On the Type-I Half-logistic Distribution and Related Contributions: A Review

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Abstract

The half-logistic (HL) distribution is a widely considered statistical model for studying lifetime phenomena arising in science, engineering, finance, and biomedical sciences. One of its weaknesses is that it has a decreasing probability density function and an increasing hazard rate function only. Due to that, researchers have been modifying the HL distribution to have more functional ability. This article provides an extensive overview of the HL distribution and its generalization (or extensions). The recent advancements regarding the HL distribution have led to numerous results in modern theory and statistical computing techniques across science and engineering. This work extended the body of literature in a summarized way to clarify some of the states of knowledge, potentials, and important roles played by the HL distribution and related models in probability theory and statistical studies in various areas and applications. In particular, at least sixty-seven flexible extensions of the HL distribution have been proposed in the past few years. We give a brief introduction to these distributions, emphasizing model parameters, properties derived, and the estimation method. Conclusively, there is no doubt that this summary could create a consensus between various related results in both theory and applications of the HL-related models to develop an interest in future studies.

Keywords: generalized half logistic distribution, half logistic distribution, parameter estimation.

1. Introduction

There have been several classical distributions in probability and statistical studies, such as exponential, Weibull, Burr, logistic, normal, gamma, beta, half logistic, and Pareto distribu-
tions, among others. Some of these models were motivated by practical applications (derived from the data), while others were developed due to mathematical interest. Most of the related models have some closed-form and manageable properties and several computer packages to ease utilization. These models have been utilized in various studies such as reliability, life testing, physics, biomedical sciences, engineering, finance, and economics, among others.

The half Logistic (HL) distribution was proposed by Balakrishnan (1985). It may be defined as a continuous distribution derived from the absolute value of a random variable following the logistic distribution; if \( Y \) is a random variable with the logistic distribution, then \( X = |Y| \) is a random variable with the HL distribution. The HL model is one of the probability models that has received considerable attention from practitioners from various statistical studies and application areas. Similar to other classical distributions, the HL distribution appears to have some weaknesses and limitations when dealing with complex data. For instance, the HL distribution can accommodate decreasing probability density function (PDF) and increasing hazard rate function (HRF). It is however unable to cover non-monotone PDF or HRF. This inability is a critical situation concerning the studies of a complex phenomenon that occurs in reliability studies, human mortality analysis, lifetime testing, engineering modeling, physics, electronics, communication, medical sciences, public health, and biomedical sciences. The HRF behavior can be an upside-down bathtub, a bathtub, or other non-monotone.

Practitioners addressed such issue by extending existing models and developing various techniques for combining two or more models to produce new flexible ones that produce more accurate results than the parent model. For instance, Tahir and Cordeiro (2016); Ahmad, Hamedani, and Butt (2019) provided a review regarding various techniques for generating new models and the list of models developed through several techniques. Tomy, Jose, and Veena (2020) gave a summary regarding exponential distribution-related studies and applications. Several researchers provided an in-depth examination of Weibull distributions, Weibull-related models, and applications. One can see Almalki and Nadarajah (2014); Luko (1999); Arthur and Hallinan (1993); Quinn and Quinn (2010); Lai, Murthy, and Xie (2006), among others. Alshkaki (2020) gave some review and tabulated the list of names of some generated models related to Kumaraswamy distributions and their generalizations. Selim (2020) provided a brief survey on several extensions and modifications of the Kumaraswamy and beta distributions proposed in the literature. Recently, Tomy and Satish (2021) provided an overview of recent developments in trigonometric transformation of statistical distributions. Rahman, Al-Zahrani, Shahbaz, and Shahbaz (2020) provided an overview of recent developments in transmuted statistical distributions, among other things.

The HL distribution was extended (or generalized) through different available techniques due to its importance in real-life studies. Concerning the techniques and mixture of models, in some cases, the constructed model may appear to have more complicated properties such as the closed-form expressions of the quantile function (Q), HRF, skewness and kurtosis, entropy, stress strength reliability parameter (R), structural properties, extreme value analysis, characterization of models from well-established theorems, dealing with different methods of parameter estimation, testing results numerically or by simulation, and goodness-of-fits test. These complications are no longer significant due to the availability and utilization of several advanced computational and mathematical ideas, power series, sampling techniques, and computer packages available in R, Mathematica, Python, OX, Matlab, and Maple, among others. Several extensions of the HL distribution are based on the exponentiation technique Gupta, Gupta, and Gupta (1998), beta-generator of distributions (beta-G) with \( G(.) \) any valid CDF by Eugene, Lee, and Famoye (2002), Kumaraswamy-G Cordeiro and de Castro (2011), Topp-Leone-G Al-Shomrani, Arif, Shawky, Hanif, and Shahbaz (2016), Mc.Donald-G Alexander, Cordeiro, Ortega, and Sarabia (2012), T-X families Alzaatreh, Lee, and Famoye (2013), new extended beta Muhammad and Liu (2021a), exponentiated T-X families Alzaghahel, Famoye, and Lee (2013), sine-G Kumar, Singh, and Singh (2015b), exponentiated sine-G Muhammad, Alshanbari, Alanzi, Liu, Sami, Chesneau, and Jamal (2021a), Gamma-G Type-III Torabi and
Montazeri Hedesh (2012), new extended Topp–Leone-G Mustapha, Lixia, Badamasi, Isyaku, Mouna, Hexin, and Sani (2022), transmuted-G Shaw and Buckley (2009), and other transformations, convolution, and compounding methods. Tahir and Cordeiro (2016) provided a comprehensive review regarding various techniques for generating new flexible models.

On the other hand, several researchers have studied different aspects of inferential procedures regarding the HL distribution. The aspect of parameter estimation is significant for any probability distribution. Therefore, various estimation methods are frequently provided and studied through various viewpoints, such as censored data, uncensored data, group data, and record values. The traditional estimation methods include the maximum likelihood estimation (MLE), method of moment estimation (MME), least-square estimation (LSE), and weighted least-squares estimation (WLSE). Other important methods include the Bayes estimation (BE) under different objective priors and different loss functions, maximum product spacing estimation (MPSE), percentile estimation (PE), approximate maximum likelihood estimation (AMLE), modified maximum likelihood estimation (MMLE), modified method of moments estimation (MMME), maximum product spacing estimation (MPSE), Anderson-Darling estimation (ADE) and Cramér-on Mises estimation (CVME), among others. In most of the inferential studies, the estimators’ performances were examined by conducting Monte Carlo simulation studies via the bias, mean square error, and coverage probability with 95% confidence level, among others. The results are supported by the application of relevant real data for illustrative purposes. Here, we emphasize the techniques and some significant results presented in various articles through the HL distribution.

