.2$) and ($\theta = 0.5,\ \alpha = 1.3,\ \phi_1 = 1.7,\ \phi_2 = 2.3,\ p_1 = 0.6,\ p_2 = 0.3$). For each n, CMLEs, its bias and MSEs were calculated and reported through boxplots. The simulation results are illustrated in Figures 7, 8, 9, 10, 11 and 12. That is, for the set of parameter values, $\theta = 0.2,\ \alpha = 1.1,\ \phi_1 = 1,\ \phi_2 = 2,\ p_1 = 0.5\ p_2 = 0.2$, boxplots are illustrated from Figures 7 to 9 and for the set of parameter values, $\theta = 0.5,\ \alpha = 1.3,\ \phi_1 = 1.7,\ \phi_2 = 2.3,\ p_1 = 0.6,\ p_2 = 0.3$, boxplots are from Figures 10 to 12. To make the results more comparable and interpretable, $\log 10-$scale of the y-axis for plots in Figures 7 and 10 is used. The plotted figures show that for the CMLEs, their bias and MSE corresponding to each parameter decrease as the sample size increases.

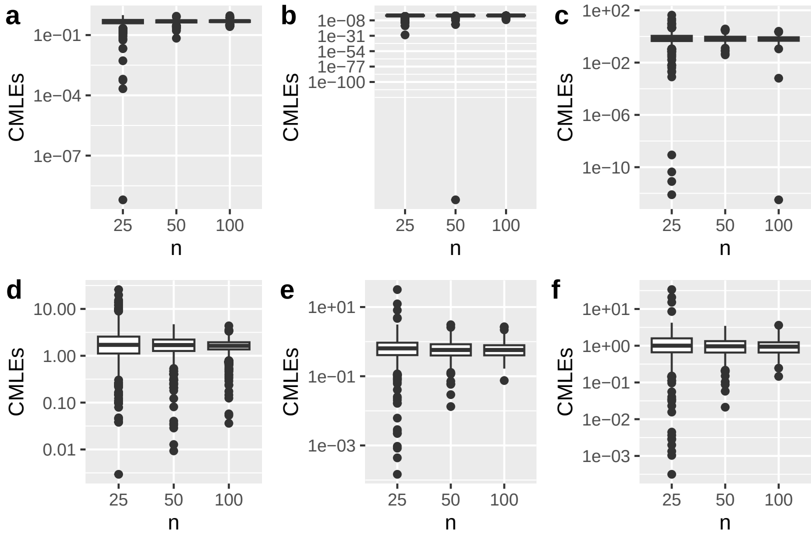

Figure 7: Sample size (x axis) against CMLEs (y axis) boxplots for parameters (a)θ, (b)α, (c) ϕ_1, (d) ϕ_2, (e) p_1 and (f) p_2

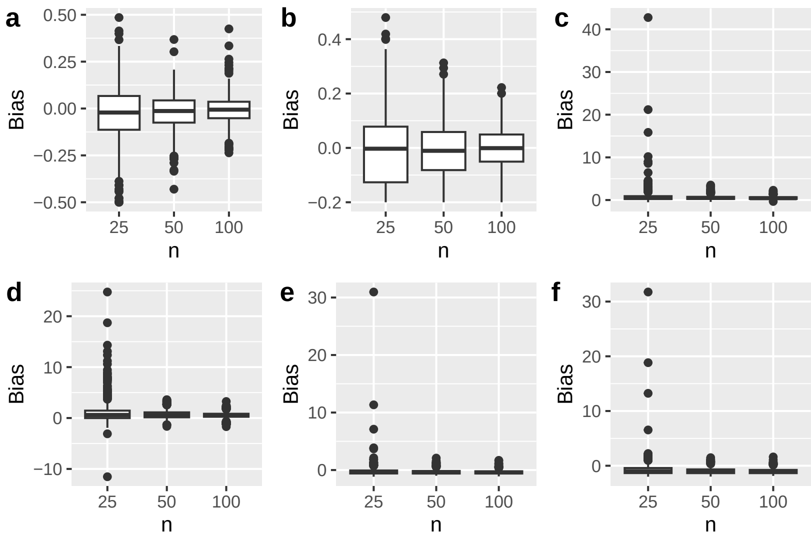

Figure 8: Sample size (x axis) against bias (y axis) boxplots for parameters (a)θ, (b)α, (c) ϕ_1, (d) ϕ_2, (e) p_1 and (f) p_2

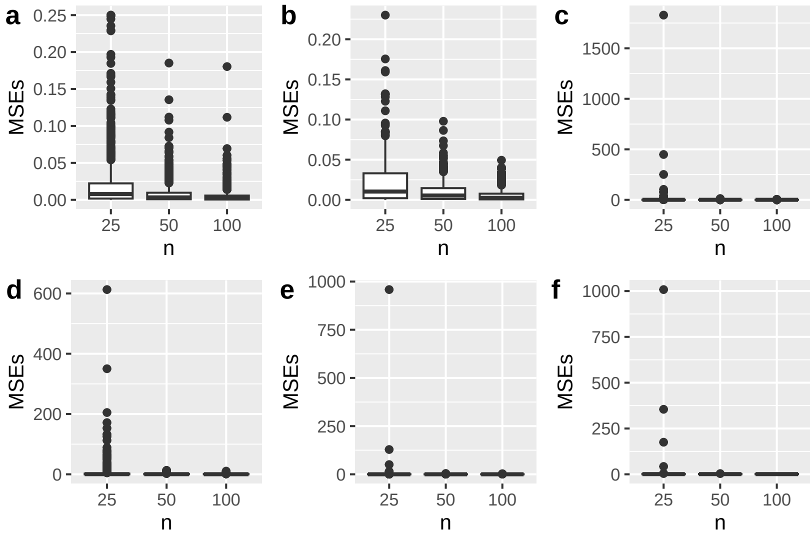

Figure 9: Sample size (x axis) against MSEs (y axis) boxplots for parameters (a)θ, (b)α, (c) ϕ_1, (d) ϕ_2, (e) p_1 and (f) p_2

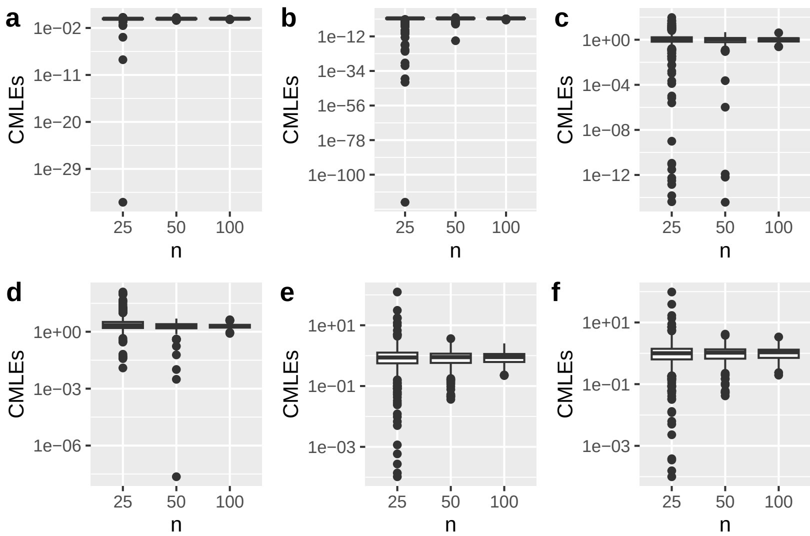

Figure 10: Sample size (x axis) against CMLEs (y axis) boxplots for parameters (a)θ, (b)α, (c) ϕ_1, (d) ϕ_2, (e) p_1 and (f) p_2

