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# Properties and Applications of the Type I Half-logistic Nadarajah-Haghighi Distribution

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**Abstract**

A new three-parameter distribution called the type I half-logistic Nadarajah-Haghighi ($TIHL_{NH}$) is proposed. We discussed some important mathematical and statistical properties of the new model such as an explicit form of its $r^{th}$ moment, mean deviations, quantile function, Bonferroni and Lorenz curves. The Shannon entropy and Renyi entropy are computed, the expression for the Kullback-Leibler divergence measure is provided. The model parameters estimation was approached by the maximum likelihood estimation (MLE), and the information matrix is obtained. The finite sample properties of the MLEs are investigated numerically by simulation studies; by examining the bias and mean square error of the estimators, and the results was satisfactory. We used two real data applications to demonstrate the superior performance of the $TIHL_{NH}$ in terms of fit over some other existing lifetime models.

*Keywords*: type I half logistic-$G$, Nadarajah-Haghighi distribution, moments, entropy, maximum likelihood estimation.

## 1. Introduction

Statistical models provide a significant contribution to the studies of natural life phenomenon. One of the significant part of statistical studies is the modeling the lifetime data by a lifetime distribution. The exponential, half logistic, Rayleigh, and Weibull distributions etc., are the commonly used classical models in reliability studies, biomedical sciences and life testing because they have many close form properties with simple algebraic representation. Unfortunately, there are still various important problems which are often encountered in practice where a real data does not follow any of the classical probability distribution; moreover, if the failure rate is non-monotone, the majority of the classical models are incapable, also, the rapid development in the applied fields of studies such as biomedical sciences, communication, engineering, etc., required to provide for more alternative models as well as flexible models for better exploration of complex data and non-monotone failure rates. To tackle these obstacles, over the years an attempt has been made to define new class of probability distributions that generalizes well-known probability distributions and at the same time provide high flexibility in modeling lifetime data in practice. For example, the beta-$G$ by Eugene, Lee, and Famoye (2002), new extension of the beta-G Muhammad and Liu (2021a), gamma-$G$ due to Zografos and Balakrishnan (2009), Kumaraswamy-$G$ Cordeiro and de Castro (2011), Weibull-$X$ Alzaatreh and Ghosh (2015), exponentiated sine-G Muhammad, Alshanbari, Alanzi, Liu, Sami, Chesneau, and Jamal (2021a), odd-generalized exponential-G Tahir, Cordeiro, Alizadeh, Mansoor, Zubair, and Hamedani (2015), transmuted Weibull-G Alizadeh, Rasekhi, Yousof, and Hamedani (2018), generalized transmuted-G Alizadeh, Merovci, and Hamedani (2017), complementary geometric transmuted-G Afify, Cordeiro, Nadarajah, Yousof, Ozel, Nofal, and Altun (2016), generalized gamma-G Alzaatreh, Carl, and Famoye (2016), new Weibull-G Tahir, Zubair, Mansoor, Cordeiro, Alizadehk, and Hamedani (2016), extended cosine-G Muhammad, Bantan, Liu, Chesneau, Tahir, Jamal, and Elgarhy (2021b) Lindley –G Cakmakyapan and Gamze (2016), Hamedani, Yousof, Rasekhi, Alizadeh, and Najibi (2018) type I general exponential-G, and Poisson odd-generalized exponential-G Muhammad (2016b) among others.

In the past few years, a new family of distributions called type I half-logistic family has been proposed in order to extend the class of lifetime distributions Cordeiro, Alizadeh, and Diniz Marinho (2016). The type I half logistic Burr X (TIHLBX) distribution is one of the recently studied member of this family Shrahili, Elbatal, and Muhammad (2019). The cumulative distribution function (cdf) of the type I half-logistic (TIHL-G) family of distributions is given by

$$F(x;\delta,\lambda) = \int_0^{-\log[1-G(x;\delta)]} \frac{2\lambda e^{-\lambda x}}{(1+e^{-\lambda x})^2}dx = \frac{1-[1-G(x;\delta)]^\lambda}{1+[1-G(x;\delta)]^\lambda}, \tag{1}$$

where $G(x;\delta)$ is the baseline cdf depending on a parameter vector $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_n),\ \delta_i \in \mathbb{R}$, and $\lambda > 0$. For each baseline $G$, we can have type I half-logistic-$G$ by considering (1). The corresponding density (pdf) functions to (1) is

$$f(x;\delta,\lambda) = \frac{2\lambda g(x;\delta)\,[1-G(x;\delta)]^{\lambda-1}}{\left(1+[1-G(x;\delta)]^\lambda\right)^2}, \tag{2}$$

where $g(x;\delta)$ is the baseline pdf.

In the last few years, Nadarajah and Haghighi (2011) introduced a new two-parameter distribution as an extension of the exponential distribution called Nadarajah-Haghighi (NH), its cumulative distribution function (cdf) and probability density function (pdf) are

$$G(x) = 1 - e^{1-(1+\theta x)^\alpha}, \quad x > 0, \tag{3}$$

and

$$g(x) = \alpha\theta(1+\theta x)^{\alpha-1}e^{1-(1+\theta x)^\alpha}, \quad x > 0, \tag{4}$$

respectively, where $\theta,\ \alpha > 0$ are the scale and shape parameters respectively, when $\alpha = 1$, (3) become the exponential distribution. The NH distribution possesses only decreasing density and increasing, decreasing and constant hazard rates. Due to the fact that type I half logistic generator of distributions provide additional flexibility to a baseline distribution, in this study, we aim at providing a new flexible extension of the NH called the Type I half logistic Nadarajah-Haghighi (TIHL$_{NH}$) distribution with capability of accommodating both monotone and non-monotone failure rate, also with the ability to accommodate both decreasing and unimodal densities.

In statistical literature, the extensions of probability models have greatly contributed to distribution theory, leading to several important mathematical and statistical tools useful in both theory and practice. Here, we want provide an extensive studies of the mathematical and statistical properties of the TIHL$_{NH}$ to present closed form and convenient representation of the model properties with the aid of several mathematical techniques, computational algorithms and computer packages for numerical computations. Different probability models serve different purposes and represent different data generation procedures. The TIHL$_N H$ serves as a new tool for generating different kind of data with different characteristics. In addition, to demonstrate how TIHL$_{NH}$ provide a better fit in comparison with the NH and other related models in some practical applications. We hoped that the new proposed TIHL$_{NH}$ may serve as an important additional tool in both theoretical studies and practice in probability and applied statistics, among others.

The rest of the paper is arranged as follows: In section 2, we derived the $TIHL_{NH}$ distribution and presented some of its mathematical and statistical properties. In section 3, the maximum likelihood estimation is discussed. In section 4, the potentiality of the new model is illustrated by the use of two real data set. Conclusions in section 5.

## 2. The new model and properties

In this section, we introduce the new three parameter type I half-logistic Nadarajah-Haghighi ($TIHL_{NH}$) distribution. Let the baseline cumulative distribution of (1) be the NH distribution given by (3), therefore, the cdf of the $TIHL_{NH}$ distribution can be written as

$$F(x;\alpha,\theta,\lambda) = \int_0^{-\log\left[1-\left(1-e^{1-(1+\theta x)^{\alpha}}\right)\right]} \frac{2\lambda e^{-\lambda t}}{(1+e^{-\lambda t})^2} dt = \frac{1-e^{\lambda[1-(1+\theta x)^{\alpha}]}}{1+e^{\lambda[1-(1+\theta x)^{\alpha}]}},\ x,\alpha,\theta,\lambda > 0. \quad (5)$$

The corresponding probability density, survival function and hazard rate function of the $TIHL_{NH}$ are respectively given by

$$f(x;\alpha,\theta,\lambda) = \frac{2\alpha\theta\lambda\ (1+\theta x)^{\alpha-1}\ e^{\lambda[1-(1+\theta x)^{\alpha}]}}{\left(1+e^{\lambda[1-(1+\theta x)^{\alpha}]}\right)^2}, \quad (6)$$

$$s(x;\alpha,\theta,\lambda) = \frac{2e^{\lambda[1-(1+\theta x)^{\alpha}]}}{1+e^{\lambda[1-(1+\theta x)^{\alpha}]}}, \quad (7)$$

$$h(x;\alpha,\theta,\lambda) = \frac{\alpha\theta\lambda(1+\theta x)^{\alpha-1}}{1+e^{\lambda[1-(1+\theta x)^{\alpha}]}}. \quad (8)$$

We denote $X \sim TIHL_{NH}(\phi)$, a random variable $X$ with pdf given by (6), where $\phi = (\alpha,\theta,\lambda)$.

**Interpretation 1.** *Let $(X,T)$ be a random vector with joint density function $f(x,t)$ defined on $\mathbb{R}^2$. Suppose that the conditional cumulative distribution of $X$ given $T = t$ is $\boldsymbol{K}(x|t)$ and $T \sim \boldsymbol{E}(t)$. Then the following defines the unconditional survival function of $X$, $s(x) = \int \bar{\boldsymbol{K}}(x|t)\boldsymbol{E}(t)dt$. The survival function $s(x)$ is obtained by compounding the survival function $\bar{\boldsymbol{K}}(x|t) = 1 - \boldsymbol{K}(x|t)$ and the density of $\boldsymbol{E}(t)$. Suppose that the survival function $\bar{\boldsymbol{K}}(x|t) = e^{-t\left(\frac{1-e^{\lambda(1-(1+\theta x)^{\alpha})}}{2e^{\lambda(1-(1+\theta x)^{\alpha})}}\right)}$ where $x,\alpha,\theta,\lambda,t > 0$, and $T$ assumed to have exponential distribution with mean 1, then $X$ has survival function given by (7).*

*Proof.* For all $x,t,\alpha,\theta,\lambda > 0$, the survival function is given as

$$s(x) = \int \bar{\mathbf{K}}(x|t)\mathbf{E}(t)dt = \int_0^{\infty} e^{-t\left(\frac{1-e^{\lambda(1-(1+\theta x)^{\alpha})}}{2e^{\lambda(1-(1+\theta x)^{\alpha})}}\right)} e^{-t}dt$$
$$= \int_0^{\infty} e^{-t\left(\frac{1-e^{\lambda(1-(1+\theta x)^{\alpha})}}{2e^{\lambda(1-(1+\theta x)^{\alpha})}}+1\right)} dt = \left(\frac{1-e^{\lambda(1-(1+\theta x)^{\alpha})}}{2e^{\lambda(1-(1+\theta x)^{\alpha})}}+1\right)^{-1} = \frac{2e^{\lambda[1-(1+\theta x)^{\alpha}]}}{1+e^{\lambda[1-(1+\theta x)^{\alpha}]}}.$$

□

**Proposition 2.1.** *The asymptotic of the cdf of $TIHL_{NH}$ in (5) for a very small $x$ i.e, $x \to 0^+$ is*

$$\lim_{x\to 0^+} F(x) \sim (1/2)(\lambda[(1+\theta x)^{\alpha}-1]).$$

*Proof.* By the transitivity property of asymptotic,

$$\lim_{x\to 0^+} F(x) \sim (1/2)(1-e^{\lambda[1-(1+\theta x)^{\alpha}]}) \sim (1/2)(\lambda[(1+\theta x)^{\alpha}-1]).$$

□

**Theorem 2.2.** *The probability density function of the $TIHL_{NH}$ in (6) is (i) decreasing if $\alpha \le 1$, (ii) log-concave if $\alpha \ge 1$.*