In this work, we aim to provide an overview regarding the HL distribution, as well as synthesize and extend the body of literature in a summarized way to clarify the state of knowledge, importance, and role of the HL distribution in probability and statistical studies. As a result, we hope to provide a summary that will help researchers identify the need for and develop interest in using the newly constructed models related to the HL distribution in their various fields of research. This review could create a consensus between various related results in both theory and applications of the HL distribution to develop interest in future studies. This can produce more accurate and reliable results, allowing for better real-life exploration and human progress.

2. The HL distribution and some related contributions

This section gives a brief review of the HL distribution and some extensions. The related contributions by researchers are extensively summarized.

2.1. The HL distribution

The HL distribution is an important model in reliability, biomedical studies, engineering, finance, computer science, communication, insurance, and life-testing. The distribution is defined on the positive real number and has a monotone increasing HRF. Some properties of the HL distribution and related works are discussed.

**Definition 1.** If $X$ is a random variable following the HL distribution with a scale parameter $\alpha > 0$, then $X$ has the CDF ($F(x)$) and corresponding PDF ($f(x)$) given by

\[
F(x) = \frac{1 - e^{-\alpha x}}{1 + e^{-\alpha x}}, \quad f(x) = \frac{2\alpha e^{-\alpha x}}{(1 + e^{-\alpha x})^2}, \quad \alpha, x > 0,
\]

respectively.

In some literature, the scale parameter $\alpha$ appears in the denominator of the variable. The
survival function (SF) \(s(x)\) and (HRF) \(h(x)\) of the HL distribution are given by

\[
s(x) = \frac{2e^{-\alpha x}}{1 + e^{-\alpha x}}, \quad h(x) = \frac{\alpha}{1 + e^{-\alpha x}}, \quad x > 0,
\]

respectively.

Figure 1 provided the plots of the PDF and HRF of the HL distribution. We can notice that the PDF is always a decreasing function since

\[
\frac{\text{d} \log f(x)}{\text{d}x} = -\alpha (1 - e^{-\alpha x}) / (1 + e^{-\alpha x}) < 0
\]

for all \(\alpha > 0\). The HRF is always an increasing function since

\[
\frac{\text{d} \log h(x)}{\text{d}x} = \alpha e^{-\alpha x} / (1 + e^{-\alpha x}) > 0
\]

for all \(\alpha > 0\).

![Figure 1: Plots of the PDF and HRF of the HL distribution for some parameter values](image)

The quantile function of the HL distribution is given by

\[
Q(u) = -\frac{1}{\alpha} \log \left( \frac{1 - u}{1 + u} \right), \quad u \in (0, 1).
\]

The median of the HL distribution is \(\text{Med} = 1.0986 / \alpha\) and the mode is \(\text{Mode} = \alpha / 2\). Olapade (2003) provided some characterization results based on some homogeneous differential equation and log-transformation from the HL distribution.

The following result is about several equivalent expressions of the moments of the HL distribution.

**Theorem 2.1.** Let \(X\) be a random variable with the HL distribution with scale parameter \(\alpha > 0\). Then the \(r\)th moment of \(X\) can be expressed as

1. First expression (see Balakrishnan and Wong (1991)):

\[
E[X^r] = r! 2 \eta(r), \quad \alpha = 1,
\]

where \(\eta(r)\) is the Dirichlet eta function. In particular, we have \(E[X] = \log 4\) and \(E[X^2] = \pi^2 / 3\).

2. Second expression (see Oliveira, Santos, Xavier, Trindade, and Cordeiro (2016)):

\[
E[X^r] = r! 2 (1 - 2^{1-r}) \zeta(r), \quad \alpha = 1,
\]

where \(\zeta(r)\) is the Riemann zeta function.

3. Third expression:

\[
E[X^r] = 2 \left( -\frac{1}{\alpha} \right)^r \sum_{i=0}^{\infty} (i+1) \mathcal{B}_{r,0}(i+1,1), \quad \alpha > 0,
\]

where \(\mathcal{B}_{p,r}(t,z) = \partial^p t^r \mathcal{B}(t,z) / (\partial p \partial z^r)\) and \(\mathcal{B}(t,z) = \int_0^1 (1 - v)^{-1} v^{t-1} dv\) is a beta function, \(t, z > 0\), available in [https://www.wolframalpha.com/input/?i=d](https://www.wolframalpha.com/input/?i=d)
In addition, according to Balakrishnan and Wong (1991), the moment generating function of \(X\) is given by
\[ M_X(t) = 2J_1(1 + t, 1 - t), \quad \alpha = 1, \]
where \(J_p(a, b) = \int_0^\infty [u^{a-1}/(1 + u)^{a+b}]du\) is the type-II incomplete beta function.

In some literature, some researchers studied the two-parameter HL distribution with scale and location parameters defined with the CDF given by
\[ F(x) = \frac{1 - e^{-\frac{x-\theta}{\alpha}}}{1 + e^{-\frac{x-\theta}{\alpha}}}, \quad x > \theta, \alpha, \theta > 0. \]

The shape parameter was added to the HL distribution by the exponentiation method of Gupta et al. (1998), and called the exponentiated (EHL) (or generalized) HL distribution (see Kantam, Ramakrishna, and Ravikumar (2013)). The CDF of the EHL distribution is generally defined by
\[ F(x) = \left(1 - e^{-\frac{x-\theta}{\alpha}}\right)^\beta, \quad x > \theta, \alpha, \theta, \beta > 0, \]
In most cases, the location parameter \(\theta = 0\), and the scale parameter is multiplied by the variable. If \(\beta = 1\), it is the HL distribution. The moment of the EHL distribution is given in Oliveira et al. (2016) and Muhammad (2016). When \(\alpha = 1\), some properties of the HL distribution can be deduced from the EHL distribution. Seo and Kang (2015b) demonstrated some relationships between the EHL distribution and the exponential distribution, type-I Pareto distribution, uniform distribution, Weibull distribution, and Gumbel distribution via log transformation.

2.2. HL and related contributions

Here, we discussed pioneers’ contributions regarding the HL, GHL and some other related distributions in theoretical studies and practice.

Balakrishnan (1985) established certain recurrence relations for the moments and product moments of order statistics from the HL distribution. Later, the results were generalized by Saran and Pande (2012). Balakrishnan and Puthenpura (1986) included a table of values with the coefficients of the best linear unbiased estimators (BLUEs) of location and scale parameters of the HL distribution and the values of the variances and covariance of these estimators. Balakrishnan and Wong (1994) extended their previous results in Balakrishnan (1985); Balakrishnan and Puthenpura (1986) and tabulated the BLUEs when the samples are singly and doubly type-II censored from the HL distribution. Also, they presented the values of the variances and covariance of these estimators; and provided two examples in which tabulated values are applied. Sultan, Mahmoud, and Saleh (2006) derived the recurrence relations for the single and product moments of progressively type-II right censored order statistics from the HL distribution, and discussed the MLEs of its location and scale parameters by considering Balakrishnan, Balakrishnan, and Aggarwala (2000). Rao and Kantam (2012a) offered a chart of extreme values and a mean analysis based on the HL distribution. Liu and Balakrishnan (2020) developed an algorithm for computing single and product moments of order statistics from the HL distribution based on complete and type-II censored samples, as well as recurrence relations that enable the computation of means, variances and covariances of all order statistics for all sample sizes in a simple and efficient way. In addition, the findings demonstrated that the results are crucial in defining the BLUE of the scale parameter HL distribution, as well as the parameters of the more general location-scale.