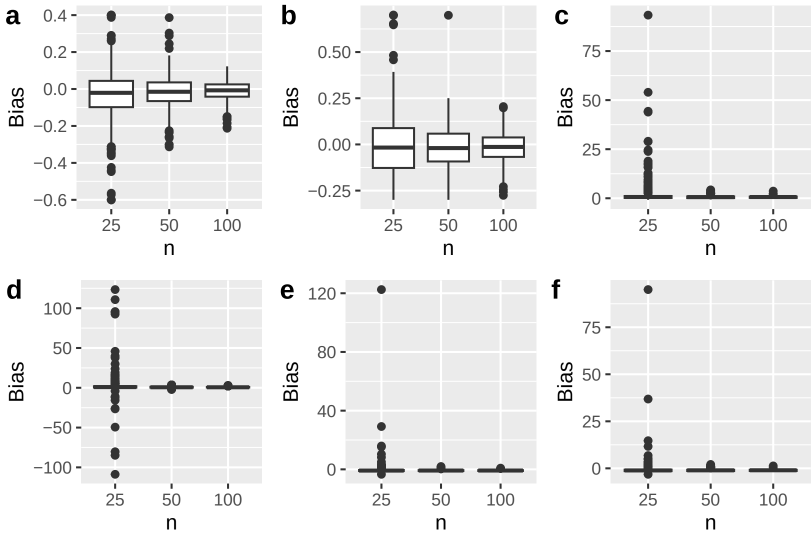

Figure 11: Sample size (x axis) against bias (y axis) boxplots for parameters (a)θ, (b)α, (c) ϕ_1, (d) ϕ_2, (e) p_1 and (f) p_2

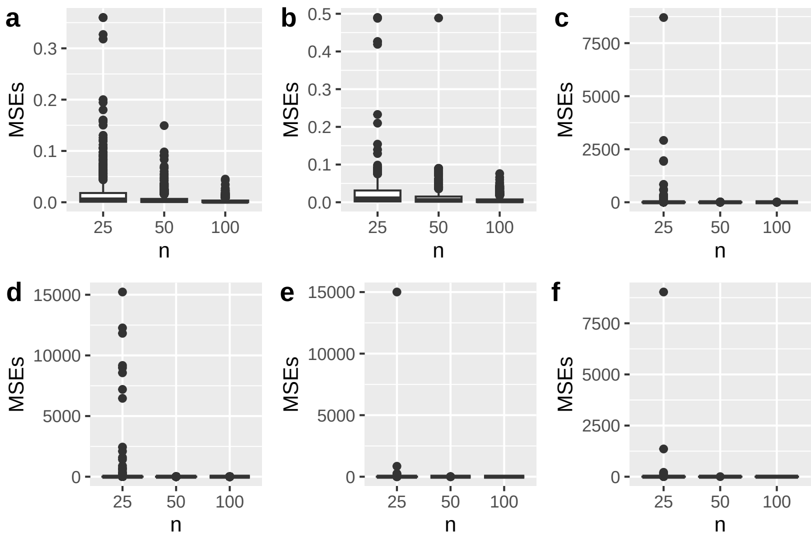

Figure 12: Sample size (x axis) against MSEs (y axis) boxplots for parameters (a)θ, (b)α, (c) ϕ_1, (d) ϕ_2, (e) p_1 and (f) p_2

4. The SPGL distribution and the corresponding BINAR(1) process

Sarmanov (Sarmanov 1966) introduced the Sarmanov family of distributions. Suppose X_1 and X_2 be two random variables each with pmf $P(X_1 = x_1)$ and $P(X_2 = x_2)$ and with supports defined on $A_1 \subseteq \mathbb{R}$ and $A_2 \subseteq \mathbb{R}$ respectively. Now let $q_i(x_i), i = 1, 2$ are bounded non-constant functions such as

$$\sum_{x_i=-\infty}^{\infty} q_i\left(x_i\right) f_i\left(x_i\right) = 0.$$

Then, the joint pmf for the Sarmanov family,

$$P\left(x_1 = x_1, x_2 = x_2\right) = P_1\left(x_1 = x_1\right) P_2\left(x_2 = x_2\right)\left[1 + \omega q_1\left(x_1\right) q_2\left(x_2\right)\right],$$

where the factor $\omega q_1(x_1)q_2(x_2)$ is a measure of departure of two variables X_1, X_2 from independence and ω is a real number that satisfies the condition

$$\left[1 + \omega q_1\left(x_1\right) q_2\left(x_2\right)\right] \geq 0, \text{ for all } x_1, x_2. \tag{4.1}$$

Depending on the choices for the functions $q_i(x)$, $i = 1, 2$, we can derive different cases. Here we use $q_i(x_i) = e^{-x_i} - L_i(1)$ $x_i = 0, 1, ...$ as mentioned in Ting Lee (1996), where $L_i(1)$ is the value of the Laplace transform of the marginal distribution evaluated at $s = 1$, that is

$$L(s) = E\left(e^{-sX}\right) = \sum_{x=0}^{\infty} e^{-sX} P(x),$$

where $P(.)$ is the marginal distribution. Hence, we derive the SPGL distribution having TPPGL as marginals. The Laplace transform for the TPPGL distribution,

$$\begin{aligned} L(s) &= \mathrm{E}(e^{-sx}) \\ &= \frac{\theta^\alpha}{(1+\theta)^{\alpha+1}} e^{-s}(1 - \frac{e^{-s}}{1+\theta})^{-\alpha}(e^s(2+\theta) - 1) \\ &= \frac{\theta^\alpha}{\theta+1} \frac{2+\theta-e^{-s}}{(1+\theta-e^{-s})^\alpha} \end{aligned}$$

and at s = 1,

$$L(1) = \frac{\theta^\alpha}{\theta+1} \frac{2+\theta-e^{-1}}{(1+\theta-e^{-1})^\alpha}.$$

Then the joint pmf for the Sarmanov bivariate distribution takes the form,

$$\begin{aligned} P\left(x_1 = x_1, x_2 = x_2\right) &= P_1\left(x_1 = x_1\right) P_2\left(x_2 = x_2\right) \\ &\quad \left[1 + \omega\left(e^{-x_1} - L_1(1)\right)\left(e^{-x_2} - L_2(1)\right)\right], \end{aligned}$$

where $L_i(.)$ is the Laplace transform for the i^{th} marginal, $i = 1, 2$.

Proposition 4. *The joint pmf of X_1 and X_2 each having the TPPGL marginals is,*

$$\begin{gathered} P(X_1 = x_1, X_2 = x_2) = \tfrac{\theta_1^\alpha \Gamma(x_1+\alpha-1)}{\Gamma(\alpha) x_1! (\theta_1+1)^{x_1+\alpha+1}} \left(x_1 + (\alpha-1)(\theta_1+2)\right) \times \\ \tfrac{\theta_2^\alpha \Gamma(x_2+\alpha-1)}{\Gamma(\alpha) x_2! (\theta_2+1)^{x_2+\alpha+1}} \left(x_2 + (\alpha-1)(\theta_2+2)\right) \times \\ \left[1 + \omega(e^{-X_1} - \tfrac{\theta_1^\alpha}{(\theta_1+1)e^{1-\alpha}} \tfrac{e(2+\theta_1)-1}{(e(1+\theta_1)-1)^\alpha})(e^{-X_2} - \tfrac{\theta_2^\alpha}{(\theta_2+1)e^{1-\alpha}} \tfrac{e(2+\theta_2)-1}{(e(1+\theta_2)-1)^\alpha})\right], \end{gathered} \tag{4.2}$$

where $x_1, x_2 = 0, 1, 2, ..., \theta_1, \theta_2 > 0, \alpha > 1$ and ω satisfies (4.1).