*Proof.* (i) We consider $f'(x)$ below, and it is shown that $f'(x)$ is negative for all $\alpha \le 1$.

$$f'(x) = \frac{2\alpha\theta^2\lambda(1+\theta x)^{\alpha-2}e^{\lambda(1-(1+\theta x)^{\alpha})}}{(1+e^{\lambda(1-(1+\theta x)^{\alpha})})^3} \times \left[(\alpha-1)(1+e^{\lambda(1-(1+\theta x)^{\alpha})}) - \alpha\lambda(1+\theta x)^{\alpha-1}(1-e^{\lambda(1-(1+\theta x)^{\alpha})})\right].$$

(ii) Let $w = (1+\theta x)^{\alpha}$, it implies that $w \ge 1$ for $x > 0$, thus, $x = (w^{\frac{1}{\alpha}}-1)/\lambda$. Therefore, we can write $TIHL_{NH}$ pdf in term of $w$ as

$$B(w) = f((w^{\frac{1}{\alpha}}-1)/\lambda) = 2\alpha\theta\lambda\frac{w^{\frac{\alpha-1}{\alpha}}e^{\lambda(1-w)}}{[1+e^{\lambda(1-w)}]^2}, \quad w \ge 1.$$

The result is obtained by considering the second derivative of $\log B(w)$ as

$$(\log B(w))'' = -\frac{(\alpha-1)}{\alpha w^2} - \frac{2\lambda e^{\lambda(1-w)}}{1+e^{\lambda(1-w)}}\left[1-\frac{e^{\lambda(1-w)}}{1+e^{\lambda(1-w)}}\right],$$

where $(\log B(w))'' < 0$ at $\alpha \ge 1$. This proof is analogous to that of Lemonte (2013). □

**Theorem 2.3.** *The hazard rate function of the $TIHL_{NH}$ given by (8) is monotonic increasing if $\alpha \ge 1$.*

*Proof.* Follow from theorem 2.2 (ii), log-concave property of $f(x)$. □

The limiting behavior of the $f(x)$ and $h(x)$ are: $\lim_{x\to 0} f(x) = \alpha\theta\lambda/2, \lim_{x\to\infty} f(x) = 0$, $\lim_{x\to 0} h(x) = \alpha\theta\lambda/2$, $\lim_{x\to\infty} f(x) = 0$ for $\alpha < 1$, $\lim_{x\to\infty} f(x) = \infty$ for $\alpha > 1$, and $\lim_{x\to\infty} f(x) = \theta\lambda$ for $\alpha = 1$.

Its clear from the figure 1 (left) that unlike NH, the $TIHL_{NH}$ can accommodate both decreasing and unimodal densities. In addition, NH possesses only monotone and constant failure rate, from the figures 1 (right), 2 and 3 we can see that $TIHL_{NH}$ has increasing, decreasing, unimodal, bathtub and decreasing-increasing-decreasing failure rate functions.

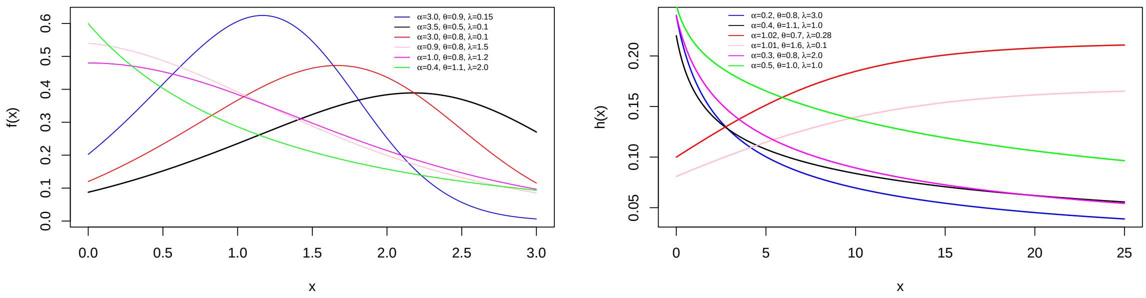


**Figure 1:** Plots of some possible shapes of the pdf (left) and hrf (right) of the $TIHL_{NH}$ for some value of parameters

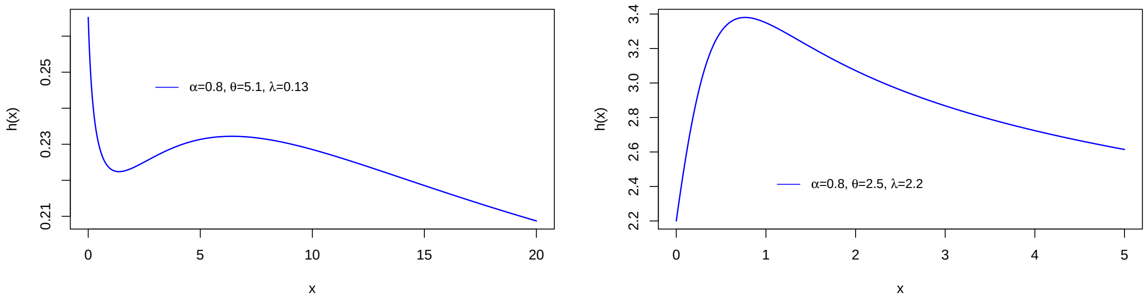


**Figure 2:** Plots of some possible shapes of the hrf of the $TIHL_{NH}$ for some value of parameters

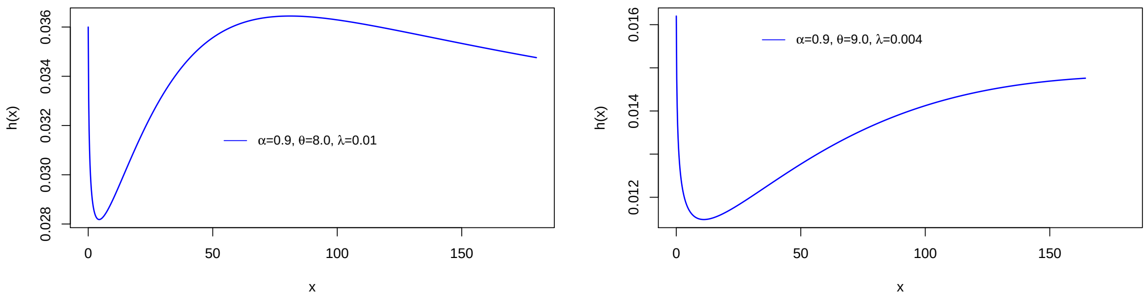


**Figure 3:** Plots of some possible shapes of the hrf of the $TIHL_{NH}$ for some value of parameters

## 2.1. Quantile function

The quantile function of the $TIHL_{NH}$ is computed by inverting equation (5) as

$$Q(u) = \theta^{-1}\left(\left[1-\lambda^{-1}\ln(\frac{1-u}{1+u})\right]^{\frac{1}{\alpha}}-1\right), \tag{9}$$

where $u \in (0,1)$. Therefore, we can get the value of the median and other percentiles of $X$ from (9). Moreover, equation (9) can be used to generate random data that follow $TIHL_{NH}$ by setting $u \sim U(0,1)$, where $U(0,1)$ is the uniform distribution. The median of $X$ is obtained as $Med(X) = \theta^{-1}\left(\left[1-\lambda^{-1}\ln(\frac{1}{3})\right]^{\frac{1}{\alpha}}-1\right)$, the behavior of the median is discussed in table 1, indicating that the median is decreasing as $\alpha$, $\theta$, and $\lambda$ increases. Further, equation (9) can be use to determine the behavior of the skewness and kurtosis of the $TIHL_{NH}$ with respect to the parameters by using the Bowley's skewness ($Bs$) and the Moors' kurtosis ($Mk$) defined below; notice that these measures are independent of $\theta$.

$$Bs = \frac{Q(\frac{3}{4}) - 2Q(\frac{2}{4}) + Q(\frac{1}{4})}{Q(\frac{3}{4}) - Q(\frac{1}{4})} \quad and \quad Mk = \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) - Q(\frac{3}{8}) + Q(\frac{1}{8})}{Q(\frac{6}{8}) - Q(\frac{2}{8})}.$$

Figure 4 shows that both the skewness and kurtosis of the $TIHL_{NH}$ distribution are decreasing in $\alpha$ and increasing in $\lambda$.

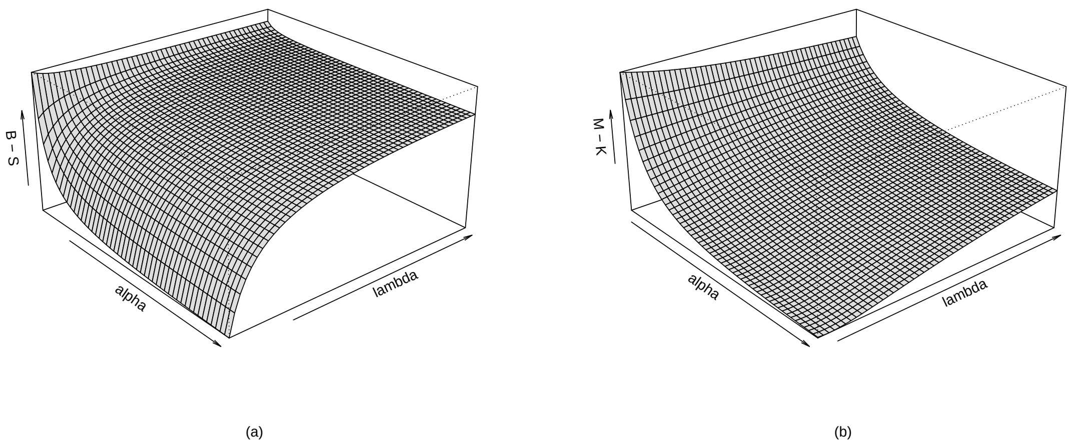


**Figure 4:** Plots of (a) Bowley skewness and (b) Moores kurtosis of the $TIHL_{NH}$ distribution

## 2.2. Moments

In this subsection, the moments of the $TIHL_{NH}$ distribution are derived and analyzed.