Balakrishnan and Wong (1991) showed that the MLEs of the HL distribution parameters are not explicit for the scale parameter based on either complete or right censored samples.
The author next showed how to construct an explicit estimator by approximating the likelihood function. This estimator’s bias and variance are explored, demonstrating that it is as efficient as the BLUE. By calculating the PDF of the scaled HL distribution based on progressively type-II censored samples, Balakrishnan and Asgharzadeh (2005) developed a straightforward method for constructing an explicit estimator. Some wide range of sample sizes and progressive censoring schemes were considered. Then the bias and variance of the estimator were examined. It is shown that the estimator is as efficient as the MLE. Later, it was shown that the asymptotic normality-based coverage probability of pivotal values is insufficient, especially when the effective sample size is small. As a result, it was suggested that, for the generation of confidence intervals, unconditional simulated percentage points of the pivotal values can be used. The bias reduction for the MLEs of the parameters in the HL distribution was described in Giles (2012), and it was found that the bias-corrected estimator of the location parameter outperforms its bootstrapped-based counterpart significantly. By estimating the likelihood function and obtaining the asymptotic variances of MLE and the approximate MLE, Kim and Han (2010) calculated the scale parameter of the HL distribution under progressively type-II censored sample and derived an explicit estimator. Also, they used the Laplace approximation and significance sampling approaches to calculate the BE.

In the setting of the HL distribution, the MLEs and BEs of the scale parameter under the multiple type-II hybrid censoring scheme were investigated in Jeon and Kang (2020a). The least variance unbiased estimators, simple estimators, and ranked-set sample estimators were studied in Adatia (2000). Based on a progressively type-II censored sample, AL-Hussaini, Abdel-Hamid, and Hashem (2015b) investigated Bayesian interval prediction under progressive stress accelerated life testing. The prediction bounds of future order statistics are determined by assuming that the lifetime of a unit under usage condition stress follows the HL distribution with a scale parameter obeying the inverse power law. Seo and Kang (2015c) provided a new approach for estimating the scale parameter of the HL distribution based on progressively type-II censored samples that is based on a pivotal quantity, and the proposed method gives a simpler estimate equation as the likelihood equation. Kang, Cho, and Han (2009) established a double hybrid censoring technique and derived some AMLEs of the scale parameter for the HL distribution. The scale parameter was calculated using two different methods of Taylor series expansion and the approximate MLE method. The suggested approach additionally discusses the MLE and LSE of the scale parameter.

The HL distribution was used to develop the reliability test strategies in Rosaiah, Kantam, and Rao (2009). According to the amount of failures from each group, Rao and Rao (2016) considered a two-stage group acceptance sampling strategy based on life tests that follow the HL distribution, where the choice on lot acceptance can be taken in the first or second stage. When group size and test length are established, the suggested plan’s design parameters are determined to fulfill the consumer’s risk at the stated unreliability, such as the number of groups necessary and the acceptance number for each of two stages. In terms of average sample quantity and operating parameters, a special case of the proposed plan is compared to the proposed plan. Seo and Kang (2015a) calculated the entropy of the HL distribution using the scale parameter and type-II censored samples, as well as providing a MLE and an approximate confidence interval for entropy. The hierarchical structure of the gamma prior distribution, which creates a noninformative prior, has been used to construct hierarchical BE methods. Panichkitkosolkul and Saothayanum (2012) used a conventional bootstrap confidence interval, percentile bootstrap confidence interval, and bias-corrected percentile bootstrap confidence interval to create the bootstrap confidence intervals for the process capability index. Prasad, Sowmya, and Mahesh (2016) addressed a control mechanism based on the mean value function of the HL distribution based on a non-homogeneous Poisson process that was planted on the cumulative observations of interval domain failure data and utilized MLE to estimate the model’s unknown parameters. For skewed distributions, Rao and Kantam (2012b) mentioned variable control charts for the average and range of subgroups. These are constructed using
the percentiles of sampling distributions of sample mean and sample interval, as well as the skewness correction process based on the coefficient of skewness, and the coverage probabilities in both ways are compared using simulation.

The Bayesian prediction boundaries of order statistics based on progressively type-II censored competing risks data from a general class of distributions were investigated in AL-Hussaini, Abdel-Hamid, and Hashem (2015a), and the results were subsequently applied to the HL distribution. Future order statistics prediction intervals were also calculated using three alternative progressive type-II censoring strategies. Based on a gradually type-II censored sample, Wang (2008) provided an exact confidence interval and an exact test for the scale parameter of the HL distribution. Kang and Park (2005) used several type-II censored samples to construct AMLEs of this scale parameter. Adatia (1997) gave an approximation of the size and location parameters of the HL distribution based on a few order statistics such as \((k)\) for reasonably large doubly censored samples. Also, for sampling sizes of \(N = 10(1)40\) and \(k = 2(1)4\), tabulated the ranks, coefficients, variances, and relative efficiencies of the BLUEs and approximate BLUEs of the parameters based on a selected few order statistics \((k)\). The variances obtained were then compared to those of the AMLEs. Seo and Kang (2017) introduced a new approach for estimating the scale parameters of the two-parameter HL distribution based progressively type-II censoring scheme based on a pivotal quantity based on an unbiased estimator of the location parameter based on a pivotal quantity. In comparison to the maximum likelihood equation, the proposed method gave a simpler estimate equation. Using a pivotal quantity, the shortest-length prediction intervals for the censored failure times are then calculated.