Remark 4.1. If $\alpha = 2$, then the joint pmf of the SPGL$(\theta_1, \theta_2, \alpha, \omega)$ reduces to that of the Sarmanov-based bivariate Poisson-Lindley (SBPL) distribution (Gomez-Deniz *et al.* (2012) and Zamani, Faroughi, and Ismail (2015)).

Therefore, the mean and variance $X_i,\ i = 1, 2,$

$$\begin{aligned} \mathrm{E}(X_i) &= \frac{1 + (\alpha - 1)(\theta_i + 1)}{\theta_i(1 + \theta_i)} \\ \text{and} \\ \mathrm{Var}(X_i) &= \frac{\alpha(1 + \theta_i)}{\theta_i^2} - \frac{2 + \theta_i}{(1 + \theta_i)^2}. \end{aligned}$$

The covariance between X_1 and X_2,

$$\mathrm{Cov}(X_1, X_2) = \omega u_1 u_2,$$

where $u_i = \mathrm{E}(X_i e^{-X_i}) - \mathrm{E}(X_i)L(1)$. For SPGL distribution,

$$\begin{aligned} u_i &= \frac{\theta_i^\alpha(1 - \alpha + e(\alpha(2 + \theta_i) - \theta_i - 1))}{e^{1-\alpha}(1 + \theta_i)(e(\theta_i + 1) - 1)^{\alpha+1}} - \frac{1 + (\alpha - 1)(\theta_i + 1)}{\theta_i(1 + \theta_i)} \frac{\theta_i^\alpha}{\theta_i + 1} \frac{2 + \theta_i - e^{-1}}{(1 + \theta_i - e^{-1})^\alpha} \\ &= \frac{(1 - e)\left(\alpha\left(\theta_i + 1\right)^2 \left(e\left(\theta_i + 2\right) - 1\right) - \theta_i\left(\theta_i + 2\right)\left(e\left(\theta_i + 1\right) - 1\right)\right)}{\left(\left(\theta_i + 1\right)^2 \left(e\left(\theta_i + 1\right) - 1\right)^{\alpha+1}\right) \mathrm{e}\theta_i^\alpha}. \end{aligned}$$

4.1. Estimation of parameters of the SPGL distribution

The ML method is used for the estimation of unknown parameters. Let $(X_{1i}, X_{2i}),\ i = 1, 2, ..., n$ be the observations of a random sample from SPGL$(\theta_1, \theta_2, \alpha, \omega)$. Then the log-likelihood function,

$$U(\lambda) = \sum_{i=1}^{n} \log(P(X_1 = x_{1i}, X_2 = x_{2i})), \tag{4.3}$$

where $\boldsymbol{\lambda} = (\theta_1, \theta_2, \alpha, \omega)$ and $P(X_1 = x_{1i}, X_2 = x_{2i})$ is the pmf of SPGL distribution defined in (4.2). (4.3) have to be maximized to find the estimators for $\boldsymbol{\lambda}$. The asymptotic normality and consistency properties of the ML estimators follow from the well-known Lindberg-Levy central limit theorem. Consequently, the ML estimators of $\boldsymbol{\lambda}$, $\hat{\boldsymbol{\lambda}}$ is such that, $(\hat{\boldsymbol{\lambda}} - \boldsymbol{\lambda}) \sim$ N(0,I^{-1}), where I^{-1} is the inverse of the Fisher information matrix. Here we have used the quasi-Newton approach, the BFGS algorithm available in the `optim` function of the R programming language to find the ML estimators for the unknown parameters.

4.2. Simulation study for the SPGL distribution

The MLEs obtained for the unknown parameters of the SPGL distribution are assessed through a simulation study. Therefore, N=1000 samples each of sizes n=25, 50,100, 200, 400, 800 are taken for two sets of parametric values ($\theta_1 = 0.5,\ \theta_2 = 0.4,\ \alpha = 1.5,\ \omega = 0.9$) and ($\theta_1 = 1.4,\ \theta_2 = 1.7,\ \alpha = 2,\ \omega = 0.1$). For each n, MLEs, bias and MSEs were calculated and are illustrated through boxplots. The simulation results are plotted in Figures 13, 14, 15, 16, 17 and 18. That is, for the set of parameter values, $\theta_1 = 0.5,\ \theta_2 = 0.4,\ \alpha = 1.5,\ \omega = 0.9$, the simulation results are plotted from Figures 13 to 15, and for $\theta_1 = 1.4,\ \theta_2 = 1.7,\ \alpha = 2,\ \omega = 0.1$, the plotted figures are from Figures 16 to 18. To make the results more comparable and interpretable, $\log 10-$scale of the y-axis for plots in Figures 13 and 16 is used.

The plotted figures show that for the MLEs, their bias and MSEs corresponding to each parameter decrease as the sample size increases.

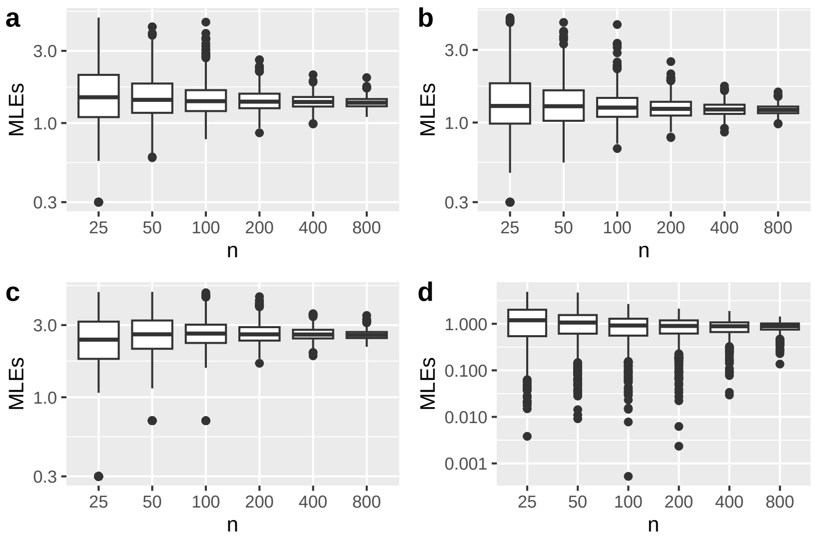

Figure 13: Sample size (x axis) against MLEs (y axis) boxplots for parameters (a)θ_1, (b)θ_2, (c) α and (d) ω