**Theorem 2.4.** *The $r^{th}$ moment of the $TIHL_{NH}$ is given by*

$$\mu_r^{'}(x) = \sum_{j=0}^{\infty}\sum_{i=0}^{r} \frac{2\lambda\binom{-2}{j}\binom{r}{i}(-1)^{r-i}e^{\lambda(j+1)}}{\theta^r\left[\lambda(j+1)\right]^{\frac{r}{\alpha}+1}}\Gamma(\frac{r}{\alpha}+1, \lambda(j+1)). \tag{10}$$

*Proof.* The $r^{th}$ ordinary moment of $X$ can be obtained as

$$\begin{aligned} \mu_r^{'}(x) = E(X^r) &= \int_0^{\infty} x^r f(x,\phi)dx \qquad (11)\\ &= 2\lambda\theta\alpha \sum_{j=0}^{\infty}\binom{-2}{j}e^{\lambda(j+1)}\int_0^{\infty} x^r(1+\theta x)^{\alpha-1}\, e^{-\lambda(j+1)(1+\theta x)^{\alpha}}\, dx, \end{aligned}$$

by setting $t = \lambda(j+1)(1+\theta x)^{\alpha}$ , $\mu_r^{'}(x)$ reduces to

$$\mu_r^{'}(x) = 2\lambda\theta^{-r}\sum_{j=0}^{\infty}\binom{-2}{j}e^{\lambda(j+1)}\int_{\lambda(j+1)}^{\infty}\left[\frac{t^{\frac{1}{\alpha}}}{\left[\lambda(j+1)\right]^{\frac{1}{\alpha}}} - 1\right]^r e^{-t}dt.$$

Using the binomial series in power of $r$, the integral becomes

$$\mu_r^{'}(x) = \sum_{j=0}^{\infty}\sum_{i=0}^{r} \frac{2\lambda\binom{-2}{j}\binom{r}{i}(-1)^{r-i}e^{\lambda(j+1)}}{\theta^r\left[\lambda(j+1)\right]^{\frac{r}{\alpha}+1}}\Gamma(\frac{r}{\alpha}+1, \lambda(j+1)).$$

Where $\Gamma(a,n) = \int_n^{\infty} x^{a-1}e^{-x}dx$ is the upper incomplete gamma function. ☐

It is clear from (10) that the $r^{th}$ moments of the $TIHL_{NH}$ has simple convergent series representation with less summations, thus, it can be determine numerically with the aid of computer capabilities.

The $r^{th}$ ordinary moment $\mu_r^{'}$ in (10) or (11) can be used to obtain the moments by substituting $r = 1, 2, 3 \ldots$, where possible. The variance ( $\sigma^2$), skewness ($\gamma^3$), kurtosis($\gamma^4$) and coefficient of variation (CV) of $X$ could be obtain using $\sigma^2 = \mu_2^{'} - \mu_1^{'2}$, $\gamma^3 = \frac{\mu_3^{'} - 3\mu_2^{'}\mu_1^{'} + 2\mu_1^{'3}}{(\mu_2^{'} - \mu_1^{'2})^{3/2}}$, $\gamma^4 = \frac{\mu_4^{'} - 4\mu_3^{'}\mu_1^{'} + 6\mu_2^{'}\mu_1^{'2} + 3\mu_1^{'4}}{(\mu_2^{'} - \mu_1^{'2})^2}$ and $CV = \sqrt{\frac{\mu_2^{'}}{\mu_1^{'2}} - 1}$.

The numerical values of some moments, variance ( $\sigma^2$), skewness ($\gamma^3$), kurtosis($\gamma^4$), and coefficient of variation (CV) of the $TIHL_{NH}$ are given in table 1. The numerical values indicated that as the parameters $\alpha$, $\theta$, and $\lambda$ increases the given moments, variance, and skewness are decreasing, while the kurtosis is decreasing-increasing-decreasing and the coefficient of variation is decreasing than increasing. For the missing values of $\mu_{25}^{'}$ and $\mu_{100}^{'}$ for some parameters in table 1, the numerical integral was unable to obtained even with higher error tolerance, also, the for the series representation it is necessary to used higher precision which required `Rmpfr (mpfr)` package Mächler (2015) in R software Team (2019), but unfortunately the incomplete gamma function `gammainc` in R is yet to operate with `mpfr`, hence, Monte Carlo method can be used, thus, we consider this case as future work to consider other possible packages and address it in detail, as occurs in the Castellares and Lemonte (2019) for the moments of the generalized Gompertz, Guerra, Peña-Ramirez, Peña-Ramirez, and Cordeiro (2020) for the moments of the beta Burr XII, Cordeiro and Bager (2015) for the moments of some Kumaraswamy generalized distributions, among others.

Now, we compute the conditional moments of the $TIHL_{NH}$ which are useful in computing the mean deviations, Bonferroni and Lorenz curves and so on. The $s^{th}$ lower incomplete moments of $X$ is defined by $v_s(t)$ and computed as

$$v_s(t) = \int_0^t x^s f(x,\phi)dx = \sum_{j=0}^{\infty}\sum_{i=0}^{s} \frac{2\lambda\binom{-2}{j}\binom{s}{i}(-1)^{s-i}e^{\lambda(j+1)}}{\theta^s\left[\lambda(j+1)\right]^{\frac{s}{\alpha}+1}}\gamma(\frac{s}{\alpha}+1, \lambda(j+1)(1+\theta t)^{\alpha}), \quad (12)$$

where $\gamma(a,t) = \int_0^t x^{a-1}e^{-x}dx$ is the lower incomplete gamma function.

### 2.3. Mean deviations, Bonferroni and Lorenz curves

The mean deviation about the mean $\delta_1(X)$ and mean deviation about the median $\delta_2(X)$ measure the amount of scatter in a population. For a random variable $X$ with $TIHL_{NH}$, the mean deviations are defined by $\delta_1(X) = \int_0^{\infty} \mid x - \mu^{'} \mid f(x)dx = 2\mu^{'}F(\mu^{'}) - 2\psi(\mu^{'})$ and $\delta_2(X) = \int_0^{\infty} \mid x - M \mid f(x)dx = \mu^{'} - 2\psi(M)$, respectively, where $F(x)$ is the distribution function of $X$, $\mu^{'} = E(X)$ is the mean of $X$ and $M = Med(X)$. To compute $\delta_1(X)$ and $\delta_2(X)$, it is enough to obtain $\psi(a)$ by considering (12) at $s = 1$. The numerical values of the $\delta_1(X)$ and $\delta_2(X)$ of the $TIHL_{NH}$ are given in table 1 showing that both $\delta_1(X)$ and $\delta_2(X)$ are decreasing as the parameters $\alpha$, $\theta$ and $\lambda$ increases.

The Bonferroni and Lorenz curves are very important in reliability, econometrics, and insurance among others. The Bonferroni and Lorenz curves are defined respectively by $B(p) = \frac{\psi(q)}{p\mu^{'}}$ and $L(p) = \frac{\psi(q)}{\mu^{'}}$, where $\psi(q)$ is computed from (12), $\mu^{'}$ can be determined from (10) with r = 1, $q = Q(p)$ is calculated from (9) and $p$ is any given probability. Figure 5 show the plots of the Bonferroni and Lorenz curves of $TIHL_{NH}$ as the probability $p$ for some values of parameters.

### 2.4. Entropy and Kullback-Leibler divergence

The entropy of a random variable $X$ is a measure of the variation of uncertainty. Here, we derived the Renyi entropy and Shannon entropy. The Renyi entropy of a random variable

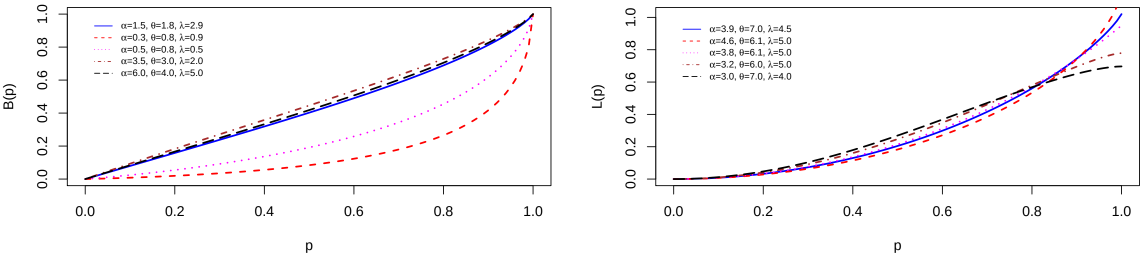


**Figure 5:** Plots of Bonferroni and Lorenz curves of the $TIHL_{NH}$ distribution for some parameter value.

**Table 1:** The numerical values of the median $(Med(X))$, some moments $(\mu_r^{'},\ r = 1, 2, 3, 4, 25, 100)$, variance ( $\sigma^2$), skewness $(\gamma^3)$, kurtosis$(\gamma^4)$, coefficient of variation (CV), and mean deviations $\delta_1(X)$ & $\delta_2(X)$ of the $TIHL_{NH}$ for some parameter values

| $(\alpha,\theta,\lambda)$ | $(0.5, 0.6, 0.7)$ | $(0.8, 0.9, 0.8)$ | $(1.2, 1.1, 1.5)$ | $(1.5, 1.6, 1.7)$ | $(2.5, 2.1, 1.9)$ | $(2.9, 2.8, 2.0)$ | $(3.0, 3.5, 2.5)$ | $(5.7, 5.0, 4.5)$ | $(7.0, 6.0, 5.0)$ |
|---|---|---|---|---|---|---|---|---|---|
| $Med(X)$ | 9.3368 | 2.1619 | 0.5279 | 0.24638 | 0.095350 | 0.058199 | 0.036885 | 0.00781 | 0.004797 |
| $\mu_1^{'}$ | 17.7914 | 2.9506 | 0.6436 | 0.29213 | 0.108890 | 0.066128 | 0.042367 | 0.00912 | 0.005614 |
| $\mu_2^{'}$ | 950.96 | 16.540 | 0.6779 | 0.13482 | 0.017882 | 0.006555 | 0.00273 | 0.00013 | $5.693\times10^{-5}$ |
| $\mu_3^{'}$ | 106093.1 | 138.86 | 0.9553 | 0.08097 | 0.003689 | 0.000812 | 0.000221 | $2.078\times10^{-5}$ | $1.743\times10^{-7}$ |
| $\mu_4^{'}$ | 20852297 | 1588.7 | 1.6689 | 0.05888 | 0.000894 | 0.000118 | $2.076\times10^{-5}$ | $4.326\times10^{-8}$ | $9.037\times10^{-7}$ |
| $\mu_{25}^{'}$ | - | - | $3.283\times10^{14}$ | 6185.156 | $2.115\times10^{-10}$ | $3.261\times10^{-16}$ | $1.398\times10^{-20}$ | $9.050\times10^{-37}$ | $1.241\times10^{-41}$ |
| $\mu_{100}^{'}$ | - | - | - | - | $4.021\times10^{-9}$ | $6.252\times10^{-34}$ | $7.074\times10^{-51}$ | $1.029\times10^{-116}$ | $4.342\times10^{-137}$ |
| $\sigma^2$ | 634.4293 | 7.8337 | 0.2637 | 0.04948 | 0.006025 | 0.002182 | 0.000931 | $4.551\times10^{-5}$ | $2.541\times10^{-5}$ |
| $\gamma^3$ | 4.16769 | 1.9990 | 1.3262 | 1.15240 | 0.919099 | 0.885717 | 0.839885 | 0.24222 | -3.36231 |
| $\gamma^4$ | 38.2825 | 16.9660 | 20.2695 | 22.5241 | 27.0273 | 27.7770 | 25.7478 | 25.2995 | 29.2310 |
| $CV$ | 1.41573 | 0.9486 | 0.7978 | 0.76140 | 0.712840 | 0.706370 | 0.720215 | 0.73990 | 0.89784 |
| $\delta_1(X)$ | 15.8011 | 2.0593 | 0.3986 | 0.17503 | 0.062170 | 0.037508 | 0.024416 | 0.00537 | 0.003322 |
| $\delta_2(X)$ | 14.0341 | 1.9591 | 0.3881 | 0.17136 | 0.061286 | 0.037007 | 0.024047 | 0.00528 | 0.003262 |

X with probability density function $f(x)$ is defined as $I_{R(\rho)} = (1-\rho)^{-1}\ln\int_{-\infty}^{\infty} f^{\rho}(x)dx$, for $\rho > 0$ and $\rho \neq 1$. First, we compute the $\int_0^{\infty} f^{\rho}(x)dx$.