On the basis of a doubly type-II censored sample, Jang, Park, and Kim (2011) investigated the MLE and BE of the scale parameter of the HL distribution. The MLE and the Bayes estimator appear to be implicit. To discuss the asymptotic variances and AMLE, a simple approach of approximating the likelihood function to produce an explicit estimator was utilized. In Tierney and Kadane (1986), the Bayes estimator was calculated using the Laplace approximation. Seo and Kang (2016b) proposed an objective Bayesian approach for estimating the parameter under a gradually type-II censoring scheme that took into account the Jeffreys and reference priors, which were also evaluated to see if they matched probability-matching criteria. Due to the nature of their posterior PDFs, the Metropolis Hasting technique was used to create Markov chain Monte Carlo samples. Under noninformative priors, Kang, Kim, and Lee (2014) proposed Bayesian hypothesis testing procedures for the equality of the scale parameters of the HL distribution. As a result, the Bayes factor is generally incorrect and results in a calibration challenge, requiring the Bayes factor to be defined up to a multiplicative constant. As a remedy to the problem, default Bayesian hypothesis testing procedures based on the fractional Bayes factor and the intrinsic Bayes factors under the reference priors were presented. The scale parameter of the HL distribution was estimated using a type-I progressively hybrid censoring strategy in Wang and Liu (2017). The highest posterior confidence intervals for the relevant highest posterior confidence intervals were produced. On the basis of the generalized adaptive progressive hybrid censored sample, Cho and Lee (2021) proposed the generalized adaptive progressive hybrid censoring scheme and the estimate of the parameter of the HL distribution. The MLE was used to estimate the parameter, and Taylor series expansion was used to approximate the MLE. The Bayes estimator is based on the squared error loss function, and it is calculated using the Tierney and Kadane approximations. The noninformative priors for the ratio of the scale parameters in the HL distribution were established by Kang, Kim, and Lee (2012). Furthermore, the second order matching prior matches the alternative coverage probabilities and the highest posterior PDF matching prior when the first and second order matching priors are provided. The one-at-a-time reference prior and Jeffereys’ prior are then revealed to be second-order matching priors. Later, using a simulation analysis and real data, the proposed reference prior matches the intended coverage probability in a frequentist sense.
Under a multiply progressive censoring method, Park and Lee (2020) used the MLE of the scale parameter of the HL distribution using Taylor series expansion. In the estimation of parameters of the HL distribution, Adatia and Saleh (2020) described the generalized ranked-set sampling technique. The novel estimators are based on ranked-set sample estimators and linear minimum variance unbiased estimators. Also tabulated are the coefficients, variances, and relative efficiencies. Based on multiple type-II censored data, Kang, Cho, Han, and Sakong (2010) constructed four modified empirical distribution function type tests for the HL distribution using AMLEs. Modified normalized sample Lorenz curve plot and new test statistics for the model were also proposed. On the basis of a future sample from the HL distribution, AL-Hussaini, Abdel-Hamid, and Hashem (2020) addressed the point and interval predictions of the 8th order statistic. The informative and future samples based on gradually type-II censored under competing risks model and assumption were discussed. In addition, six distinct progressive filtering techniques are subjected to numerical calculations. Furthermore, biases, mean squared prediction errors of maximum likelihood predictors, coverage probabilities, and average interval lengths of Bayesian prediction intervals were all taken into account. The estimate of the scale parameter for the HL distribution using nearly best unbiased estimators utilizing Blom’s approach, least square method for the BLUEs, MLE, and approximate maximum likelihood were covered in Jamjoom (2003). The BLUEs were also explored using two optimally determined order statistics. Based on the asymptotic distribution of the MLEs, log-transformed MLE, pivotal quantity, and generalized pivotal quantity, Potdar and Shirke (2017) computed the confidence interval for the scale parameter of the HL distribution. The EM algorithm is used to generate the MLEs. Also, using coverage probability, length, and the coverage-to-length ratio, they compared the performance of the confidence intervals suggested by Balakrishnan and Asgharzadeh (2005).

The generalized upper \((k)\) record values and generalized lower \((k)\) record values coming from the HL and inverse HL (IHL) distributions were considered by Thomas and Paul (2019). In addition, based on some moment relations of generalized upper \((k)\) record values and those of generalized lower \((k)\) record values, some characterization results of the HL distribution were derived, and diagnostic tools were devised to identify the HL distribution as a model to the distribution of a population. For the IHL distribution, similar characterization theorems and diagnostic tools were developed. In order to estimate the quantiles of the HL distribution, Adatia and Saleh (2004) used the generalized ranked-set sampling technique. Also taken into account are the lowest variance unbiased estimator, simple estimator, and ranked-set sample estimator. The relative efficiencies, variances, and coefficients were tabulated. The performance of the estimators of the HL distribution was compared in Asgharzadeh, Rezaie, and Abdi (2011). MLEs, AMLEs, technique of MMEs, PEs, LSEs, WLSEs, and estimators based on linear combinations of order statistics were used in this study. Awodutire, Olapade, Kolawole, and Ilori (2018) used the type-I generalized HL distribution to estimate breast cancer patients’ survival periods. The three-parameter type-I generalized HL survival model was applied to breast cancer survival data using MLEs by Awodutire, Olapade, and Kolawole (2016). In Seo and Kang (2015b), the exponentiated HL (EHL) distribution’s moment estimators and MLEs were discussed. The estimate of the entropy was determined. The approximate confidence intervals for unknown parameters are calculated using the exact expression of Fisher information. The log transformation was used to provide a relationship with the EHL and some other models. Chaturvedi, Kang, and Pathak (2016) used uniformly minimum variance unbiased estimators and MLEs to construct estimate and testing techniques for the generalized HL (GHL) distribution’s SF under type-II censoring and sampling strategy. The MLEs for the GHL distribution were provided by Seo, Lee, and Kang (2012) based on upper records values. Seo and Kang (2014) used BE and type-II censored samples, the entropy estimators of the GHL distribution were obtained. For the single and product moments of increasingly type-II right-censored order statistics from the GHL distribution, Balakrishnan and Saleh (2013) found many recurrence relations. Later, these relations were used in a logical recursive fashion, allowing all of the means, variances, and covariances of progressively type-II right-
censored order statistics from the GHL distribution to be computed for all sample sizes and progressive censoring methods. Balakrishnan and Aggarwala (1996) proposed that the single and product moments for order statistics from the right truncated GHL distribution satisfy several recurrence relations, and that the relationships can be used in a simple recursive manner to compute the single and product moments of all order statistics for any sample size and truncation parameter. The MLEs and AMLEs of the scale parameter of the GHL and HL distributions based on progressively type-II censored data were discussed in Kang and Seo (2011); Kang, Cho, and Han (2008). The proposed estimators were utilized to calculate the SF’s estimates.

For the EHL distribution, Rastogi and Tripathi (2014) looked at estimating the SF and HRF using progressive type-II censoring. Under Linex and entropy loss functions, MLEs and BEs are examined. The Lindley method is used to get BE. The BEs are computed using Markov Chain Monte Carlo and the importance sampling strategy. For the unknown parameters, credible intervals, asymptotic confidence intervals, bootstrap-p, and bootstrap-t are built. Arora, Bhimani, and Patel (2010) investigated the MLEs of the GHL distribution parameters under type-I progressive censoring with changing failure rates. For the type-I GHL distribution, Kumar, Jain, and Gupta (2015a) derived some new explicit expressions and recurrence relations for marginal and joint moment generating functions of upper record values. Also, utilizing the recurrence relation for a single moment and conditional anticipation of upper record values, the characterization result of this distribution was produced. The MLEs of the top record values were discussed, as well as their confidence intervals. Gui (2017) investigated the problem of using MLEs, inverse moment and modified inverse moment estimation, and Monte Carlo method to estimate the unknown parameters of the EHL distribution. The performance of two approaches for constructing joint confidence regions was discussed.