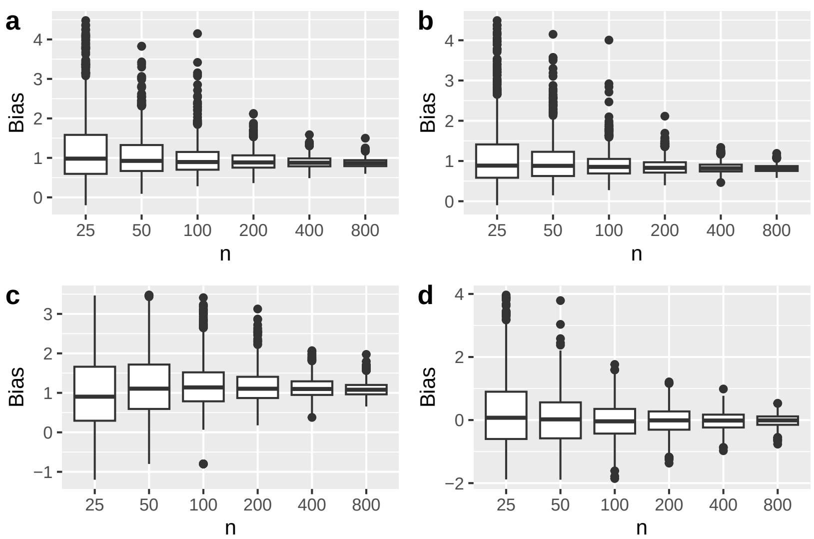

Figure 14: Sample size (x axis) against bias (y axis) boxplots for parameters (a)θ_1, (b)θ_2, (c) α and (d) ω

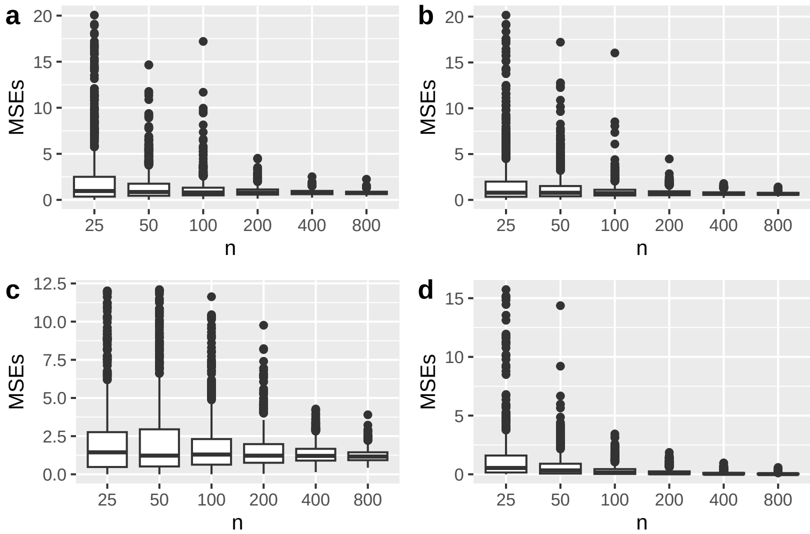

Figure 15: Sample size (x axis) against MSEs (y axis) boxplots for parameters (a)θ_1, (b)θ_2, (c) α and (d) ω

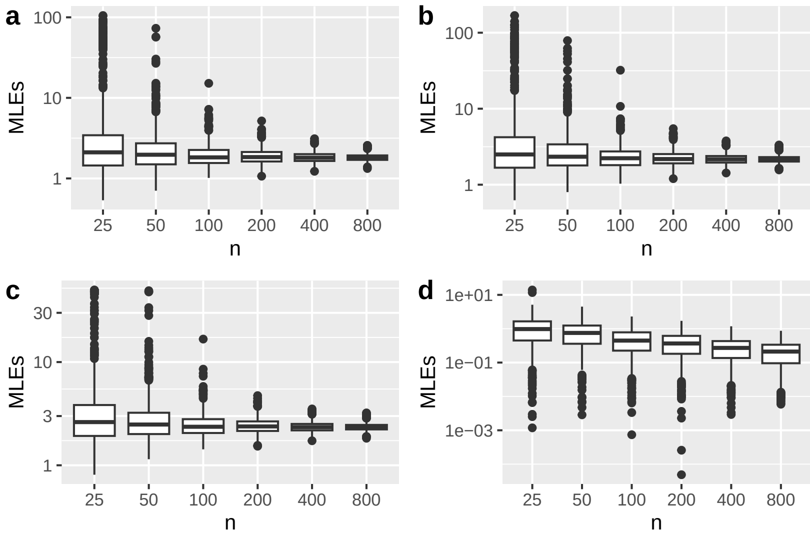

Figure 16: Sample size (x axis) against MLEs (y axis) boxplots for parameters (a)θ_1, (b)θ_2, (c) α and (d) ω

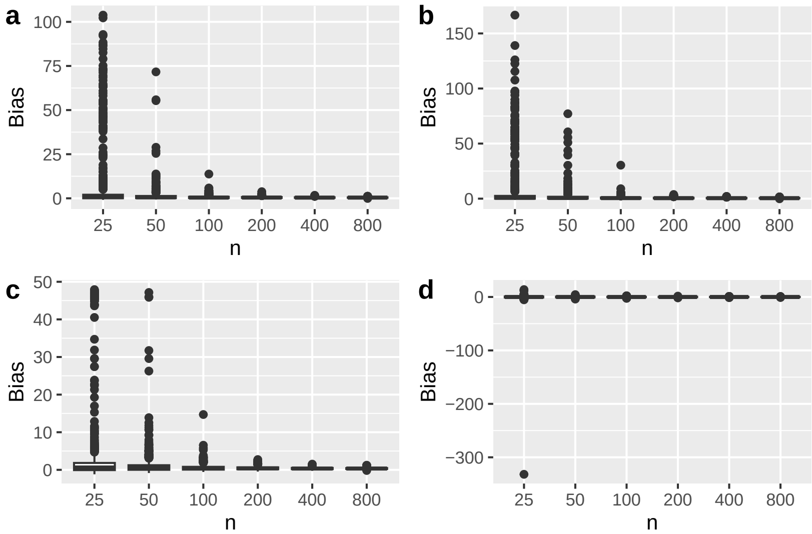

Figure 17: Sample size (x axis) against bias (y axis) boxplots for parameters (a)θ_1, (b)θ_2, (c) α and (d) ω

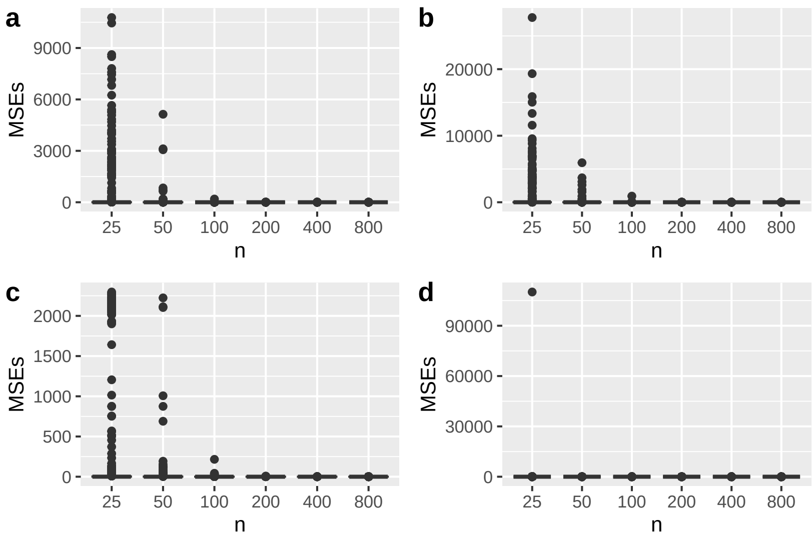

Figure 18: Sample size (x axis) against MSEs (y axis) boxplots for parameters (a)θ_1, (b)θ_2, (c) α and (d) ω

4.3. The BINAR(1) process with paired SPGL innovations

Let us define the bivariate random vector $\mathbf{Y_t} = (Y_{t,1}, Y_{t,2}); t = 1, 2, ...$ having BINAR(1) process as in section 3.3 but the innovation vector $\boldsymbol{\epsilon_t} = (\epsilon_{t,1}, \epsilon_{t,2})$ possess SPGL(θ_1, θ_2, α, ω) and the assumptions mentioned in section 3.3 holds. Then the proposition mentioned below explains some of its properties.