$$\int_0^{\infty} f^{\rho}(x)dx = \sum_{j=0}^{\infty} \binom{-2\rho}{j} 2^{\rho}\alpha^{\rho}\theta^{\rho}\lambda^{\rho} \int_0^{\infty} (1+\theta x)^{\rho(\alpha-1)}\, e^{\lambda(j+\rho)} e^{-\lambda(j+\rho)(1+\theta x)^{\alpha}}\, dx,$$

let $t = \lambda(j+\rho)(1+\theta x)^{\alpha}$, then after some algebra the integral become a mixture of upper incomplete gamma function as

$$\begin{aligned}\int_0^{\infty} f^{\rho}(x)dx &= \sum_{j=0}^{\infty} \frac{\binom{-2\rho}{j} 2^{\rho}\alpha^{\rho-1}\theta^{\rho-1}\lambda^{\rho}\, e^{\lambda(j+\rho)}}{[\lambda(j+\rho)]^{\frac{1}{\alpha}(1-\rho)+\rho}} \int_{\lambda(j+\rho)}^{\infty} t^{\frac{1}{\alpha}(1-\rho)+\rho-1} e^{-t}dt \\ &= \sum_{j=0}^{\infty} \tau_j(\alpha,\theta,\lambda,\rho)\Gamma(\frac{1}{\alpha}(1-\rho)+\rho, \lambda(j+\rho)),\end{aligned}$$

where $\tau_j(\alpha,\theta,\lambda,\rho) = \frac{\binom{-2\rho}{j} 2^{\rho}\alpha^{\rho-1}\theta^{\rho-1}\lambda^{\rho}\, e^{\lambda(j+\rho)}}{[\lambda(j+\rho)]^{\frac{1}{\alpha}(1-\rho)+\rho}}$. Therefore, the Renyi entropy is given by

$$I_{R(\rho)} = (1-\rho)^{-1}\ln\left(\sum_{j=0}^{\infty} \tau_j(\alpha,\theta,\lambda,\rho)\Gamma(\frac{1}{\alpha}(1-\rho)+\rho, \lambda(j+\rho))\right). \quad (13)$$

Table 2 provide some numerical values of the Renyi entropy for some parameter values and $\rho$. Observe that the Renyi entropy is decreasing as the parameters and $\rho$ increases.
The Shannon entropy of $X$ is defined by $E[-\log f(X)]$ it is also a particular case of the Renyi entropy when $\rho \to 1$. The Shannon entropy of $TIHL_{NH}$ is obtained by considering the following lemma 2.5.

**Lemma 2.5.** *Let $X \sim TIHL_{NH}$, and $\xi > 0$, then*

$$E[(1+\theta X)^{\xi}] = \sum_{j=0}^{\infty} \zeta_j^{*}(\alpha,\lambda,\xi)\Gamma(\frac{\xi}{\alpha}+1, \lambda(j+1)), \tag{14}$$

$$E[\log(1+\theta X)] = \sum_{j=0}^{\infty} \zeta_j^{**}(\lambda)\frac{\partial}{\partial a}\left(\frac{\Gamma(\frac{a}{\alpha}+1,\lambda(j+1))}{[\lambda(j+1)]^{\frac{a}{\alpha}}}\right)|_{a=0}, \tag{15}$$

$$E\left[\log\left(1+e^{\lambda[1-(1+\theta X)^{\alpha}]}\right)\right] = \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{2(-1)^{k+1}\binom{-2}{j}}{k\,(k+j+1)}, \tag{16}$$

*where* $\zeta_j^{*}(\alpha,\lambda,\xi) = \frac{\binom{-2}{j}2\lambda e^{\lambda(j+1)}}{[\lambda(j+1)]^{\frac{\xi}{\alpha}+1}}$, *and* $\zeta_j^{**}(\lambda) = \frac{\binom{-2}{j}2\lambda e^{\lambda(j+1)}}{[\lambda(j+1)]}$.

*Proof.* The proof of (14) as follows

$$E[(1+\theta X)^{\xi}] = \int_0^{\infty}(1+\theta x)^{\xi} f(x)dx = 2\alpha\theta\lambda\sum_{j=0}^{\infty} e^{\lambda(j+1)}\binom{-2}{j}\int_0^{\infty}(1+\theta x)^{\alpha+\xi-1}e^{-\lambda(j+1)(1+\theta x)^{\alpha}}dx$$

by letting $t = \lambda(j+1)(1+\theta x)^{\alpha}$, we get

$$E[(1+\theta X)^{\xi}] = \frac{\binom{-2}{j}2\lambda e^{\lambda(j+1)}}{[\lambda(j+1)]^{\frac{\xi}{\alpha}+1}}\int_{\lambda(j+1)}^{\infty} t^{\frac{\xi}{\alpha}}e^{-t}dt = \sum_{j=0}^{\infty}\zeta_j^{*}(\alpha,\lambda,\xi)\Gamma(\frac{\xi}{\alpha}+1,\lambda(j+1)),$$

where $\zeta_j^{*}(\alpha,\lambda,\xi) = \frac{\binom{-2}{j}2\lambda e^{\lambda(j+1)}}{[\lambda(j+1)]^{\frac{\xi}{\alpha}+1}}$.

The first step in the proof of (15) is given below, but the rest follow similar to the (14), therefore, omitted.

$$E[\log(1+\theta X)] = \frac{\partial}{\partial a}E[(1+\theta X)^{a}]\,|_{a=0} = \frac{\partial}{\partial a}\int_0^{\infty}(1+\theta x)^{a} f(x)dx\,|_{a=0}.$$

For the proof of (16), after expansion of $\log\left[1+e^{\lambda[1-(1+\theta x)^{\alpha}]}\right]$ and the denominator of $f(x)$, we have,

$$\begin{aligned} E\left[\log\left(1+e^{\lambda[1-(1+\theta X)^{\alpha}]}\right)\right] &= \int_0^{\infty}\log\left[1+e^{\lambda[1-(1+\theta x)^{\alpha}]}\right] f(x)dx \\ &= \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{2\binom{-2}{j}^{k+1}\alpha\theta\lambda e^{\lambda(k+j+1)}}{k}\int_0^{\infty}(1+\theta x)^{\alpha-1}e^{-\lambda(k+j+1)(1+\theta x)^{\alpha}}dx, \end{aligned}$$

letting $t = \lambda(k+j+1)(1+\theta x)^{\alpha}$, we obtain $E\left[\log\left(1+e^{\lambda[1-(1+\theta X)^{\alpha}]}\right)\right] = \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{2(-1)^{k+1}\binom{-2}{j}}{k\,(k+j+1)}$. □

Then, the Shannon entropy of the $TIHL_{HN}$ can be derived as

$$\begin{aligned} E[-\log f(X)] = &-\log(2\alpha\lambda\theta) - \lambda - (\alpha-1)E[\log(1+\theta X)] + \lambda E[(1+\theta X)^{\alpha}] \\ &+ 2E\left[\log\left(1+e^{\lambda[1-(1+\theta X)^{\alpha}]}\right)\right]. \end{aligned} \tag{17}$$

By applying (14)-(16) in (17) we get

$$\begin{aligned} E[-\log f(X)] = \log(2\alpha\lambda\theta)^{-1} - \lambda - (\alpha-1)\sum_{j=0}^{\infty}\zeta^{**}(\lambda)\frac{\partial}{\partial a}\left(\frac{\Gamma(\frac{a}{\alpha}+1,\lambda(j+1))}{[\lambda(j+1)]^{\frac{a}{\alpha}}}\right)|_{a=0} \\ + \lambda\sum_{j=0}^{\infty}\zeta_j^{*}(\alpha,\lambda)\Gamma(2,\lambda(j+1)) + \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{4(-1)^{k+1}\binom{-2}{j}}{k\,(k+j+1)}. \end{aligned}$$

The numerical values of the Shannon entropy given in table 2 indicated that the Shannon entropy is decreasing as the parameters increases.

Now, we shall compute the Kullback-Leibler (KL) divergence for the $TIHL_{NH}$ distributions. The KL-divergence is a fundamental equation of information theory that measures the proximity of two probability distributions, i.e it measures the distance between two density functions. It is also called the information divergence and relative entropy. For a random variables $X_1 \sim TIHL_{NH}(\alpha_1, \theta, \lambda_1)$ and $X_2 \sim TIHL_{NH}(\alpha_2, \theta, \lambda_2)$ with density functions $f_1(x)$ and $f_2(x)$, then the KL divergence measure of $f_1$ and $f_2$ is defined as $KL(f_1||f_2) = \int_0^\infty f_1 \log\left(\frac{f_1}{f_2}\right) dx$. The following lemma 2.6 is used to obtain the last expression in the simplified $KL(f_1||f_2)$ in (19).

**Lemma 2.6.** *Let* $X_1 \sim TIHL_{NH}(\alpha_1, \theta, \lambda_1)$ *and* $X_2 \sim TIHL_{NH}(\alpha_2, \theta, \lambda_2)$, *then*

$$\begin{aligned} E_{f_1}\left[\log\left(1 + e^{\lambda_2[1-(1+\theta X)^{\alpha_2}]}\right)\right] &= \int_0^\infty \log\left[1 + e^{\lambda_2[1-(1+\theta X)^{\alpha_2}]}\right] f_1(x)dx \\ &= \sum_{k=1}^{\infty}\sum_{i,j=0}^{\infty} \zeta^{***}_{i,j,k}(\alpha_1, \alpha_2, \lambda_1, \lambda_2)\Gamma(\frac{\alpha_2 i}{\alpha_1} + 1, \lambda_1(j+1)), \end{aligned} \tag{18}$$

*where* $\zeta^{***}_{i,j,k}(\alpha_1, \alpha_2, \lambda_1, \lambda_2) = \frac{\binom{-2}{j} 2(-1)^{k+i+1}\lambda_1\lambda_2^i k^i e^{\lambda_2 k + \lambda_1(j+1)}}{k\, i!\, [\lambda_1(j+1)]^{\frac{\alpha_2 i}{\alpha_1}+1}}$.