When the life test is shortened at a pre-specified time, Rao and Naidu (2014) established acceptance sampling plans for the EHL distribution percentiles. To ensure that the stated life percentile is obtained under a given customer’s risk, a minimum sample size is required. The operational characteristic values (and curves) of the sample plans, as well as the producer’s risk, were then given. Liu, Shi, Bai, and Zhan (2018) investigated the reliability estimation of a multicomponent system, dubbed the N-M-cold-standby redundancy system, using a progressive type-II censoring sample for the GHL distribution with various shape parameters. There are $N$ subsystems in the system, each made up of $M$ statistically independent distributed strength components, and only one of these subsystems works under the influence of the stresses at any given moment, with the others acting as standbys. When one of the working subsystems fails, one of the backups takes over. When all of the subsystems fail, the system fails. The uniformly minimum variance unbiased estimator, MLE, BE under squared error loss function utilizing the Gauss hypergeometric function were used to measure the system’s dependability. The approximate greatest PDF credible interval and asymptotic confidence interval were created. On the basis of increasingly type-II censored data under various loss functions, Kim, Kang, and Seo (2011b) presented BE of the shape parameter and SF in the GHL distribution. Based on a progressively type-II censored sample from the GHL distribution, Azimi, Yaghmaei, and Fasihi (2013) investigated the Bayesian and E-Bayesian (expectation of the BE) estimators for the parameter and SF under the LINEX and squared-error loss functions. The moment of order statistics and MLE based on complete observations of the five parameter type-I GHL distribution were explored in Bello, Awodutire, Sule, and Lawal (2020). On the basis of comprehensive and censored data, Torabi and Bagheri (2010) estimated the parameters for an extended GHL distribution.

Under the type-II hybrid censoring, Seo, Kim, and Kang (2013) calculated the MLEs and AMLEs of unknown parameters in the GHL distribution, as well as the estimated confidence. If record values have an EHL distribution, Seo and Kang (2016a) developed more efficient methods for estimating shape parameters in the presence of a scale parameter from Bayesian and non-Bayesian viewpoints. Also offered are estimate methods based on pivotal quantities,
and a robust estimation method was established in the Bayesian approach by constructing some hierarchical structure of the parameter of interest. Chaturvedi and Kumari (2017) explored robust Bayesian analyses for the shape parameter of the GHL distribution using a $\epsilon$-contamination class of priors, taking into account type-II censoring and the sampling technique in Bartholomew (1963). To ensure the product’s quality over time, Rao and Naidu (2016) created a resubmitted lot for group acceptance sampling plan. By assuming that the product lifetime follows the EHL distribution, the plan parameters were determined by setting the experiment termination time and the number of testers to meet both the producer and customer risks at the same time. If the lifetime is shortened at a pre-assigned time with a known index parameter, Rao and Naidu (2015) considered group acceptance sampling plans for the EHL distribution. When both the producer’s and consumer’s risks correspond to the specified quality standards, the number of groups and acceptance number of the suggested sampling plan are determined. The percentiles were used to calculate the termination time and the number of components in each group. Azimi, Yaghmaei, and Babanezhad (2012) developed BEs for the parameters of the GHL distribution under progressively type-II censored samples using the LINEX loss function, precautionary loss function, and entropy loss function. The median ranks approach (Benard’s approximation) was used to estimate the parameters of the GHL distribution in Mohan, Rajasekharam, and Anjaneyulu (2016). Kim, Kang, Han, and Seo (2011a) analyzed the parameters of the GHL distribution using profile likelihood estimation and approximated the maximum profile likelihood estimates for the scale parameter using progressively type-II censored data. Because of the complex form of their usual MLE, Hu and Gui (2019) estimated the scale parameter of the EHL distribution under progressively type-II censoring and general progressively type-II censoring. The scale parameter was approximated by the MLE based on Taylor expansion and estimation based on pivotal quantities. After that, based on pivotal quantities, asymptotic confidence intervals were calculated.

The estimation of the stress-strength reliability using paired observations with ties from bivariate EHL distribution was studied in Xavier and Jose (2022). The significance sampling technique is used to assess MLE and BE with a squared error loss function and a gamma prior. For the EHL distribution, Raqab, Bdair, Rastogi, and Al-Aboud (2021) looked at frequentist and Bayesian ways of estimating unknown parameters. Point estimators and accompanying confidence intervals for the parameters are obtained using the expectation-Maximization, Lindley’s approximation, and Metropolis-Hastings algorithms. If the lifetime of the units follows the EHL distribution, Rao, Rosaiah, and Naidu (2020a) established a multiple postponed state sampling plan for a time-truncated life test. The ideal parameters of the proposed plan, such as the number of subsequent lots required to decide whether to accept or reject the current lot, sample size, and rejection and acceptance numbers, were determined using the two-point technique. The suggested plan’s execution was demonstrated with examples, and a table is created for various combinations of consumer and producer risks. Kumar and Farooqi (2017) used the type-I GHL distribution to establish some clear formulas and recurrence relations for the marginal and joint moment generating functions of generalized order statistics. The conditional expectation of generalized order statistics was also used to characterize the distribution. Rao, Rosaiah, and Naidu (2020b) calculated the number of groups required for each of two steps of EHL distribution in order to reduce the average sample size while fulfilling the producer and consumer risks. In addition, single-stage group sampling plans were evaluated as special cases of the stated plan and were compared to the suggested plan in terms of average sample size and operating characteristics. The likelihood approach, parametric percentile bootstrap, and bootstrap-t confidence intervals were explained in Almarashi (2020) when the lifetime of units under normal conditions follows the GHL lifetime distribution based on progressive type-II censored schemes. The ideal parameters and control limits are determined using an attribute $np$ control chart built utilizing re-sampling techniques for monitoring non-conforming items under the EHL distribution described by Tauveer, Azam, Aslam, and Shujaat Navaz (2020). The control constants were established by considering
the goal in-control ARL at a normal process, and the operational formulas for in-control and out-of-control average run lengths (ARLs) were derived.

The stress-strength reliability parameter estimation from the GHL distribution under type-I and type-II censoring was studied in Chaturvedi and Nandchahal (2017). The same aim was discussed using a generalization of the power transformed HL (PTHL) distribution in Xavier and Jose (2021b). Jeon and Kang (2020b) used the multiple type-I hybrid censoring approach to estimate the scale parameter of the EHL distribution, assuming the shape parameter was known. The maximum likelihood was calculated using two alternative Taylor series expansions, as well as BEs for the squared error and general entropy loss functions. The asymptotic confidence interval and the highest posterior density interval were obtained. The highest posterior density interval and the asymptotic confidence interval were calculated. The MLEs of the parameters of the Poisson HL (PHL) distribution were investigated using progressive type-II censoring and the predictive of future order statistics using one and two-sample Bayesian prediction techniques in Abdel-Hamid (2016). The prediction intervals’ greatest conditional density and coverage probabilities were also estimated. Furthermore, Muhammad and Liu (2021b) investigated the characterization of the PHL distribution using truncated moments of a random variable function and its link to other models using log transformation. The stress-strength reliability parameter based on complete samples based on random variables with the PHL distribution was extensively explored in Muhammad, Wang, Li, Yan, and Chang (2020).