Proposition 5. *Suppose the bivariate random vector $\mathbf{Y_t} = (Y_{t,1}, Y_{t,2}); t = 1, 2, ...$ follows the BINAR(1)SPGL, then*

a) for i=1,2, the mean, variance and DI of $Y_{t,i}$,

$$\begin{aligned}
\mathrm{E}(Y_{t,i}) &= \frac{1+(\alpha-1)(\theta_i+1)}{(1-p_i)\theta_i(1+\theta_i)}, \\
\mathrm{Var}(Y_{t,i}) &= \frac{\alpha(\theta_i+1)^2(\theta_i+\theta_i p_i+1)-\theta_i^2(\theta_i+\theta_i p_i+p_i+2)}{\theta_i^2(\theta_i+1)^2\left(1-p_i^2\right)} \\
&\textit{and} \\
\mathrm{DI(Y_{t,i})} &= 1+\frac{1-\alpha}{((\alpha-1)\theta_i+\alpha)(1+p_i)}+\frac{1}{\theta_i(1+p_i)}+\frac{1}{(\theta_i+1)(1+p_i)},
\end{aligned}$$

implying $Y_{t,i}, i = 1, 2$ are over-dispersed.

b) the conditional mean and the conditional variance of components of the process for i=1,2,

$$\mathrm{E}\left(Y_{t,i} \mid Y_{t-1,i}\right) = p_i X_{t-1} + \frac{(1+(\alpha-1)(\theta_i+1))}{(1-p_i)\theta_i(1+\theta_i)}$$

and

$$\mathrm{Var}\left(Y_{t,i} \mid Y_{t-1,i}\right) = p_i(1-p_i)X_{t-1} + \frac{\alpha(1+\theta_i)}{\theta_i^2} - \frac{2+\theta_i}{(1+\theta_i)^2}.$$

c) the covariance of $\epsilon_{t,1}$ and $\epsilon_{t,2}$,

$$\begin{aligned}
\mathrm{Cov}(\epsilon_{t,1}, \epsilon_{t,2}) &= \omega\frac{(1-e)\left(\alpha\left(\theta_1+1\right)^2\left(e\left(\theta_1+2\right)-1\right)-\theta_1\left(\theta_1+2\right)\left(e\left(\theta_1+1\right)-1\right)\right)}{\left(\left(\theta_1+1\right)^2\left(e\left(\theta_1+1\right)-1\right)^{\alpha+1}\right)e\theta_1^\alpha} \\
&\quad\times\frac{(1-e)\left(\alpha\left(\theta_2+1\right)^2\left(e\left(\theta_2+2\right)-1\right)-\theta_2\left(\theta_2+2\right)\left(e\left(\theta_2+1\right)-1\right)\right)}{\left(\left(\theta_2+1\right)^2\left(e\left(\theta_2+1\right)-1\right)^{\alpha+1}\right)e\theta_2^\alpha}.
\end{aligned}$$

d) the covariance of two components of the BINAR(1)SPGL process, is obtained as

$$\begin{aligned}
\mathrm{Cov}(Y_{t,1}, Y_{t,2}) &= \frac{\omega}{(1-p_1p_2)}\frac{(1-e)\left(\alpha(\theta_1+1)^2(e(\theta_1+2)-1)-\theta_1(\theta_1+2)(e(\theta_1+1)-1)\right)}{((\theta_1+1)^2(e(\theta_1+1)-1)^{\alpha+1})e\theta_1^\alpha} \\
&\quad\times\frac{(1-e)\left(\alpha(\theta_2+1)^2(e(\theta_2+2)-1)-\theta_2(\theta_2+2)(e(\theta_2+1)-1)\right)}{((\theta_2+1)^2(e(\theta_2+1)-1)^{\alpha+1})e\theta_2^\alpha}.
\end{aligned}$$

e) the conditional joint pmf of the BINAR(1)SPGL process can be obtained by using (3.3) except that $P(\epsilon_{t,1} = \epsilon_1, \epsilon_{t,2} = \epsilon_2)$ is swaped by (4.2).

Remark 4.2. If $\alpha = 2$, then the conditional joint pmf of the BINAR(1)SPGL process reduces to that of the BINAR(1) process with SBPL innovation (BINAR(1)SBPL) by Khan *et al.* (2020).

Also, the stationary condition for the model is that $0 < p_i < 1$ and $\mathrm{E}(Y_{t,i})$, $\mathrm{Var}(Y_{t,i})$ for $i = 1, 2$ and $\mathrm{Cov}(Y_{t,1}, Y_{t,2})$ does not depend on t and $\mathrm{Var}(Y_{t,i})$ is finite under the conditions mentioned.

4.4. Estimation of parameters of the BINAR(1)SPGL process

We here use the method of CML to estimate the parameters. Suppose $\{Y_{t,i}, t = 1,\ 2, ..., n, i = 1,\ 2\}$ is a random sample of size n taken from the BINAR(1)SPGL process. Then the conditional log-likelihood function is such that,

$$\ell'(\Theta') \;=\; \sum_{t=2}^{n} \log\left[P\left(\boldsymbol{y_t} \mid \boldsymbol{y_{t-1}}\right)\right], \tag{4.4}$$

where $\Theta' = (\theta_1, \theta_2, \alpha, \omega, p_1, p_2)$ is the unknown parametric vector to be estimated and $P\left(\boldsymbol{y_t} \mid \boldsymbol{y_{t-1}}\right)$ is obtained by substituting the joint pmf of $(\epsilon_{t,1}, \epsilon_{t,2})$ by (4.2) in (3.3) . The CML estimators are obtained by maximizing (4.4). Also, the consistency and asymptotic normality of the CML estimators under standard regularity conditions are demonstrated in Andersen (1970) and Bu and McCabe (2008). That is, the CML estimator for Θ', $\hat{\Theta}'$ is such that, $(\hat{\Theta}' - \Theta') \sim$ N(0,I^{-1}), where I^{-1} is the inverse of the Fisher information matrix (Freeland and McCabe (2004)). Here we have used the quasi-Newton approach, the BFGS algorithm available in `optim` function of the R programming language to find the CML estimators for the parameters in the BINAR(1)SPGL process.