*Proof.* After expanding the logarithmic expression, then follow similar to (10). □

Here, we denote by $E_{f_1}[k(x)] = \int_0^\infty k(x)f_1(x)dx$, the expectation a function $K(x)$ base on $f_1$. Now, we compute the $KL(f_1||f_2)$ as follows

$$\begin{aligned} \int_0^\infty f_1 \log\left(\frac{f_1}{f_2}\right) dx &= \int_0^\infty f_1(x)\log f_1(x)dx - \int_0^\infty f_1(x)\log f_2(x)dx \\ &= E_{f_1}[\log f_1(X)] - E_{f_1}[\log f_2(X)] \\ &= \log(\alpha_1\lambda_1/\alpha_2\lambda_2) + [\lambda_1 - \lambda_2] + (\alpha_1 - \alpha_2)E_{f_1}[\log(1+\theta X)] \\ &- \lambda_1 E_{f_1}[(1+\theta X)^{\alpha_1}] + \lambda_2 E_{f_1}[(1+\theta X)^{\alpha_2}] \\ &- 2E_{f_1}\left[\log\left(1 + e^{\lambda_1[1-(1+\theta X)^{\alpha_1}]}\right)\right] + 2E_{f_1}\left[\log\left(1 + e^{\lambda_2[1-(1+\theta X)^{\alpha_2}]}\right)\right]. \end{aligned} \tag{19}$$

Therefore, by considering lemma 2.5 for the third to six terms and lemma 2.6 for the last expression of (19), we get

$$\begin{aligned} KL(f_1||f_2) = \log(\frac{\alpha_1\lambda_1}{\alpha_2\lambda_2}) + [\lambda_1 - \lambda_2] + (\alpha_1 - \alpha_2)\sum_{j=0}^{\infty}\zeta^{**}(\lambda_1)\frac{\partial}{\partial a}\left(\frac{\Gamma(\frac{a}{\alpha_1} + 1, \lambda_1(j+1))}{[\lambda_1(j+1)]^{\frac{a}{\alpha_1}}}\right)|_{a=0} \\ - \sum_{j=0}^{\infty}\lambda_1\zeta^*(\alpha_1, \lambda_1)\Gamma(2, \lambda_1(j+1)) + \sum_{j=0}^{\infty}\lambda_2\zeta^*(\alpha_1, \lambda_1, \alpha_2)\Gamma(\frac{\alpha_2}{\alpha_1} + 1, \lambda_1(j+1)) \\ - \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{4(-1)^{k+1}\binom{-2}{j}}{k\,(k+j+1)} + \sum_{k=1}^{\infty}\sum_{i,j=0}^{\infty} 2\zeta^{***}_{i,j,k}(\alpha_1, \alpha_2, \lambda_1, \lambda_2)\Gamma(\frac{\alpha_2 i}{\alpha_1} + 1, \lambda_1(j+1)). \end{aligned}$$

## 3. Parameter estimation

In this section, parameters estimation of the $TIHL_{NH}(\phi)$ distribution is accomplished by the maximum likelihood method (MLE) and investigated by simulation studies.

**Table 2:** The numerical values of the Shannon and Renyi entropies of $TIHL_{NH}$ for some parameter values

| $(\alpha,\theta,\lambda)$ | *Shannon Entropy* | $(\alpha,\theta,\lambda)$ | *Shannon Entropy* | $(\rho,\alpha,\theta,\lambda)$ | $I_R(\rho)$ | $(\rho,\alpha,\theta,\lambda)$ | $I_R(\rho)$ |
|---|---|---|---|---|---|---|---|
| (0.1, 0.5, 0.6) | 14.2018 | (1.0, 0.9, 1.0) | 1.4122 | (0.1, 0.4, 0.3, 0.5) | 10.2020 | (0.8, 0.9, 0.9, 1.0) | 1.6952 |
| (0.3, 0.6, 0.8) | 5.3002 | (1.0, 1.0, 1.0) | 1.3069 | (0.2, 0.6, 0.3, 0.5) | 6.0462 | (0.9, 1.2, 1.0, 1.0) | 1.0295 |
| (0.5, 0.6, 0.8) | 3.6149 | (1.0, 1.0, 1.2) | 1.1245 | (0.4, 0.7, 0.3, 0.5) | 4.6857 | (1.1, 1.2, 1.4, 1.3) | 0.3895 |
| (0.6, 0.6, 0.7) | 3.3213 | (1.5, 1.0, 1.2) | 0.4922 | (0.7, 0.9, 0.5, 0.8) | 2.5802 | (2.0, 1.3, 1.4, 1.4) | 0.0578 |
| (0.7, 0.7, 0.7) | 2.7858 | (1.5, 3.0, 1.2) | -0.6064 | (0.8, 0.9, 0.5, 0.8) | 2.5196 | (3.0, 1.5, 4.0, 2.0) | -1.5752 |
| (0.9, 0.7, 0.7) | 2.2316 | (5.0, 4.0, 6.0) | -3.6380 | (0.8, 0.9, 0.8, 0.9) | 1.9246 | (6.0, 5.0, 7.0, 9.0) | -4.9034 |
| (0.9, 0.8, 0.7) | 2.0981 | (9.0, 7.0, 10.0) | -5.2500 | (0.8, 0.9, 0.9, 0.9) | 1.8068 | (10.0, 9.0, 12.0, 18.0) | -6.7590 |

## 3.1. Maximum likelihood method (MLE)

Let $X_1, X_2, ..., X_n$ be a random sample of size $n$ from $TIHL_{NH}(\phi)$, where $\phi = (\alpha,\theta,\lambda)^T$. Let the estimator of $\phi$, be $\widehat{\phi}$, the log likelihood function for the vector of parameters $\phi$ can be written as

$$\log L = n\log 2 + n\log\alpha + n\log\theta + n\log\lambda + n\lambda + (\alpha-1)\sum_{i=1}^{n}\log(1+\theta x_i)$$
$$-\lambda\sum_{i=1}^{n}(1+\theta x_i)^{\alpha} - 2\sum_{i=1}^{n}\log\left(1+e^{\lambda(1-(1+\theta x_i)^{\alpha})}\right). \tag{20}$$

The log-likelihood can be maximized by solving the nonlinear equations obtained by differentiating (20) as

$$\frac{\partial\log\ L}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\log z_i - \lambda\sum_{i=1}^{n} z_i^{\alpha}\log z_i + 2\lambda\sum_{i=1}^{n}\frac{e^{\lambda(1-z_i^{\alpha})}z_i^{\alpha}\log z_i}{1+e^{\lambda(1-z_i^{\alpha})}}, \tag{21}$$

$$\frac{\partial\log\ L}{\partial\theta} = \frac{n}{\theta} + (\alpha-1)\sum_{i=1}^{n}\frac{x_i}{z_i} - \alpha\lambda\sum_{i=1}^{n} z_i^{\alpha-1}x_i + 2\alpha\lambda\sum_{i=1}^{n}\frac{z_i^{\alpha-1}x_i e^{\lambda(1-z_i^{\alpha})}}{1+e^{\lambda(1-z_i^{\alpha})}}, \tag{22}$$

$$\frac{\partial\log\ L}{\partial\lambda} = \frac{n}{\lambda} + n - \sum_{i=1}^{n} z^{\alpha} - 2\sum_{i=1}^{n}\frac{e^{\lambda(1-z_i^{\alpha})}(1-z_i^{\alpha})}{1+e^{\lambda(1-z_i^{\alpha})}}, \tag{23}$$

where $z_i = 1+\theta x_i$. These equations can be solved by mathematical packages such as `R` and `Mathematica`. Now, we discuss the existence of the MLEs under some possible conditions. Analogous discussions for this proofs can be found in Jafari and Tahmasebi (2016); Muhammad (2016a, 2017a,c); Muhammad and Liu (2019, 2021b).

**Proposition 3.1.** *Let $K_1(\alpha;\lambda,\theta,\boldsymbol{x})$, be the right hand side of equation (21), where $\lambda$ and $\theta$ are true values of the parameters, then $K_1(\alpha;\lambda,\theta,\boldsymbol{x}) = 0$ has at least one root.*

*Proof.* From the (21), $\lim_{\alpha\to 0} K_1 = \infty$, and in the $\lim_{\alpha\to\infty} K_1$ we rewrite the last term in (21), then the limit by L'hopital's is $2\lambda\lim_{\alpha\to\infty}\sum_{i=1}^{n}\frac{(1+\theta x_i)^{\alpha}\log(1+\theta x_i)}{e^{-\lambda(1-(1+\theta x_i)^{\alpha})}+1} = 0$, therefore, $\lim_{\alpha\to\infty} K_1 = -\infty$. Hence, $K_1$ is a function passes from non-negative to negative, so $K_1 = 0$ has at least one root. □

**Proposition 3.2.** *Let $K_2(\theta;\lambda,\alpha,\boldsymbol{x})$, be the right hand side of equation (22), where $\alpha > 1$ and $\lambda$ are true values of the parameters, then $K_2(\theta;\lambda,\alpha,\boldsymbol{x}) = 0$ has at least one root.*

*Proof.* From the (22), $\lim_{\theta\to 0} K_2 = \infty$, and in the $\lim_{\theta\to\infty} K_2$ we rewrite the last term in (22) and its limit is $2\alpha\lambda\lim_{\theta\to\infty}\sum_{i=1}^{n}\left[\frac{x_i(1+\theta x_i)^{\alpha-1}}{e^{-\lambda(1-(1+\theta x_i)^{\alpha})}+1}\right]|_{\alpha>1}, = 0$, thus, $\lim_{\theta\to\infty} K_2 = -\infty$. Hence, $K_2$ is a function that passes from non-negative to negative, so $K_2 = 0$ has at least one root. □

**Proposition 3.3.** *Let $K_3(\lambda;\alpha,\theta,\boldsymbol{x})$, be the right hand side of equation (23), where $\alpha$ and $\theta$ are true values of the parameters, then $K_3(\lambda;\alpha,\theta,\boldsymbol{x}) = 0$ has at least one root.*

*Proof.* From the (23), $\lim_{\lambda\to 0} K_3 = \infty$, and $\lim_{\lambda\to\infty} K_1 = n - \sum_{i=1}^{n} z_i^{\alpha} < 0$, since $z_i = (1+\theta x_i) > 1$, $\forall \theta, \alpha, x_i$. Thus, $K_3$ is a function passes from non-negative to negative, hence $K_3 = 0$ has at least one root. □

Now, we can obtain an approximate confidence interval for the parameters. In the theorem 3.5 we established the asymptotic normality distribution for the MLEs, but we require the following lemma 3.4 which is useful in the computation of the elements of Fisher information matrix $I_n$.

**Lemma 3.4.** *Let $\beta_1 \in \mathbb{R}$ and $\beta_2, \beta_3, \beta_4, \beta_5 \in \mathbb{N} \cup \{0\}$, let $X \sim TIHL_{NH}(\phi)$ with pdf in (6), let*

$$C(\beta_1,\beta_2,\beta_3,\beta_4,\beta_5) = E\left[\frac{(1+\theta X)^{\beta_1}\, X^{\beta_2}\, e^{\beta_3\lambda(1-(1+\theta X)^{\alpha})}\, \log^{\beta_4}(1+\theta X)}{[1+e^{\lambda(1-(1+\theta X)^{\alpha})}]^{\beta_5}}\right],$$

*then,*

$$C(\beta_1,\beta_2,\beta_3,\beta_4,\beta_5) = \sum_{j=0}^{\infty}\sum_{i=0}^{\beta_2} \kappa_{i,j}\, \frac{\partial^{\beta_4}}{\partial t^{\beta_4}} \left[\frac{\Gamma(\frac{\beta_1+t+i}{\alpha}+1, \lambda(\beta_3+j+1))}{[\lambda(\beta_3+j+1)]^{\frac{t}{\alpha}}}\right]_{t=0},$$

*where* $\kappa_{i,j} = \frac{\binom{-(\beta_5+2)}{j}\binom{\beta_2}{i} 2\lambda(-1)^{\beta_2-i}\, e^{\lambda(\beta_3+j+1)}}{\theta^{\beta_2}[\lambda(\beta_3+j+1)]^{\frac{\beta_1+i}{\alpha}+1}}$.