Aldahlan (2019) considered the entire sample when discussing MLE, LSE, WLSE, PE, and CVME procedures for the shape and scale parameters of the HL Lomax (HLL) distribution. For the HL-truncated exponential (HLTE) distribution, Gul and Mohsin (2021) derived recurrence relations for the single moments and product moments of order statistics for various values of scale parameter. The recurrence relations are used to tabulate the means, variances, and covariances in samples of size \( n = 1(1)10 \) and to determine the first \( k \) moments of all order statistics quickly and easily. Swathi and Anjaneyulu (2020) proposed utilizing the median rank approach (Benard’s approximation) to estimate the location and scale parameters of the HL Rayleigh (HLR) distribution. For both parameters in the whole sample,
the average estimate, variance, standard deviation, mean absolute deviation, mean square error, simulated error, and relative absolute bias were also computed. Swathi, Lakshmi, and Anjaneyulu (2020) examined the performance of the change point technique in the case of the HLR distribution, taking average run lengths into account, and determined that the HLR model is good at identifying tiny shifts.

3. HL and related models

As mentioned in Section 1, practitioners and statisticians proposed a variety of models and model generators that allow the use of existing classical models to provide more flexible ones capable of accommodating variations of an assumed random phenomenon. Thus, it leads to several explorations of mathematical ideas, techniques, algorithms, and simulations in various fields of study. The generated models possess some advantages in modeling non-monotonic HRFs and skewed PDFs; thus, they explain better how the lifetime phenomenon occurs in various studies. Many authors in the literature consider HL-type models to be extended or generalized through various techniques. In most of the studies, several important properties of the newly proposed models are explored, and the estimation of the unknown parameters of the model is also discussed. Finally, the applications of the model in other fields of study are presented by fitting the model to real data and comparing it to other existing models.

This section focused on the number of parameters, some valuable properties, and estimation methods regarding the model discussed by the authors when introducing the distribution. We provide some abbreviations for some terms discussed in this work in Table 1. In Tables 2 and 3, we also present the list of some available probability models related to the HL distribution proposed in the literature.

<table>
<thead>
<tr>
<th>Abbreviations of some important properties and estimation methods</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moments (M)</td>
</tr>
<tr>
<td>Quantile (Q)</td>
</tr>
<tr>
<td>Vector of parameter (ξ)</td>
</tr>
<tr>
<td>Order statistics (Os)</td>
</tr>
<tr>
<td>Mean deviation (Md)</td>
</tr>
<tr>
<td>Bonferroni curve (Bc)</td>
</tr>
<tr>
<td>Lorenz curve (Lc)</td>
</tr>
<tr>
<td>Percentile estimation for censored data (PEc)</td>
</tr>
<tr>
<td>Percentile estimation (PE)</td>
</tr>
<tr>
<td>Residual entropy (ReE)</td>
</tr>
<tr>
<td>Shannon entropy (SE)</td>
</tr>
<tr>
<td>q-Entropy (qE)</td>
</tr>
<tr>
<td>Havrda and Charvát entropy (HCE)</td>
</tr>
<tr>
<td>Tsallis entropy (TE)</td>
</tr>
<tr>
<td>Log transformation (Lt)</td>
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<tr>
<td>Moment of residual life (MRL)</td>
</tr>
<tr>
<td>Characterization (Ch)</td>
</tr>
<tr>
<td>Stress–strength reliability parameter (R)</td>
</tr>
<tr>
<td>Probability weighted moments (PWM)</td>
</tr>
<tr>
<td>Stochastic ordering (So)</td>
</tr>
<tr>
<td>Maximum likelihood estimation (MLE)</td>
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<tr>
<td>Maximum likelihood for censored data (MLEc)</td>
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<tr>
<td>Maximum likelihood for group data (MLEg)</td>
</tr>
<tr>
<td>Modified maximum likelihood estimation (MMLE)</td>
</tr>
<tr>
<td>Bayes estimation (BE)</td>
</tr>
<tr>
<td>Bayes estimation for censored data (BEc)</td>
</tr>
<tr>
<td>Least-square estimation (LSE)</td>
</tr>
<tr>
<td>Least-square estimation for censored data (LSEc)</td>
</tr>
<tr>
<td>Weighted least-squares estimation (WLSE)</td>
</tr>
<tr>
<td>Weighted least-squares estimation for censored data (WLSEc)</td>
</tr>
<tr>
<td>Renyi entropy (RE)</td>
</tr>
<tr>
<td>Mellin transform (Mt)</td>
</tr>
<tr>
<td>Arimoto entropy (AE)</td>
</tr>
<tr>
<td>Kullback-Leibler divergence (KLD)</td>
</tr>
<tr>
<td>Method of moments estimation (MME)</td>
</tr>
<tr>
<td>Modified method of moments estimation (MMME)</td>
</tr>
<tr>
<td>Maximum product of spacings estimation (MPSE)</td>
</tr>
<tr>
<td>Anderson–Darling estimation (ADE)</td>
</tr>
<tr>
<td>Crammer–von Mises estimation (CVME)</td>
</tr>
</tbody>
</table>
Table 2: Some probability models related to the HL distribution