4.5. Simulation study for the BINAR(1)SPGL process

The CMLEs obtained for the unknown parameters of the BINAR(1)SPGL process is assessed through a simulation study. Hence N=1000 samples each of sizes n=25, 50, 100 are taken for two sets of parametric values ($\theta_1 = 0.7,\ \theta_2 = 0.6,\ \alpha = 1.1,\ \omega = 0.2,\ p_1 = 0.1,\ p_2 = 0.7$) and ($\theta_1 = 0.1,\ \theta_2 = 0.2,\ \alpha = 1.5,\ \omega = 0.8,\ p_1 = 0.4,\ p_2 = 0.9$). For each n, bias and MSEs were calculated. The simulation results are given as boxplots through the Figures 19, 20, 21, 22, 23 and 24.
That is, for the first parameter set, the figures are given from 19 to 21, and for the second parameter set, the figures are given from 22 to 24. To make the results more comparable and interpretable, log 10−scale of the y-axis for plots in Figures 19 and 22 is used. The plotted

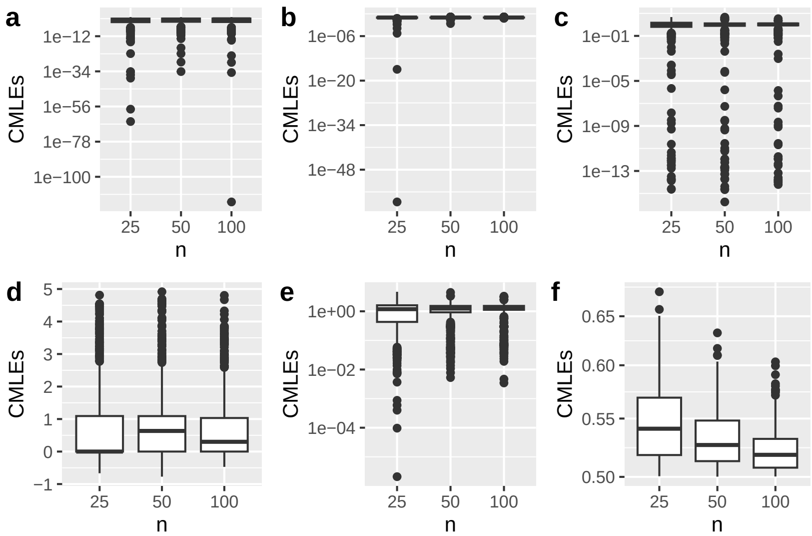

Figure 19: Sample size (x axis) against CMLEs (y axis) boxplots for parameters (a)θ_1, (b)θ_2, (c) α, (d) ω, (e) p_1 and (f) p_2

figures show that the CMLEs, their bias and MSE corresponding to each parameter decrease as the sample size increases.

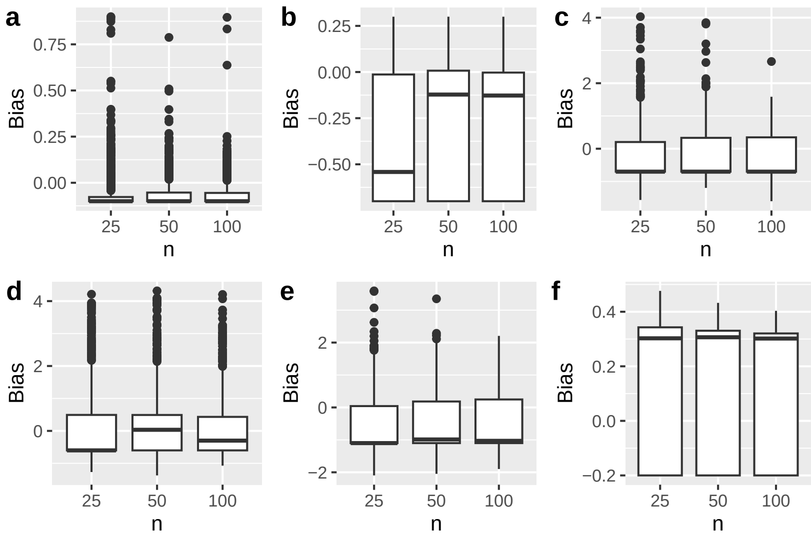

Figure 20: Sample size (x axis) against bias (y axis) boxplots for parameters (a)θ_1, (b)θ_2, (c) α, (d) ω, (e) p_1 and (f) p_2

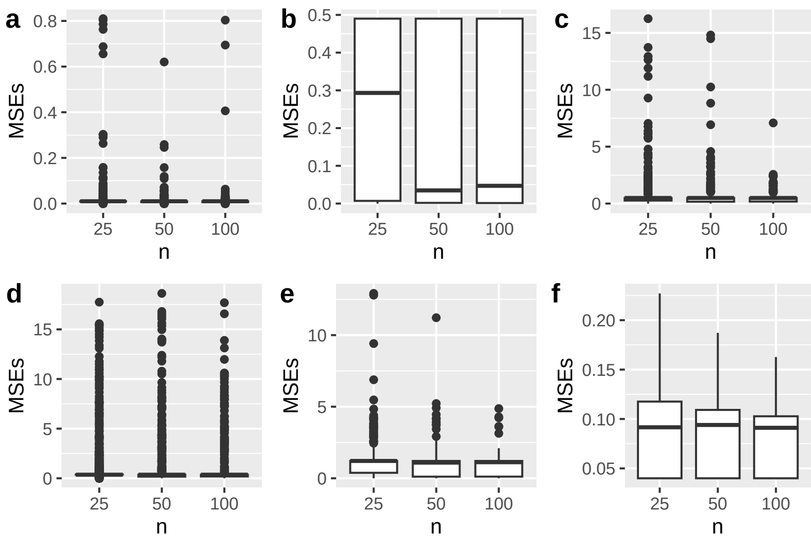

Figure 21: Sample size (x axis) against MSE (y axis) boxplots for parameters (a)θ_1, (b)θ_2, (c) α, (d) ω, (e) p_1 and (f) p_2

5. Empirical study

In this section we do some testing procedures for the new bivariate distributions and the BINAR(1)s, that is, they are compared with the corresponding models when $\alpha = 2$ for some real datasets. Also, the empirical importance of the proposed two BINAR(1) processes is proved using a real data set.

5.1. Football data

Here we consider the real bivariate count data which is reported in Lee and Cha (2015), the Italian Series A football match score played between two Italian football giants "ACF Fiorentina " and "Juventus" during the period 1996 to 2011. For comparison purposes, we do two tests of hypothesis for both the discussed bivariate distributions against the corresponding situations when $\alpha = 2$. That is, for both cases, we test the hypothesis $H_0 : \alpha = 2 \;\; v/s \;\; H_1 : \alpha \neq 2$.

For the first case, to test H_0 : BBPL distribution versus H_1 : BPGL distribution, we use the

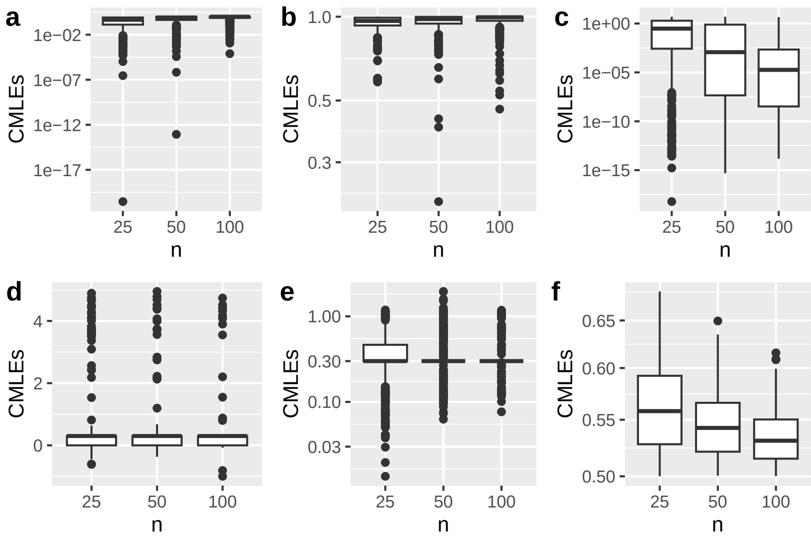

Figure 22: Sample size (x axis) against CMLEs (y axis) boxplots for parameters (a)θ_1, (b)θ_2, (c) α, (d) ω, (e) p_1 and (f) p_2