*Proof.* we obtain the second step below after rewriting the $\log^{\beta_4}(1+\theta x)$ in partial derivative form and the expansion of the denominator of $f(x)$.

$$\begin{aligned} C(\beta_1,\beta_2,\beta_3,\beta_4,\beta_5) &= E\left[\frac{(1+\theta X)^{\beta_1}\, X^{\beta_2}\, e^{\beta_3\lambda(1-(1+\theta X)^{\alpha})}\, \log^{\beta_4}(1+\theta X)}{[1+e^{\lambda(1-(1+\theta X)^{\alpha})}]^{\beta_5}}\right] \\ &= \int_0^{\infty} \frac{(1+\theta x)^{\beta_1}\, x^{\beta_2}\, e^{\beta_3\lambda(1-(1+\theta x)^{\alpha})}\, \log^{\beta_4}(1+\theta x)}{[1+e^{\lambda(1-(1+\theta x)^{\alpha})}]^{\beta_5}} f(x)dx \\ &= 2\alpha\theta\lambda \sum_{j=0}^{\infty} \binom{-(\beta_5+2)}{j} \frac{\partial^{\beta_4}}{\partial t^{\beta_4}} \int_0^{\infty} x^{\beta_2}(1+\theta x)^{\beta_1+t+\alpha-1} e^{\lambda(\beta_3+j+1)(1-(1+\theta x)^{\alpha})} dx|_{t=0}. \end{aligned}$$

Let, $u = \lambda(\beta_3+j+1)(1+\theta x_i)^{\alpha}$, then,

$$\begin{aligned} C(\beta_1,\beta_2,\beta_3,\beta_4,\beta_5) &= 2\lambda \sum_{j=0}^{\infty} \binom{-(\beta_5+2)}{j} \frac{e^{\lambda(\beta_3+j+1)}}{\theta^{\beta_2}[\lambda(\beta_3+j+1)]^{\frac{\beta}{\alpha}+1}} \\ &\times \frac{\partial^{\beta_4}}{\partial t^{\beta_4}} \int_{\lambda(\beta_3+j+1)}^{\infty} \frac{u^{\frac{\beta_1+t}{\alpha}}}{[\lambda(\beta_3+j+1)]^{\frac{t}{\alpha}}} \left[\frac{u^{\frac{1}{\alpha}}}{[\lambda(\beta_3+j+1)]^{\frac{1}{\alpha}}} - 1\right]^{\beta_2} e^{-u} du|_{t=0}. \end{aligned}$$

Applying $\left[\frac{u^{\frac{1}{\alpha}}}{[\lambda(\beta_3+j+1)]^{\frac{1}{\alpha}}} - 1\right]^{\beta_2} = \sum_{i=0}^{\beta_2} \frac{\binom{\beta_2}{i}(-1)^{\beta_2-i} u^{\frac{i}{\alpha}}}{[\lambda(\beta_3+j+1)]^{\frac{i}{\alpha}}}$ we have,

$$\begin{aligned} C(\beta_1,\beta_2,\beta_3,\beta_4,\beta_5) &= \sum_{j=0}^{\infty}\sum_{i=0}^{\beta_2} \frac{\binom{-(\beta_5+2)}{j}\binom{\beta_2}{i} 2\lambda(-1)^{\beta_2-i}\, e^{\lambda(\beta_3+j+1)}}{\theta^{\beta_2}[\lambda(\beta_3+j+1)]^{\frac{\beta_1+i}{\alpha}+1}} \\ &\times \frac{\partial^{\beta_4}}{\partial t^{\beta_4}} \int_{\lambda(\beta_3+j+1)}^{\infty} \frac{u^{\frac{\beta_1+t+i}{\alpha}}}{[\lambda(\beta_3+j+1)]^{\frac{t}{\alpha}}} e^{-u} du|_{t=0} \\ &= \sum_{j=0}^{\infty}\sum_{i=0}^{\beta_2} \kappa_{i,j}\, \frac{\partial^{\beta_4}}{\partial t^{\beta_4}} \left[\frac{\Gamma(\frac{\beta_1+t+i}{\alpha}+1, \lambda(\beta_3+j+1))}{[\lambda(\beta_3+j+1)]^{\frac{t}{\alpha}}}\right]_{t=0}, \end{aligned}$$

where $\kappa_{i,j} = \frac{\binom{-(\beta_5+2)}{j}\binom{\beta_2}{i} 2\lambda(-1)^{\beta_2-i}\, e^{\lambda(\beta_3+j+1)}}{\theta^{\beta_2}[\lambda(\beta_3+j+1)]^{\frac{\beta_1+i}{\alpha}+1}}$. □

**Theorem 3.5.** *The maximum-likelihood estimators* $\widehat{\phi}$ *are consistent estimators, and* $\sqrt{n}(\widehat{\phi}-\phi)$ *is asymptotically multivariate normal with mean vector* 0 *and the variance–covariance matrix* $I_n^{-1}$, *where* $I_n = -n^{-1}E[\partial^2 \log L/\partial\phi\partial\phi^T] = -n^{-1}E[J_n(\phi)]$, *and the elements of* $J_n(\phi)$ *are given in the appendix, while the elements of* $I_n$ *are*

$$
\begin{aligned}
I_{\lambda\lambda} &= \frac{1}{\lambda^2} + 2C(0,0,1,0,1) - 4C(\alpha,0,1,0,1) + 2C(2\alpha,0,1,0,1) \\
&\quad - 2C(0,0,2,0,2) + 4C(\alpha,0,2,0,2) - 2C(2\alpha,0,2,0,2) \\
I_{\alpha\alpha} &= \frac{1}{\alpha^2} + \lambda C(\alpha,0,0,2,0) - 2\lambda C(\alpha,0,1,2,1) + 2\lambda^2 C(2\alpha,0,1,2,1) - 2\lambda^2 C(2\alpha,0,2,2,2) \\
I_{\theta\theta} &= \frac{1}{\theta^2} + (\alpha-1)C(-2,2,0,0,0) + 2\alpha\lambda(\alpha-1)C(\alpha-2,2,0,0,0) \\
&\quad - 2\lambda\alpha(\alpha-1)C(\alpha-2,2,1,0,1) + 2\alpha^2\lambda^2 C(2(\alpha-1),2,1,0,1) - 2\alpha^2\lambda^2 C(2(\alpha-1),2,2,0,2) \\
I_{\lambda\alpha} &= -C(\alpha,0,0,1,0) + 2\lambda C(\alpha,0,1,1,1) - 2\lambda C(2\alpha,0,1,1,1) + 2C(\alpha,0,1,1,1) \\
&\quad - 2\lambda C(\alpha,0,2,1,2) + 2\lambda C(2\alpha,0,2,1,2) \\
I_{\lambda\theta} &= \alpha C(\alpha-1,1,0,0,0) - 2\alpha\lambda C(\alpha-1,1,1,0,1) + 2\alpha\lambda C(2\alpha-1,1,1,0,1) \\
&\quad - 2\alpha C(\alpha-1,1,1,0,1) + 2\alpha\lambda C(\alpha-1,1,2,0,2) - 2\alpha\lambda C(2\alpha-1,1,2,0,2) \\
I_{\alpha\theta} &= -C(-1,1,0,0,0) + 2\lambda C(\alpha-1,1,0,0,0) + 2\lambda C(\alpha-1,1,0,1,0) \\
&\quad - 2\lambda C(\alpha-1,1,1,0,1) - 2\lambda\alpha C(\alpha-1,1,1,1,1) + 2\lambda^2\alpha C(2\alpha-1,1,1,1,1) \\
&\quad - 2\alpha\lambda^2 C(2\alpha-1,1,2,1,2).
\end{aligned}
$$

*Proof.* Follow from the regularity conditions stated in Cox and Hinkley (1979) and the lemma 3.4. □

An $100(1-\xi)$ asymptotic confidence interval for each parameter $\phi_r$ is given by $ACI_r = \left(\widehat{\phi}_r - w_{\frac{\xi}{2}}\sqrt{\widehat{I_{rr}}}, \widehat{\phi}_r + w_{\frac{\xi}{2}}\sqrt{\widehat{I_{rr}}}\right)$, where $\widehat{I_{rr}}$ is the $(r,r)$ diagonal element of $I_n^{-1}$ for $r = 1,2,3$, and $w_{\frac{\xi}{2}}$ is the quantile $1-\frac{\xi}{2}$ of the standard normal distribution.

## 3.2. Simulation

In this subsection, we assessed the proposed maximum likelihood estimation (MLE) by simulation studies. We generate a moderate sample 1000 of size $n = (50, 100, 150, 200, 500)$ each of which is randomly sampled from $TIHL_{NH}$ for some selected values of $\alpha, \theta$ and $\lambda$. The bias and mean square error (MSE) of the estimates are computed using `R` 3.5.2. The results are presented in figure 6 to 10 indicated that the MLEs shows consistency in the estimation, and the MSE and bias of the estimators decreases as the sample size increases.