<table>
<thead>
<tr>
<th>Model</th>
<th>Year</th>
<th>No. of parameters</th>
<th>Properties</th>
<th>Estimation method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type-I half-logistic and related contributions</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1. Type-III generalized HL distribution Olapade (2008)</td>
<td>2008</td>
<td>1</td>
<td>M, Q, Os</td>
<td></td>
</tr>
<tr>
<td>2. Four-parameter type-I generalized HL distribution Olapade (2011)</td>
<td>2011</td>
<td>4</td>
<td>M, Q, Lt, Ch</td>
<td></td>
</tr>
<tr>
<td>3. Generalized (or exponentiated) HL distribution Kantam et al. (2013)</td>
<td>2013</td>
<td>2</td>
<td></td>
<td>MLE, MMLE</td>
</tr>
<tr>
<td>4. Exponential-half logistic Rao, Nagendra, and Rosaih (2013)</td>
<td>2013</td>
<td>2</td>
<td></td>
<td>MLE, MLER</td>
</tr>
<tr>
<td>5. HL generated-G distribution AL-Hussaini and Hussein (2014)</td>
<td>2014</td>
<td>ξ</td>
<td>M, Q, Os</td>
<td>MLE, BE</td>
</tr>
<tr>
<td>6. Type-I generalized HL distribution Olapade (2014)</td>
<td>2014</td>
<td>1</td>
<td>M, Q, Os, Ch, Lt</td>
<td></td>
</tr>
<tr>
<td>7. Power HL distribution Krishnaran (2015)</td>
<td>2016</td>
<td>2</td>
<td>M, Q, Lt, Ch</td>
<td>MLE, LSE, MME</td>
</tr>
<tr>
<td>8. Beta HL distribution Jose and Manoharan (2016)</td>
<td>2016</td>
<td>3</td>
<td>Q</td>
<td></td>
</tr>
<tr>
<td>9. Type-I HL distribution Oliveira et al. (2016)</td>
<td>2016</td>
<td>2</td>
<td>M, Md, RE</td>
<td></td>
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<tr>
<td>16. Type-I HL Gompertz distribution Ogunm, Osieh, and Audu (2017)</td>
<td>2017</td>
<td>2</td>
<td>M, Q, Os</td>
<td></td>
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<tr>
<td>18. Exponentiated generalized standardization HL distribution Cordeiro, de Andrade, Bourguignon, and Gomes-Silva (2017)</td>
<td>2017</td>
<td>2</td>
<td>M, Q, Os, R, RE, MRL</td>
<td></td>
</tr>
<tr>
<td>21. Type-I HL distribution Anwar and Bibi (2018)</td>
<td>2018</td>
<td>2</td>
<td>M, Q, Os, Md, SE, RE, MRL</td>
<td></td>
</tr>
<tr>
<td>23. Oll exponentiated HL Burr XII distribution Alkhalife and Alify (2018)</td>
<td>2018</td>
<td>2</td>
<td>Q, Os</td>
<td></td>
</tr>
<tr>
<td>24. Exponentiated HL exponential distribution Almasari, Khalil, Elgarby, and ESchehtry (2018)</td>
<td>2018</td>
<td>3</td>
<td>M, Q, Os, MRL, PWM</td>
<td></td>
</tr>
<tr>
<td>27. Poisson generalized HL distribution Muhammad and Liu (2019)</td>
<td>2019</td>
<td>3</td>
<td>M, Q, Md, Os, Ch, Lt, R, MRL, Bc, Lc, SE, RE, PWM, KLD</td>
<td>MLE, LSE, WSLE, CVME, ADE</td>
</tr>
<tr>
<td>28. Type-II HL exponential distribution Elgarby, ul Haq, and Perveen (2019)</td>
<td>2019</td>
<td>2</td>
<td>M, Q, Md, Os, RE, PWM</td>
<td></td>
</tr>
<tr>
<td>29. Type-I HL inverted Kumaraswamy distribution ZeinElfen, Chesneau, Jamal, and Elgarby (2019)</td>
<td>2019</td>
<td>3</td>
<td>M, Q, Md, Os, R</td>
<td>MLE, LSE, WSLE, CVME, ADE</td>
</tr>
</tbody>
</table>
## Table 3: Some probability models related to the HL distribution