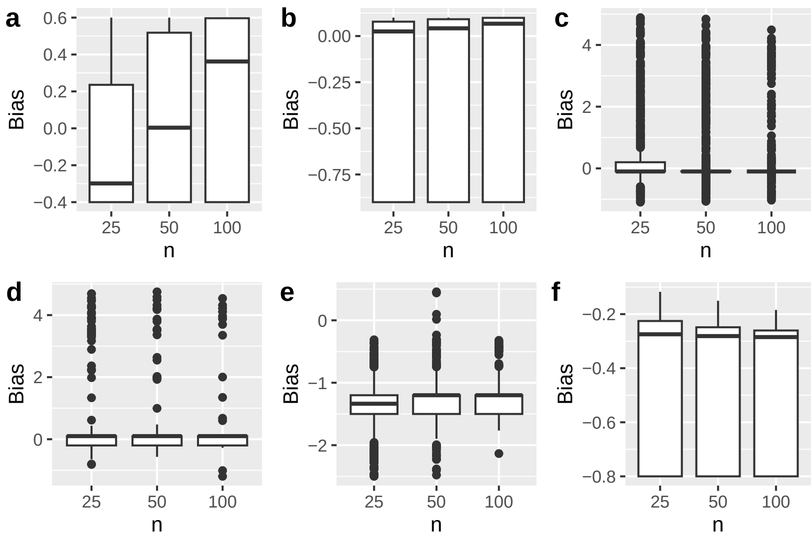

Figure 23: Sample size (x axis) against bias (y axis) boxplots for parameters (a)θ_1, (b)θ_2, (c) α, (d) ω, (e) p_1 and (f) p_2

likelihood ratio test statistic whose value is 6.77631 (p-value $= 0.0092$). Hence, the model BBPL is rejected in favour of the proposed model BPGL at any level > 0.0092.

For the second case, to test H_0 : BSPL distribution versus H_1 : SPGL distribution, we use the likelihood ratio test statistic whose value is 10.6496 (p-value $= 0.0011$). Hence, the model BSPL is rejected in favour of the proposed model SPGL at any level > 0.0011.

5.2. Criminal data

Here, the empirical importance of the proposed BINAR(1)s are studied in detail using real bivariate time series count data. The data set we used concerns the criminal records of drug activities (CDRUGS) and shooting activities (CSHOTS) in the 12th police car beat in Pittsburgh for the period from January 1990 to December 2001 downloaded from the Pittsburgh police departments in the file PghCarBeat.csv (available from the website: http://www.forecastingprinciples.com/index.php/data). The means(variances) of CDRUGS and CSHOTS are 5.1736 (13.1794) and 5.7569 (14.2412), respectively, proving clear over-

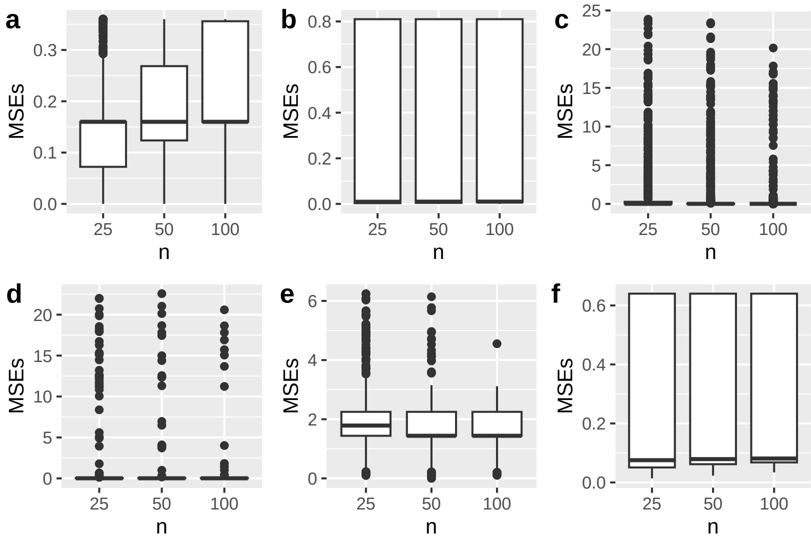

Figure 24: Sample size (x axis) against MSE (y axis) boxplots for parameters (a)θ_1, (b)θ_2, (c) α, (d) ω, (e) p_1 and (f) p_2

dispersion. The plots of the autocorrelation function (ACF) and partial autocorrelation function (PACF) of the CDRUGS data set are given in Figure 25 and ACF and PACF plots of the CSHOTS data set are given in Figure 26. The PACF plot makes it clear that this data can be used since only the first lag is significant. Also, Figure 27 denotes the cross-correlation function (CCF) plot and it clearly indicates that there is a significant cross-correlation in lag two between both the time series.