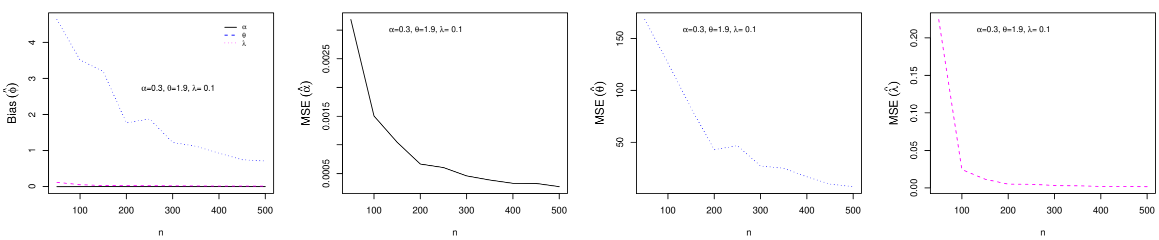


**Figure 6:** Simulation results from $TIHL_{NH}$ for parameter $\alpha = 0.3,\ \beta = 1.9,\ \lambda = 0.1$

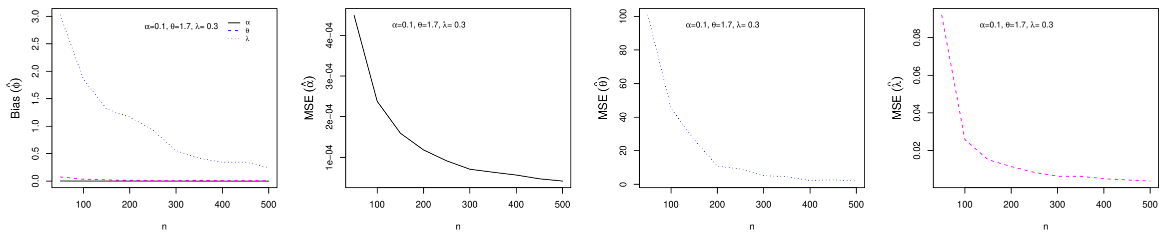


**Figure 7:** Simulation results from $TIHL_{NH}$ for parameter $\alpha = 0.1,\ \beta = 1.7,\ \lambda = 0.3$

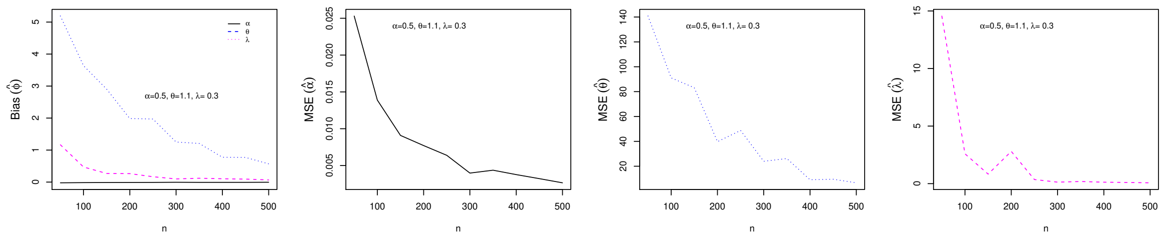


**Figure 8:** Simulation results from $TIHL_{NH}$ for parameter $\alpha = 0.5,\ \beta = 1.1,\ \lambda = 0.3$

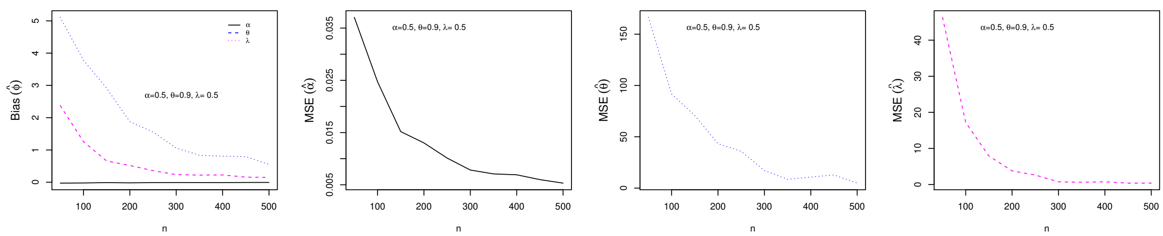


**Figure 9:** Simulation results from $TIHL_{NH}$ for parameter $\alpha = 0.5,\ \beta = 0.9,\ \lambda = 0.5$

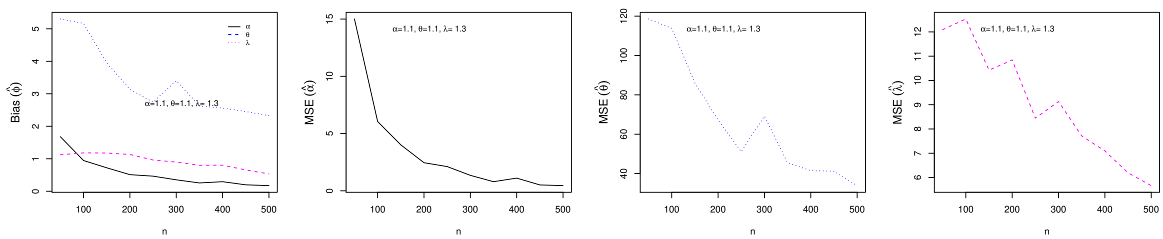


**Figure 10:** Simulation results from $TIHL_{NH}$ for parameter $\alpha = 1.1,\ \beta = 1.1,\ \lambda = 1.3$

# 4. Applications

In this section, we illustrate the superiority of the new distribution as compared with some other existing distributions using two real data applications. The models parameters are estimated by maximum likelihood technique. The competing distributions include the Nadarajah-Haghighi (NH) Nadarajah and Haghighi (2011), exponentiated Nadarajah-Haghighi (ENH) Lemonte (2013); Abdul-Moniem (2015), Kumaraswamy Nadarajah-Haghighi (KwNH) Lima (2015), generalized half logistic Poisson (GHLP) Muhammad (2017b), half logistic Poisson

(HLP) Muhammad and Yahaya (2017), Beta Nadarajah-Haghighi (BNH) Cícero, Alizadeh, and Cordeiro (2016), generalized exponential (GE) Gupta and Kundu (1999), beta exponential (BE) Nadarajah and Kotz (2006), beta Erlang-truncated exponential (BETE) Shrahili, Elbatal, Muhammad, and Muhammad (2021), generalized half logistic (GHL) Kantam, Ramakrishna, and Ravikumar (2013).

We compare the fitted models by the model selection criteria known as the Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC). Moreover, the goodness of fit statistics known as the Anderson-Darling (A), Cramer-von Mises (W), and Kolmogorov Smirnov (KS) are used to identify which distribution describes the data better. The model with the smallest value of these measures fit the data better than the other models.

### 4.1. First real data

The first data consist of the daily new deaths due to COVID-19 in the California, USA from March 12, 2020 to September 30, 2020, extracted from `https://www.worldometers.info/coronavirus/usa/california/`: 1, 1, 1, 5, 4, 3, 4, 1, 10, 6, 11, 14, 17, 12, 25, 12, 14, 35, 30, 24, 41, 44, 28, 33, 54, 64, 61, 25, 46, 44, 53, 55, 80, 86, 89, 105, 28, 48, 73, 120, 103, 71, 91, 32, 58, 85, 75, 89, 80, 75, 24, 71, 92, 74, 81, 91, 64, 26, 61, 97, 89, 80, 104, 55, 79, 32, 103, 86, 106, 69, 71, 31, 19, 43, 102, 82, 97, 74, 27, 47, 72, 61, 63, 72, 66, 29, 23, 95, 97, 71, 47, 72, 27, 30, 85, 79, 74, 65, 67, 24, 48, 69, 96, 79, 64, 33, 32, 42, 104, 82, 98, 63, 29, 19, 75, 118, 150, 137, 102, 73, 26, 46, 138, 125, 127, 121, 91, 12, 57, 119, 155, 156, 134, 90, 27, 92, 169, 175, 113, 191, 136, 38, 108, 196, 169, 148, 188, 103, 67, 87, 182, 160, 186, 151, 75, 19, 98, 179, 164, 134, 166, 146, 18, 104, 149, 142, 140, 144, 67, 35, 80, 144, 157, 167, 152, 65, 22, 33, 72, 154, 99, 172, 71, 52, 75, 152, 105, 90, 99, 73, 31, 53, 123, 117, 88, 132, 51, 21, 34, 150, 107.

The MLEs with their standard error in parenthesis and these measures based on each model are provided in the table 3, an approximate information matrix ($\mathbf{I}_n$) for the $TIHL_{NH}$ of the first data is obtained using the first five hundred terms of each $C(.,.,.,.,.)$; the computation of the $C(.,.,.,.,.)$ required `numDeriv, matlib` and `Rmpfr` packages in `R 3.5.3`. The results show that $TIHL_{NH}$ provides a better representation of the data better than the other distributions. Thus, $TIHL_{NH}$ can be considered as an alternative model for studying COVID-19 data and other statistical analysis in various fields of applied statistics. Figure 11 shows the plots of the histogram with the fitted $TIHL_{NH}$, ENH and NH models (left), and empirical cdf with fitted $TIHL_{NH}$, ENH and NH cdfs (right) for the California data. Figure 12 is quantile-quantile plots of the $TIHL_{NH}$, ENH and NH model for the California COVID-19 data.

**Table 3:** MLEs, model selection measures and goodness of fit measures of the competing models for the first data

| Model | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | $\hat{\lambda}$ | $\hat{a}$ | $\hat{b}$ | $L$ | AIC | BIC | CAIC | KS | AD | CVM |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $TIHL_{NH}$ | 8.0991(2.1267) | - | 0.0012(0.0005) | 1.0509(0.3450) | - | - | -1046.4 | 2098.8 | 2108.71 | 2098.92 | 0.0480 | 0.5921 | 0.0923 |
| ENH | 40.1720(1.8230) | 1.2320(0.1017) | - | $2.097\times10^{-4}(1.35\times10^{-5})$ | - | - | -1047.51 | 2101.02 | 2110.93 | 2101.14 | 0.0712 | 0.6834 | 0.1114 |
| NH | 50.3370(5.089) | - | - | $1.535\times10^{-4}(1.63\times10^{-5})$ | - | - | -1050.45 | 2104.90 | 2111.51 | 2104.96 | 0.1162 | 0.6971 | 0.1137 |
| KwNH | 5.2740(0.0202) | 0.0039(0.0006) | - | - | 0.9396(0.0176) | 0.2392(0.0702) | -1046.02 | 2100.04 | 2113.25 | 2100.24 | 0.0518 | 0.6110 | 0.0983 |
| BNH | 7.5441(0.2219) | 0.0023(0.00045) | - | - | 0.9700(0.1514) | 0.2988(0.1259) | -1045.97 | 2099.95 | 2113.16 | 2100.15 | 0.0536 | 0.6261 | 0.1022 |
| GHLP | $2.11\times10^{-2}(1.26\times10^{-3})$ | 1.3990(0.1252) | - | $1.236\times10^{-7}(2.480\times10^{-3})$ | - | - | -1055.73 | 2117.46 | 2127.37 | 2117.58 | 0.0952 | 1.6335 | 0.2674 |
| HLP | $5.926\times10^{-4}(5.805\times10^{-5})$ | - | - | 42.9800(2.906) | - | - | -1078.06 | 2160.12 | 2166.72 | 2160.18 | 0.1812 | 2.4722 | 0.4088 |
| GE | 1.7752(0.1760) | - | - | 0.0178(0.0014) | - | - | -1062.81 | 2129.62 | 2136.23 | 2129.69 | 0.1218 | 2.6975 | 0.4472 |
| BE | - | - | - | 0.0028(0.0005) | 1.7861(0.1663) | 7.6492(1.1412) | -1060.93 | 2127.86 | 2137.77 | 2127.98 | 0.1157 | 2.4367 | 0.4032 |
| BETE | 0.0348(2.7391) | - | 0.0883(7.2763) | - | 1.7875(0.1675) | 7.3683(1.6036) | -1060.93 | 2129.89 | 2143.08 | 2130.07 | 0.1146 | 2.4375 | 0.4033 |

$$\mathbf{I}_n = \begin{pmatrix} 0.69837072 & 455.9375 & -0.08891666 \\ 455.93750965 & 3252307.3777 & 235.66760974 \\ -0.08891666 & 235.6676 & 1.79285964 \end{pmatrix},$$

and

$$\mathbf{I}_n^{-1} = \begin{pmatrix} 1.59724534 & -0.00023186 & 0.10969332 \\ -0.00023186 & 0.00000034 & -0.00005673 \\ 0.10969332 & -0.00005673 & 0.57066530 \end{pmatrix}$$

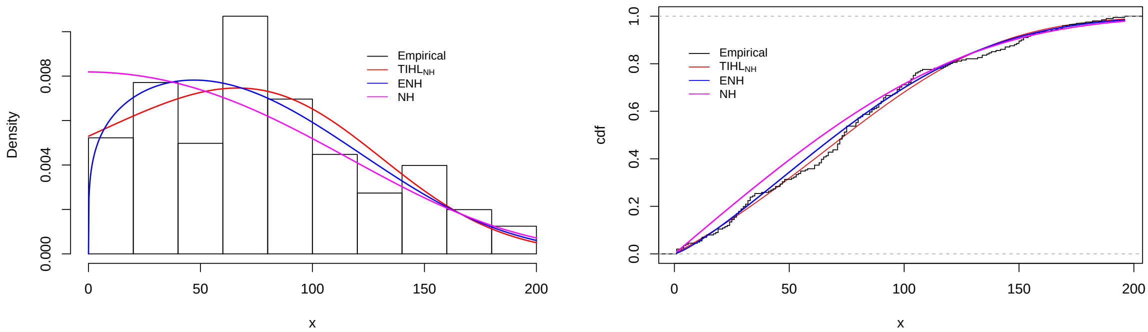


**Figure 11:** Plots of the histogram with the fitted $TIHL_{NH}$, ENH and NH (left), and empirical cdf with fitted $TIHL_{NH}$, ENH and NH cdfs (right) for the first data