<table>
<thead>
<tr>
<th>Model</th>
<th>Year</th>
<th>No. of parameters</th>
<th>Properties</th>
<th>Estimation method</th>
</tr>
</thead>
<tbody>
<tr>
<td>35 Type-I HL Burr X distribution Shrahili, Elbatal, and Muhammad (2019)</td>
<td>2019</td>
<td>3</td>
<td>M, Q, Md, Le, SE, RE, MRL</td>
<td>MLE</td>
</tr>
<tr>
<td>36 Gamma power HL distribution Arshad, Chesneau, Ghanazi, Jamal, and Manooe (2019)</td>
<td>2019</td>
<td>3</td>
<td>M, Q, Os, Lt, So, Le, R</td>
<td>MLE</td>
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<tr>
<td>37 Kumaraswamy type-I HL G family of distributions Elsehetry and El-Sherpieny (2019)</td>
<td>2019</td>
<td>3, ξ</td>
<td>M, Q, Md, Os, PWM</td>
<td>MLE</td>
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<tr>
<td>38 Type-II Kumaraswamy HL G family of distributions El-Sherpieny and Elsehetry (2019)</td>
<td>2019</td>
<td>2, ξ</td>
<td>M, Q, Md, Os, PWM</td>
<td>MLE</td>
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<tr>
<td>39 Exponentiated HL Lomax distribution Jamal, Ahmed, Shah, and Altun (2019)</td>
<td>2019</td>
<td>2, ξ</td>
<td>M, Q, Os, RE, Be, Le, PWM</td>
<td>MLE</td>
</tr>
<tr>
<td>40 Generalized power transformed HL distribution Xavier and Jose (2021b)</td>
<td>2020</td>
<td>3</td>
<td>M, Q, Mt, R</td>
<td>MLE, PE, MME</td>
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<tr>
<td>41 Bivariate exponentiated HL distribution Alotaibi, Rezk, Ghosh, and Dey (2020)</td>
<td>2020</td>
<td>4</td>
<td>Ch</td>
<td>MLE</td>
</tr>
<tr>
<td>42 Type-II HL Lomax distribution Hassan, Eliygar, and Mohamed (2020)</td>
<td>2020</td>
<td>3</td>
<td>M, Q, So, RE, Be, Le, MRL, PWM</td>
<td>MLE, SLE, WSLE, ADE</td>
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<tr>
<td>43 Generalized odd HL-G family of distributions El-Morshedy, Alshammari, Tyagi, Elbatal, Hamed, and Eliwa (2021)</td>
<td>2020</td>
<td>3</td>
<td>M, Q, Lt</td>
<td>MLE, LSE, WSLE, CVME, ADE, MPSE</td>
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<tr>
<td>44 Extended cosine generalized HL distribution Hashempour and Alizadeh (2021)</td>
<td>2021</td>
<td>2</td>
<td>M, Q, Lt</td>
<td>MLE, LSE, CVME, ADE, MPSE</td>
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<tr>
<td>45 New extended beta HL distribution Muhammad and Liu (2021a)</td>
<td>2021</td>
<td>4</td>
<td>M, Q, Os, Md, Be, LE, RE</td>
<td>MLE, BE</td>
</tr>
<tr>
<td>46 Exponentiated HL distribution El-Morshedy and Elgarhy (2021)</td>
<td>2021</td>
<td>2</td>
<td>M, Q, Lt</td>
<td>MLE, LSE, CVME, ADE, MPSE</td>
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<tr>
<td>47 Type-I HL Lomax distribution Al-Babtain (2020)</td>
<td>2020</td>
<td>2</td>
<td>M, Q, Lt</td>
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<tr>
<td>48 Generalized odd HL-G family of distributions Al-Marzouki and Alrajhi (2020)</td>
<td>2020</td>
<td>2, ξ</td>
<td>M, Q, Md, Lt</td>
<td>MLE, LSE, WSLE, CVME, ADE, MPSE</td>
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<tr>
<td>49 Type-I HL Rayleigh distribution Alotaibi, Rezk, Ghosh, and Dey (2020)</td>
<td>2020</td>
<td>2, ξ</td>
<td>M, Q, Lt</td>
<td>MLE, LSE, WSLE, CVME, ADE, MPSE</td>
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<tr>
<td>50 Discrete HL distribution Barbiero and Hitaj (2020)</td>
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<td>2</td>
<td>M, Q, Lt</td>
<td>MLE, LSE, WSLE, CVME, ADE, MPSE</td>
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<tr>
<td>51 Exponentiated HL Lomax distribution Jamal, Reyad, Ahmed, Shah, and Altun (2019)</td>
<td>2019</td>
<td>3</td>
<td>M, Q, Os, RE, Be, Le, MRL, PWM</td>
<td>MLE, SLE, WSLE, ADE</td>
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<tr>
<td>52 Complementary Poisson generalized half logistic distribution Muhammad and Liu (2021b)</td>
<td>2021</td>
<td>3</td>
<td>M, Q, Md, Os, Mt, Lt, MRL, Be, Le, SE, RE</td>
<td>MLE, MLEc, LSE, CVME</td>
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<tr>
<td>54 Topp-Leone odd exponential HL-G family of distributions Al-Marzouki and Alrajhi (2020)</td>
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<td>3, ξ</td>
<td>M, Q, Os, RE, PWM</td>
<td>MLE</td>
</tr>
<tr>
<td>55 Odd exponentiated HL exponential distribution Alotasibi, Rezk, Ghosh, and Dey (2020)</td>
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<td>2</td>
<td>M, Q, Lt</td>
<td>MLE, LSE, WSLE, CVME, ADE, MPSE</td>
</tr>
<tr>
<td>56 Discrete HL distribution Barbiero and Hitaj (2020)</td>
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<td>M, Q, Lt</td>
<td>MLE, LSE, WSLE, CVME, ADE, MPSE</td>
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<tr>
<td>57 Type-II HL Lomax distribution Hassan, Eliygar, and Mohamed (2020)</td>
<td>2020</td>
<td>3</td>
<td>M, Q, Os, RE, Be, Le, MRL, PWM</td>
<td>MLE, SLE, WSLE, CVME, ADE, MPSE</td>
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<td>58 HL inverse Rayleigh distribution Shrahili, Elbatal, and Elgarhy (2021)</td>
<td>2021</td>
<td>2</td>
<td>M, Q, Lt</td>
<td>MLE, LSE, WSLE, CVME, ADE, MPSE</td>
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<tr>
<td>59 Exponentiated HL distribution Elsehetry and El-Sherpieny (2019)</td>
<td>2019</td>
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<td>M, Q, Ms, Os, Lt, Lt, So, PWM</td>
<td>MLE</td>
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<tr>
<td>60 Generalized odd HL-G family of distributions Elsehetry and El-Sherpieny (2019)</td>
<td>2019</td>
<td>2, ξ</td>
<td>M, Q, Ms, Lt</td>
<td>MLE, LSE, WSLE, CVME, ADE, MPSE</td>
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<tr>
<td>61 Marshall-Olkin odd exponential HL G family of distributions Elsehetry and El-Sherpieny (2019)</td>
<td>2019</td>
<td>3</td>
<td>M, Q, Ms, Lt, so, PWM</td>
<td>MLE</td>
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<tr>
<td>62 Generalized odd HL-G family of distributions Elsehetry and El-Sherpieny (2019)</td>
<td>2019</td>
<td>2, ξ</td>
<td>M, Q, Ms, Lt</td>
<td>MLE, LSE, WSLE, CVME, ADE, MPSE</td>
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<tr>
<td>63 New two-parameter modified HL distribution Muhammad and Liu (2021b)</td>
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<td>2</td>
<td>M, Q, Os, Be</td>
<td>MLE, SLE, WSLE, CVME, ADE, MPSE</td>
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<tr>
<td>64 Marshall-Olkin odd exponential HL G family of distributions Elsehetry and El-Sherpieny (2019)</td>
<td>2019</td>
<td>3</td>
<td>M, Q, Ms, Lt</td>
<td>MLE, LSE, WSLE, CVME, ADE, MPSE</td>
</tr>
<tr>
<td>65 Type-I HL Rayleigh distribution Jamal, Reyad, Ahmed, Shah, and Altun (2019)</td>
<td>2019</td>
<td>3</td>
<td>M, Q, Os, RE, Be, Le, MRL, PWM</td>
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<td>66 Generalized odd HL-G family of distributions Elsehetry and El-Sherpieny (2019)</td>
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<td>M, Q, Ms, Lt</td>
<td>MLE, LSE, WSLE, CVME, ADE, MPSE</td>
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<td>67 Exponentiated HL Lomax distribution Jamal, Reyad, Ahmed, Shah, and Altun (2019)</td>
<td>2019</td>
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<td>M, Q, Ms, Lt</td>
<td>MLE, LSE, WSLE, CVME, ADE, MPSE</td>
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<tr>
<td>68 Discrete HL distribution Barbiero and Hitaj (2020)</td>
<td>2020</td>
<td>1</td>
<td>M, Q, Lt</td>
<td>MLE, LSE, WSLE, CVME, ADE, MPSE</td>
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<td>69 Complementary Poisson generalized half logistic distribution Muhammad and Liu (2021b)</td>
<td>2021</td>
<td>3</td>
<td>M, Q, Md, Os, Mt, Lt, MRL, Be, Le, SE, RE</td>
<td>MLE, MLEc, LSE, CVME</td>
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<tr>
<td>70 Marshall-Olkin odd exponential HL G family of distributions Elsehetry and El-Sherpieny (2019)</td>
<td>2019</td>
<td>3</td>
<td>M, Q, Ms, Lt</td>
<td>MLE, LSE, WSLE, CVME, ADE, MPSE</td>
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<tr>
<td>71 Poisson-logarithmic half-logistic distribution Hashem, Kac, Pokgor, and Abdel-Hamid (2022)</td>
<td>2022</td>
<td>3</td>
<td>M, Q, Os, SE, RE</td>
<td>MLE, LSE, WSLE, ADE</td>
</tr>
</tbody>
</table>
4. Final remarks

In this paper, we have reviewed some recent developments and contributions by researchers with respect to the HL distribution and their extensions. The information used in this work was gathered from the available resources in the various literature. Several results in probability theory, statistics, and applications are summarized. We briefly introduced several extensions of the HL model, emphasizing the number of model parameters, properties derived, and estimation methods. Several generators have been recently developed in the literature; therefore, several new extensions of HL can be developed in the future. It is clear that most of the newly constructed models are yet to be discussed under various estimation techniques in various viewpoints and in reliability studies, among others. This review could identify the need and develop researchers’ interest in utilizing the newly constructed models related to the HL distribution in their various fields of studies to provide more accurate and reliable results for better real-life exploration and progress.

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