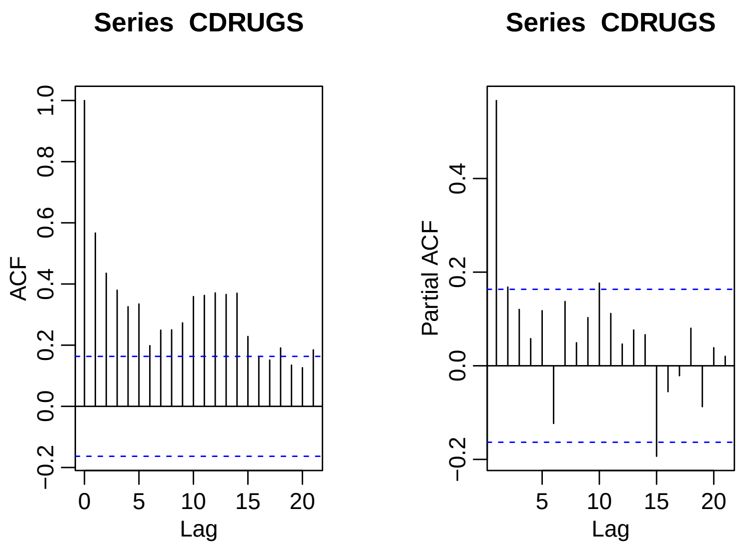

Figure 25: The ACF and PACF of the CDRUG data

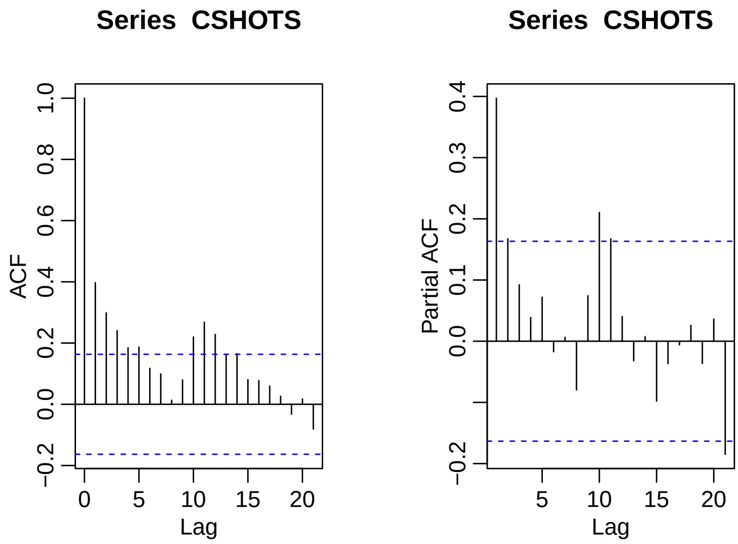

Figure 26: The ACF and PACF of the CSHOTS data

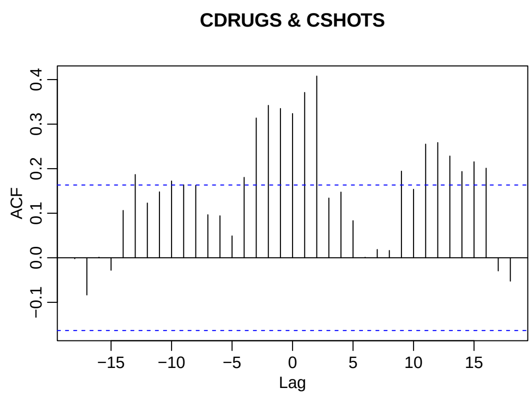

Figure 27: CCF Plots of the CDRUGS and CSHOTS datasets

First we consider testing of the hypothesis,

$$H_0 : \alpha = 2 \quad v/s \quad H_1 : \alpha \neq 2$$

for both the BINAR(1)s.

For the first case, for testing H_0 : BINAR(1)BBPL process versus H_1 : BINAR(1)BPGL process, we use the likelihood ratio test statistic whose value is 380.6648 (p-value $= 0.000$). Hence, the model BINAR(1)BBPL is rejected in favor of the proposed process BINAR(1)BPGL at any level > 0.000.

For the second case, for testing H_0 : BINAR(1)BSPL process versus H_1 : BINAR(1)SPGL process, we use the likelihood ratio test statistic whose value is 54.04063 (p-value $= 1.963 \times 10^{-13}$). Hence, the model BINAR(1)BSPL is rejected in favour of the proposed model BINAR(1)SPGL at any level $> 1.963 \times 10^{-13}$.

Now we prove the applicability of the proposed BINAR(1) models by comparing it with some other BINAR(1) models by means of model adequacy measures. Hence we compare the BINAR(1)BPGL and BINAR(1)SPGL processes with the BINAR(1) bivariate Poisson weighted exponential (BINAR(1)BPWE) process and the BINAR(1) Sarmanov Poisson weighted exponential (BINAR(1)SPWE) process by Sajjadnia, Sharafi, Khan, and Soobhug (2021), the BINAR(1)BP process and the BINAR(1) negative binomial (BINAR(1)BNB) process by Pedeli and Karlis (2011) and the BINAR(1)BSPL process. In Table 1, estimates of the parameters, Log-Likelihood(L), Akaike information criterion (AIC), Bayesian information criterion (BIC), and Root Mean Square Errors (RMSEs) of both the series for all the models described above are calculated. The RMSEs represent the sum of squared differences between true values and one-step conditional expectations. As Ristic and Popovic (2019) suggested, the standardized

Table 1: Estimates and model adequacy measures of the BINAR(1) models

Model	Estimates	L	AIC	BIC	RMSEs
BINAR(1)SPGL	$\theta_1 = 0.7870$ $\theta_2 = 0.6257$ $\alpha = 2.8658$ $p_1 = 0.4076$ $p_2 = 0.3198$ $\omega = 0.9851$	713.6356	1439.2711	1457.0900	3.0358 3.4641
BINAR(1)BPGL	$\theta = 2.9967$ $\alpha = 4.9462$ $\phi_1 = 2.3869$ $\phi_2 = 3.1247$ $p_1 = 0.3574$ $p_2 = 0.2470$	724.8362	1461.6725	1479.4914	3.0754 3.4975
BINAR(1)BPWE	$\mu_1 = 2.3271$ $\mu_2 = 3.4914$ $p_1 = 0.4883$ $p_2 = 0.3344$	750.664	1509.3279	1521.2071	3.0087 3.4742
BINAR(1)SPWE	$\tau_1 = 0.3515$ $\tau_2 = 0.2949$ $p_1 = 0.4952$ $p_2 = 0.3973$ $\omega = 0.8949$	723.2353	1456.4706	1471.3196	3.0029 3.4516
BINAR(1)BP	$\lambda_1 = 2.644$ $\lambda_2 = 3.7131$ $\phi = 0.2852$ $p_1 = 0.4243$ $p_2 = 0.3249$	765.8323	1541.6647	1556.5138	3.0413 3.4654
BINAR(1)BNB	$\lambda_1 = 2.6630$ $\lambda_2 = 3.6450$ $\beta = 0.3163$ $p_1 = 0.427$ $p_2 = 0.3045$	728.7059	1467.4119	1482.2610	3.0907 3.4857
BINAR(1)BSPL	$\theta_1 = 0.5464$ $\theta_2 = 0.4494$ $p_1 = 0.431$ $p_2 = 0.3479$ $\omega = -0.9306$	720.7707	1451.5414	1466.3905	3.022 3.4563

Pearson residuals are calculated for checking the accuracy of the model, BINAR(1)SPGL. The ACF plots for the standardized residuals for both the time series are plotted in Figure

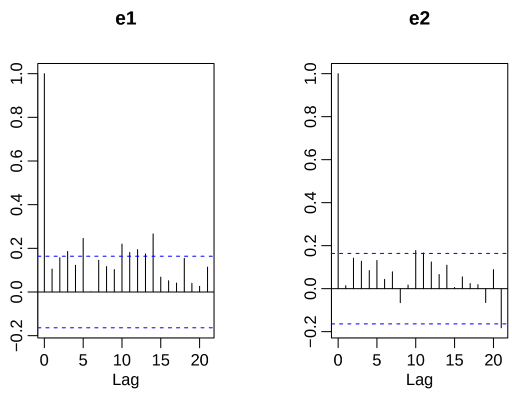

Figure 28: ACF of the standardized residuals of CDRUGS and CSHOTS data for the BINAR(1)SPGL process

28 and it clearly indicates that they are uncorrelated. Also, they have means -0.0084, -0.0106 and variances 1.0593, 0.9632, which are very close to the desired values.

6. Concluding remarks

Two novel bivariate distributions, namely, the BPGL and SPGL distributions, are introduced in this paper. Their mathematical properties are derived. The respective model parameters are estimated using the ML approach. The simulation results based on these novel bivariate models yield consistent estimates under both distributions. Moreover, the BINAR(1)BPGL and BINAR(1)SPGL processes are constructed with BPGL and SPGL paired innovations. The parameters of the BINAR(1)s are then estimated using the method of CML. The small and large sample performances of the established BINAR (1)s are studied and the simulated mean results confirm the consistency of the estimates. For the models that we have established (that is, BPGL, SPGL, BINAR(1)BPGL and BINAR(1)SPGL), we have used the likelihood ratio test for testing the aptness of these models against the models arising from the situation when $\alpha = 2$ corresponding to each of the proposed models (that is, BBPL, BSPL, BINAR(1)BBPL and BINAR(1)BSPL). We used real datasets for that purpose and we found that our models are better than the compared models. Further, the BINAR(1)s are applied to analyze the Pittsburg data and the results reveal that the BINAR(1)SPGL process provides better model adequacy measures than some other recently proposed BINAR(1) models. Hence, the BINAR(1)s with the TPPGL distribution as innovations can be considered among the commendable bivariate time series models and compete against the existing ones in the literature. Their applications are also subject to the nature of the data.

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