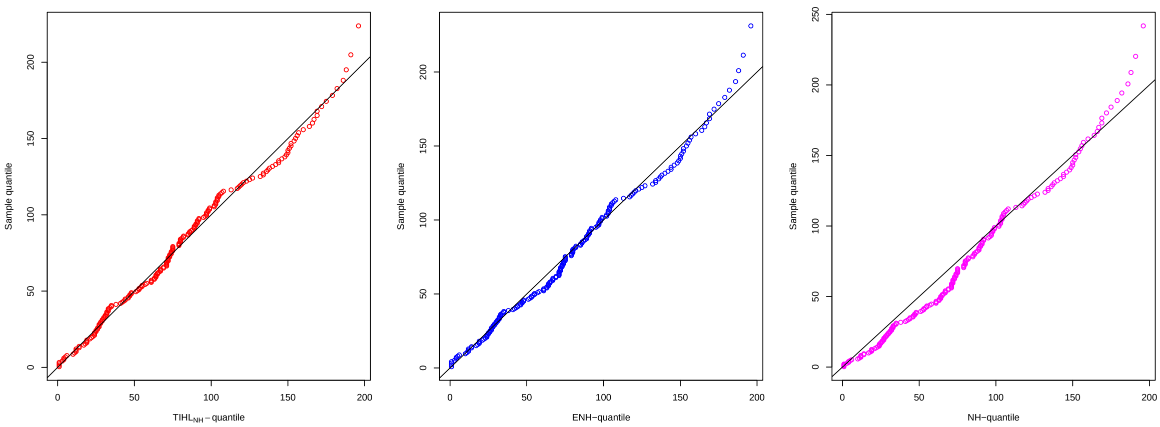


**Figure 12:** Quantile-quantile plots of the $TIHL_{NH}$ (left), ENH (middle) and NH (right) for the first data

## 4.2. Second real data

The data set can be found in Cordeiro, dos Santos Brito *et al.* (2012) also studied by Jamal and Chesneau (2019); Muhammad and Liu (2021a), it is the total milk production in the first birth of 107 cows from SINDI race. 0.4365, 0.4260, 0.5140, 0.6907, 0.7471, 0.2605, 0.6196, 0.8781, 0.4990, 0.6058, 0.6891, 0.5770, 0.5394, 0.1479, 0.2356, 0.6012, 0.1525, 0.5483, 0.6927, 0.7261, 0.3323, 0.0671, 0.2361, 0.4800, 0.5707, 0.7131, 0.5853, 0.6768, 0.5350, 0.4151, 0.6789, 0.4576, 0.3259, 0.2303, 0.7687, 0.4371, 0.3383, 0.6114, 0.3480, 0.4564, 0.7804, 0.3406, 0.4823, 0.5912, 0.5744, 0.5481, 0.1131, 0.7290, 0.0168, 0.5529, 0.4530, 0.3891, 0.4752, 0.3134, 0.3175, 0.1167, 0.6750, 0.5113, 0.5447, 0.4143, 0.5627, 0.5150, 0.0776, 0.3945, 0.4553, 0.4470, 0.5285, 0.5232, 0.6465, 0.0650, 0.8492, 0.8147, 0.3627, 0.3906, 0.4438, 0.4612, 0.3188, 0.2160, 0.6707, 0.6220, 0.5629, 0.4675, 0.6844, 0.3413, 0.4332, 0.0854, 0.3821, 0.4694, 0.3635, 0.4111, 0.5349, 0.3751, 0.1546, 0.4517, 0.2681, 0.4049, 0.5553, 0.5878, 0.4741, 0.3598, 0.7629, 0.5941, 0.6174, 0.6860, 0.0609, 0.6488, 0.2747.

The computed values of the MLEs (with their standard error in parenthesis), the model selection and goodness of fit measures of each model are provided in the table 4, an approximate information matrix ($\mathbf{I}_n$) for the $TIHL_{NH}$ of the second data is obtained using the first eight hundred terms of each $C(.,.,.,.,.)$ in similar way. The results show that $TIHL_{NH}$ represent the data better than the other competing models. Figure 13 provides the plots of the histogram with the fitted $TIHL_{NH}$, ENH and NH models (left), and empirical cdf with fitted $TIHL_{NH}$, ENH and NH cdfs (right) for the second data. Figure 14 shows the quantile-quantile plots of the $TIHL_{NH}$, ENH and NH model for the second data.

**Table 4:** MLEs, model selection measures and goodness of fit measures of the competing models for the second

| Model | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | $\hat{\lambda}$ | $\hat{a}$ | $\hat{b}$ | $L$ | AIC | BIC | CAIC | KS | AD | CVM |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $TIHL_{NH}$ | $21.120(1.93\times10^{-4})$ | - | $0.2083(3.501\times10^{-5})$ | 0.1708(0.0137) | - | - | 29.05 | -52.10 | -44.08 | -51.87 | 0.0666 | 0.2220 | 0.0321 |
| ENH | 39.4540(6.2311) | 2.3557(0.2884) | - | 0.0499(0.0085) | - | - | 21.12 | -36.23 | -28.21 | -35.99 | 0.1103 | 1.6526 | 0.2544 |
| NH | 1499.0(6.869) | - | - | $0.0010(6.236\times10^{-5})$ | - | - | 0.77 | 2.47 | 7.82 | 2.58 | 0.2934 | 1.6667 | 0.2572 |
| KwNH | 33.5664(3.8162) | 0.0133(0.0026) | - | - | 2.5919(0.2197) | 44.2524(10.3243) | 21.10 | -34.16 | -23.46 | -33.76 | 0.0875 | 1.5815 | 0.2419 |
| BNH | $11.400(1.122\times10^{-4})$ | $0.4185(1.790\times10^{-6})$ | - | - | 1.4460(0.3270) | 0.1202(0.0123) | 25.95 | -43.89 | -33.20 | -43.50 | 0.0974 | 0.7453 | 0.1097 |
| GHLP | 4.7220(0.0174) | 2.6520(0.2564) | - | $3.896\times10^{-7}(0.0076)$ | - | - | 10.32 | -14.64 | -6.62 | -14.40 | 0.1282 | 3.4143 | 0.5515 |
| GHL | 4.7771(0.3662) | - | 2.7077(0.3851) | - | - | - | 10.33 | -16.66 | -11.32 | -16.55 | 0.1291 | 3.4359 | 0.5552 |
| GE | 3.7139(0.5661) | - | - | 4.2007(0.3731) | - | - | 5.04 | -6.08 | -0.73 | -5.96 | 0.1477 | 4.3696 | 0.7257 |
| BE | - | - | - | 0.2231(0.0412) | 3.6924(0.6496) | 33.9775(0.1532) | 9.41 | -12.82 | -4.80 | -12.59 | 0.1364 | 3.6789 | 0.6014 |
| BETE | 0.1747(0.0171) | - | 9.0982(5.5931) | - | 3.6910(0.3261) | 43.7567(0.6006) | 9.42 | -10.83 | -0.14 | -10.44 | 0.1364 | 3.6780 | 0.6013 |

$$\mathbf{I}_n = \begin{pmatrix} 26.3351389 & 60.320096 & -0.6639823 \\ 60.3200958 & 427.272417 & 1.0840329 \\ -0.6639823 & 1.084033 & 0.1314848 \end{pmatrix},$$

and

$$\mathbf{I}_n^{-1} = \begin{pmatrix} 0.07915026 & -0.01244849 & 0.50233131 \\ -0.01244849 & 0.00434828 & -0.09871298 \\ 0.50233131 & -0.09871298 & 10.95599680 \end{pmatrix}$$

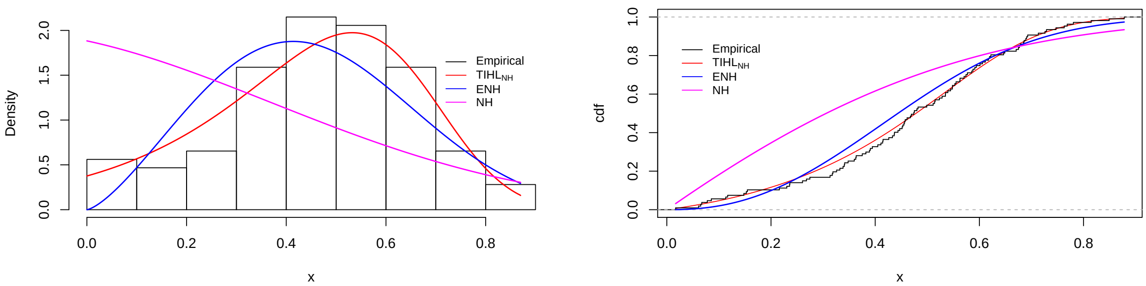


**Figure 13:** Plots of the histogram with the fitted $TIHL_{NH}$, ENH and NH (left), and empirical cdf with fitted $TIHL_{NH}$, ENH and NH cdfs (right) for the second data

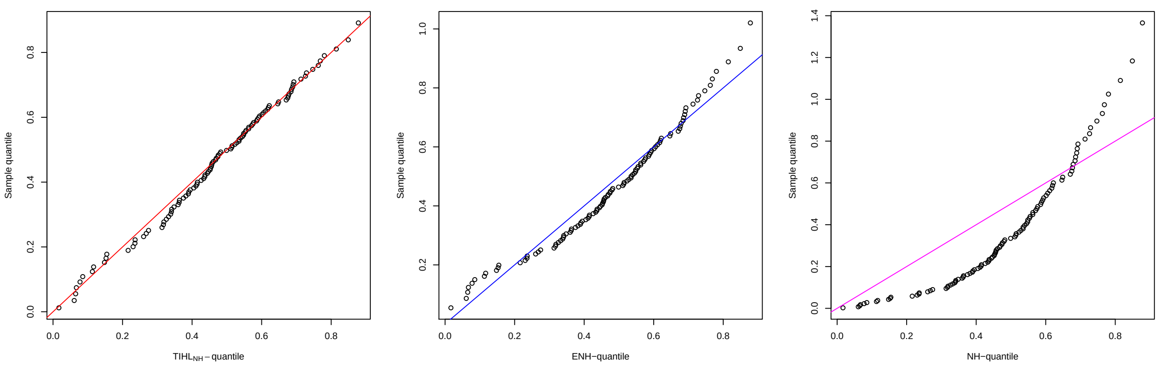


**Figure 14:** Quantile-quantile plots of the $TIHL_{NH}$ (left), ENH (middle) and NH (right) for the second data

## 5. Conclusion

In this paper, we proposed and studied a new lifetime distribution called type I half-logistic Nadarajah-Haghighi. We investigate some of its important mathematical and statistical properties such as the $r^{th}$ moment, quantile function, mean deviations, Bonferroni curve, and Lorenz curve. The Shannon entropy and Renyi entropy are discussed; the Kullback-Leibler divergence measure is computed. The model parameters estimation was conducted by the maximum likelihood method. Simulation studies is used to highlight the consistency of the MLEs using some adequate samples, we consider bias, and MSE of the estimators and the result was satisfactory. Finally, we assessed the performance of the new model by fitting it to two real data set in which one of them is a daily new deaths due to COVID-19 in the California, USA. In all the two data set the $TIHL_{NH}$ provided better representation than the other competing distributions.

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